# From kinetic to fluid descriptions of plasmas 

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## Outline:

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Materials from chapters I, II and VI.a are compiled from the books of Krall and Trivelpiece,
Akhiezer et al. and Nicholson.

## I. Equation for the one-point distribution function

Consider a plasma composed of $N$ particles ( $N_{r}$ particles of species $r$ ) at positions $x_{i}$, with velocities $v_{i}$ and total acceleration $a_{i}^{T}=\frac{q_{r_{i}}}{m_{r_{i}}}\left(E^{T}\left(x_{i}, t\right)+\frac{v_{i}}{c} \times B^{T}\left(x_{i}, t\right)\right)$. The density of the system in the 6 N -dimensional phase space

$$
\mathcal{F}\left(x_{1}, x_{2}, \cdots, x_{N}, v_{1}, v_{2}, \cdots, v_{N}, t\right)=\prod_{j=1}^{N} \delta\left(x_{i}-X_{i}(t)\right) \delta\left(v_{i}-V_{i}(t)\right)
$$

obeys Liouville equation

$$
\frac{\partial \mathcal{F}}{\partial t}+\sum_{i=1}^{N}\left(\frac{\partial \mathcal{F}}{\partial x_{i}} v_{i}+\frac{\partial \mathcal{F}}{\partial v_{i}} a_{i}^{T}\right)=0
$$

together with Maxwell's equations

$$
\begin{aligned}
& \nabla \cdot E^{T}(x, t)=4 \pi \rho_{q}^{T}(x, t) \\
& \nabla \cdot B^{T}(x, t)=0 \\
& \nabla \times E^{T}(x, t)=-\frac{1}{c} \frac{\partial B^{T}(x, t)}{\partial t} \\
& \nabla \times B^{T}(x, t)=\frac{4 \pi}{c} J^{T}(x, t)+\frac{1}{c} \frac{\partial E^{T}(x, t)}{\partial t} .
\end{aligned}
$$

Liouville equation rewrites in conservative form

$$
\frac{\partial \mathcal{F}}{\partial t}+\sum_{i=1}^{N}\left(\nabla_{x_{i}} \cdot\left(v_{i} \mathcal{F}\right)+\nabla_{v_{i}} \cdot\left(a_{i}^{T} \mathcal{F}\right)\right)=0
$$

The same equation holds for the $N$-particle probability density function $F$ which obeys $\int F d x_{1} \cdots d x_{N}, d v 1, \cdots d v_{N}=1$.

The one-particle probability distribution (or distribution function) is defined by
$n_{r} F_{r}^{(1)}\left(x_{1}, v_{1}, t\right)=N_{r} \int F\left(x_{1}, x_{2}, \cdots, x_{N}, v_{1}, v_{2}, \cdots, v_{N}, t\right) d x_{2}, \cdots, d x_{N}, d v_{2}, \cdots, d v_{N}$
with $n_{r}=\frac{N_{r}}{V}, V$ being the finite spatial volume occupied by the system.

The two-point probability density function is defined as

$$
\begin{aligned}
& n_{r} n_{s} F_{r s}^{(2)}\left(x_{1}, x_{2}, v_{1}, v_{2}, t\right)= \\
& N_{r} N_{s} \int F\left(x_{1}, x_{2}, \cdots, x_{N}, v_{1}, v_{2}, \cdots, v_{N}, t\right) d x_{3}, \cdots, d x_{N}, d v_{3}, \cdots, d v_{N}
\end{aligned}
$$

The equation satisfied by the one-body distribution function is obtained by integrating Liouville equation over the spatial and velocity coordinates of all but one particle; it reads

$$
\frac{\partial\left(n_{r} F_{r}^{(1)}\right)}{\partial t}+v_{1} \cdot \frac{\partial}{\partial x_{1}}\left(n_{r} F_{r}^{(1)}\right)+N_{r} \int a_{1}^{T} \cdot \frac{\partial F}{\partial v_{1}} d x_{2}, \cdots, d x_{N}, d v_{2}, \cdots, d v_{N}=0
$$

The acceleration on the $i^{\text {th }}$ particle can be written

$$
a_{i}^{T}=\frac{q_{r_{i}}}{m_{r_{i}}}\left(E^{E}\left(x_{i}, t\right)+\frac{v_{i}}{c} \times B^{E}\left(x_{i}, t\right)\right)+\sum_{j} a_{i j}=a_{1}^{E}+\sum_{j} a_{i j}
$$

so that the last term of the I.h.s. reads

$$
a_{1}^{E} \frac{\partial}{\partial v_{1}}\left(n_{r} F_{r}^{(1)}\right)+\frac{N_{r}-1}{N_{r}} \sum_{s} \int a_{1 s} \cdot \frac{\partial}{\partial v_{1}} n_{r} n_{s} F_{r s}^{(2)}\left(x_{1}, x_{s}, v_{1}, v_{s}, t\right) d x_{s} d v_{s}
$$

where the summation holds over the type of particle species and since all particles of the same type are identical.

Defining the two-point correlation function $G_{r s}\left(x_{1}, x_{2}, v_{1}, v_{2}, t\right)$ as

$$
F_{r s}^{(2)}\left(x_{1}, x_{2}, v_{1}, v_{2}, t\right)=F_{r}^{(1)}\left(x_{1}, v_{1}, t\right) F_{s}^{(1)}\left(x_{2}, v_{2}, t\right)+G_{r s}\left(x_{1}, x_{2}, v_{1}, v_{2}, t\right)
$$

which constitutes the first step in the Mayer cluster expansion, one has

$$
\begin{aligned}
& \int \sum_{s} a_{1 s} \cdot \frac{\partial}{\partial v_{1}} n_{r} n_{s} F_{r s}^{(2)}\left(x_{1}, x_{s}, v_{1}, v_{s}, t\right) d x_{s} d v_{s}= \\
& n_{r} \frac{\partial}{\partial v_{1}}\left(\int \sum_{s}\left(a_{1 s} n_{s} F_{s}^{(1)}\left(x_{s}, v_{s}, t\right) d x_{s} d v_{s}\right) F_{r}^{(1)}\left(x_{1}, v_{1}, t\right)\right) \\
& +\int \sum_{s} \frac{\partial}{\partial v_{1}} n_{r} n_{s} G_{r s}\left(x_{1}, x_{s}, v_{1}, v_{s}, t\right) d x_{s} d v_{s}
\end{aligned}
$$

The quantity $\int \sum_{s} a_{1 s} n_{s} F_{s}^{(1)}\left(x_{s}, v_{s}, t\right) d x_{s} d v_{s}$ is in fact the ensemble average acceleration experienced by particle 1 due to all other particles. Assuming $N_{r}$ large one finally obtains

$$
\frac{\partial F_{r}^{(1)}}{\partial t}+v_{1} \cdot \frac{\partial}{\partial x_{1}} F_{r}^{(1)}+\frac{q_{r}}{m_{r}}\left\langle E+\frac{v_{1} \times B}{c}\right\rangle \cdot \frac{\partial}{\partial v_{1}} F_{r}^{(1)}=\left.\frac{\partial F_{r}^{(1)}}{\partial t}\right|_{c}
$$

where the $\langle E\rangle$ and $\langle B\rangle$ are the self-consistent fields, sum of external and average internal fields, which satisfy average Maxwell equations

$$
\begin{aligned}
& \nabla \cdot<E>=4 \pi \sum_{s} n_{s} q_{s} \int F_{s}^{(1)} d v+4 \pi \rho_{q}^{E} \\
& \nabla \times<B>=\frac{4 \pi}{c} \sum_{s} n_{s} q_{s} \int v F_{s}^{(1)} d v+\frac{4 \pi}{c} J^{E}+\frac{1}{c} \frac{\partial<E>}{\partial t}
\end{aligned}
$$

and where the collision term reads

$$
\left.\frac{\partial F_{r}^{(1)}\left(x_{1}, v_{1}, t\right)}{\partial t}\right|_{c}=-\int \sum_{s} \frac{\partial}{\partial v_{1}} n_{s} G_{r s}\left(x_{1}, x_{s}, v_{1}, v_{s}, t\right) d x_{s} d v_{s}
$$

To simplify notations, $F_{r}^{(1)}$ will be denoted $f_{r}$ and the symbols $<.>$ will be dropped.

- From Liouville equation, it is possible to obtain an infinite chain of statistical equations for the n-point distribution functions, the so-called BBGKY hierarchy.
- This hierarchy can be closed if the ratio $g$ of the average interaction energy to the average kinetic energy $g=<q^{2} / r_{12}>/<\frac{1}{2} m v^{2}>$ is small enough. This plasma parameter can be rewritten $g \propto \frac{q^{2} n_{0}^{1 / 3}}{T} \approx \frac{1}{n_{0} r_{D}^{3}}$ where $r_{D}=\sqrt{\frac{T}{4 \pi n_{0} q^{2}}}$ is the Debye (or screening) length.
- This means that if there are enough particles in the Debye sphere around a given test particle, Coulomb interaction is efficiently shieldied and becomes short-range.

Derivation of the Debye length: consider a static electron/ion cloud in the viscinity of a test particle of charge $q_{T}$. The electric potential $\phi$ obeys $\nabla^{2} \phi=-4 \pi \sum_{i} q_{i} n_{i}-4 \pi q_{T} \delta(\mathbf{r})$. At thermodynamic equilibrium, $n_{i}=n_{i 0} \exp \left(-q_{i} \phi / T\right)$ with $q_{i} \phi / T \ll 1$. The equation for the electric potential becomes $\nabla^{2} \phi-\phi / \lambda_{D}^{2}=0$ leading to $\phi \propto \frac{q_{T}}{r} \exp \left(-r / \lambda_{D}\right)$ with $\lambda_{D}^{-2}=r_{D e}^{-2}+r_{D i}^{-2}$.

Under the assumption that three-particle correlations are negligible, that the plasma is spatially homogeneous and that the two-particle correlation function relaxes much faster than the one-point distribution function, one obtains the Lenard-Balescu equation, which is still quite involved. Further simplifications allow to turn the Lenard-Balescu equation into a Fokker-Planck equation in the form

$$
\frac{\partial f_{r}}{\partial t}=-\nabla_{v} \cdot\left[A f_{r}(v)\right]+\frac{1}{2} \nabla_{v} \nabla_{v}:\left[\mathbf{B} f_{r}(v)\right]
$$

where $A$ is the coefficient of dynamic friction and $\mathbf{B}$ the diffusion coefficient. The two effects tend to balance each other and compensate exactly for a Maxwellian distribution $F_{M}(v)=\left(\frac{m}{2 \pi T}\right)^{3 / 2} \exp \left(-\frac{m v^{2}}{2 T}\right)$. An even cruder approximation for the collision integral, is given by the Krook collision term

$$
\left.\frac{\partial f_{r}}{\partial t}\right|_{c}=-\frac{1}{\tau}\left(f-F_{M}\right)
$$

This approximation however fails to reproduce the fact that distribution tails relax much later than their central part.

A precise form of $G_{r s}$ will not be needed in what follows.

- Correlations vanish when the one-point distribution function is a Maxwellian. In the presence of collisions and in absence of external fields, an initial distribution evolves over long times towards a Maxwellian. This relaxation takes place over a time scale of order $\tau \approx l / v_{t h}$ where $l$ is the mean free path and $v_{t h}=(T / m)^{1 / 2}$ the thermal velocity.
- This relaxation time-scale is proportional to the square root of the particle mass. Since $\tau_{i i}=\sqrt{\frac{m_{i}}{m_{e}}} \tau_{e e}$ and $\tau_{e i}=\frac{m_{i}}{m_{e}} \tau_{e e}$, the electron population will relax first, then the ions and the equilibrium between ions and electrons will occur last. As a consequence, separate temperatures can be defined for the different species.
- In the case where collisions can be totally neglected, the distribution function obeys the so-called Vlasov (or collisionless Boltzmann) equation

$$
\frac{\partial f_{r}}{\partial t}+v \cdot \frac{\partial}{\partial x} f_{r}+\frac{q_{r}}{m_{r}}\left(E+\frac{v \times B}{c}\right) \cdot \frac{\partial}{\partial v} f_{r}=0
$$

This model is valid when the frequency $(\omega)$ and wavenumber $(k)$ of the typical oscillations satisfy $\omega \tau \gg 1$ and $k v_{t h} \tau \gg 1$.

- Under the Vlasov equation, the time rate of change of the total number of particle is zero $\frac{\partial}{\partial t} \int n_{r} f_{r} d x d v=0$, and if $f_{r}(x, v, t=0)>0$ for all $x$ and $v$, it will remain so for all times.
- Vlasov equation has also many equilibrium solutions, more than the exact (collisional) equation. In addition to the Maxwellian distribution, any function $f_{r}(a(x, v), b(x, v), \cdots)$ is a solution if $a(x, v), b(x, v) \ldots$ are constants of the motion of particles.


## II. Fluid description

## II.a Scale considerations

The validity of the hydrodynamic approximation a priori depends on the ratio between a typical scale of perturbation and a scale of the system above which particle get sufficiently "mixed".
(a) When there is no ambient magnetic field: $B_{0}=0$.
$l=\frac{v_{t h}}{\nu_{l}}$ : mean free path
$L=\frac{v_{t h}}{\omega}$ : distance that a particule travels during a wave period. where $\nu$ is the collision frequency and $\omega$ and $\lambda$ the typical frequency and wavelength of a wave-like perturbation.
A local hydrodynamic theory can be valid:

- (i) If $\lambda \gg l$, i.e. when there are many collisions in a wave period $(\nu \gg \omega)$
- (ii) If $\lambda \gg L$, i.e. when there are almost no collisions $(\nu \ll \omega)$ but particles move very slowly (cold plasma).

In case (i), there is no condition on $v_{\phi}=\lambda \omega$ and there is no closure problem since collisions ensure that the distribution function stays close to a Maxwellian. In that case, the heat flux (third order moment) is zero.

In case (ii) $v_{\phi} \gg v_{t h}$ : a hydrodynamic theory can only describe "rapid" waves. An adiabatic theory is here possible. If waves travel more slowly resonant particles have to be taken into account.
(b) When there is an ambient magnetic field: $B_{0} \neq 0$.

In the direction along the ambient field, the conditions are the same as in absence of magnetic field.
In the direction perpendicular to the magnetic field, a third length has to be considered, namely the particle gyration radius $r_{b}=\frac{v_{t h}}{\Omega}$ where $\Omega=\frac{q B_{0}}{m c}$ is the particle gyrofrequency.

- (i) If $r_{b} \gg l$ (i.e. $\Omega \ll \nu$ ), the magnetic field plays almost no role. Parallel and perpendicular directions are not decoupled and the conditions for the validity of a hydrodynamic approximation are the same as in the absence of magnetic field.
- (ii) If $r_{b} \ll l$ (i.e. $\Omega \gg \nu$ ), collisions are infrequent and $r_{b}$ plays the role of the mean free path in the perpendicular direction. Typical perturbation scales have to be separated into a parallel $\left(\lambda_{\|}\right)$and a perpendicular $\left(\lambda_{\perp}\right)$ wavelength and the conditions of validity for a hydrodynamic theory now read:

$$
\begin{aligned}
& \lambda_{\|} \gg l \text { if } \omega \ll \nu \\
& \text { or } \\
& \lambda_{\|} \gg L \text { if } \omega \gg \nu \\
& \lambda_{\perp} \gg r_{b} \text { if } \omega<\Omega \\
& \text { or } \\
& \lambda_{\perp} \gg L \text { if } \omega>\Omega
\end{aligned}
$$

There are thus three possibilities:
(1) $\omega>\Omega \gg \nu$

In this case the conditions read $v_{\phi \|} \gg v_{t h}$ and $v_{\phi \perp} \gg v_{t h}$; this is the case of a cold plasma.
(2) $\Omega>\omega \gg \nu$

In this case the conditions read $v_{\phi \|} \gg v_{t h}$ and $k_{\perp} r_{b} \ll 1$. Perpendicular scales have to be much larger than the inertial length.
(3) $\Omega \gg \nu>\omega$

There is no condition on $v_{\phi \|}$ but it is necessary to have $k_{\perp} r_{b} \ll 1$.

## II.a Derivation of the bi-fluid equations

In the general case, we write the full system of kinetic and Maxwell equations in the form

$$
\begin{aligned}
& \partial_{t} f_{r}+v \cdot \nabla f_{r}+\frac{q_{r}}{m_{r}}\left(e+\frac{1}{c} v \times b\right) \cdot \nabla_{v} f_{r}=\left.\partial_{t} f_{r}\right|_{c} \\
& \frac{1}{c} \partial_{t} b=-\nabla \times e \\
& \nabla \times b=\frac{4 \pi}{c} \sum_{r} q_{r} n_{r} \int v f_{r} d^{3} v+\frac{1}{c} \partial_{t} e \\
& \nabla \cdot e=4 \pi \sum_{r} q_{r} n_{r} \int f_{r} d^{3} v,
\end{aligned}
$$

where $f_{r}$ and $n_{r}$ are the distribution function and the average number density of the particle of species $r$ with charge $q_{r}$ and mass $m_{r}$. The displacement current $\frac{1}{c} \partial_{t} e$ turns out to be negligible in the present analysis.

Let us write the kinetic equation for electrons and ions as:

$$
\begin{aligned}
& \mathcal{L}_{e}\left(f_{e}\right)=\mathcal{I}_{e e}\left(f_{e}, f_{e}\right)+\mathcal{I}_{e i}\left(f_{e}, f_{i}\right) \\
& \mathcal{L}_{i}\left(f_{i}\right)=\mathcal{I}_{i e}\left(f_{i}, f_{e}\right)+\mathcal{I}_{i i}\left(f_{i}, f_{i}\right)
\end{aligned}
$$

where $\mathcal{L}_{r}\left(f_{r}\right)$ denotes the I.h.s. and $\mathcal{I}_{r s}$ is the part of the collision integral for particle $r$ that describes collision between particles $r$ and $s$. We have here assumed bilinearity of the collision operator. There is a logarithmic departure to bilinearity due to the long range nature of the Coulomb interaction. The conservation of the number of particles, momentum and energy imply that

$$
\begin{aligned}
& \left.\int \frac{\partial f_{r}}{\partial t}\right|_{c} d v_{r}=0, \quad r=e, i \\
& \left.\sum_{r} \int m_{r} v_{r} \frac{\partial f_{r}}{\partial t}\right|_{c} d v_{r}=\left.0 \quad \sum_{r} \int \frac{1}{2} m_{r} v_{r}^{2} \frac{\partial f_{r}}{\partial t}\right|_{c} d v_{r}=0
\end{aligned}
$$

We also have

$$
\int m_{r} v_{r} \mathcal{I}_{r r}\left(f_{r}, f_{r}\right) d v=0, \quad \int \frac{1}{2} m_{r} v_{r}^{2} \mathcal{I}_{r r}\left(f_{r}, f_{r}\right) d v=0
$$

- From Vlasov-Maxwell equations, derive a hierarchy of moment equations for each particle species $r$ :
density $\rho_{r}=m_{r} n_{r} \int f_{r} d^{3} v$
hydrodynamic velocity $u_{r}=\frac{\int v f_{r} d^{3} v}{\int f_{r} d^{3} v}$
pressure tensor $P_{r}=m_{r} n_{r} \int\left(v-u_{r}\right) \otimes\left(v-u_{r}\right) f_{r} d^{3} v$
heat flux tensor $Q_{r}=m_{r} n_{r} \int\left(v-u_{r}\right) \otimes\left(v-u_{r}\right) \otimes\left(v-u_{r}\right) f_{r} d^{3} v$.

$$
\begin{aligned}
& \partial_{t} \rho_{r}+\nabla \cdot\left(\rho_{r} u_{r}\right)=0 \\
& \rho_{r}\left(\partial_{t} u_{r}+u_{r} \cdot \nabla u_{r}\right)+\nabla \cdot \mathbf{P}_{r}-q_{r} n_{r}\left(e+\frac{1}{c} u_{r} \times b\right)=R_{r} \\
& \partial_{t} \mathbf{P}_{r}+u_{r} \cdot \nabla \mathbf{P}_{r}+\mathbf{P}_{r} \nabla \cdot u_{r}+\nabla \cdot \mathbf{Q}_{r}+\left[\mathbf{P}_{r} \cdot \nabla u_{r}+\frac{q_{r}}{m_{r} c} b \times \mathbf{P}_{r}\right]^{\mathcal{S}}=\mathbf{H}_{r}
\end{aligned}
$$

where $[A]^{\mathcal{S}}=A+A^{\operatorname{Tr}}$
The extra term $R_{e}=-R_{i}$ determines the forces which act upon the electrons as a result of collisions with ions. Similarly the tensor $\mathbf{H}_{e}=m_{e} n_{e} \int\left(v-u_{e}\right) \otimes\left(v-u_{e}\right) \mathcal{I}_{e i}\left(f_{e}, f_{i}\right) d v=-\mathbf{H}_{i}$ is the amount of heat obtained by the electrons per unit time as a result of their collisions with ions.

When collisions dominate, simplifications can be made that lead to the so-called MHD equations. At dominant order, the distribution function is close to a Maxwellian. The anisotropy due to the ambient magnetic field is subdominant so that the pressure tensor can be written as $\mathbf{P}_{r}=p \mathbf{I}+\Pi_{r}$, where $\mathbf{I}$ denotes the unit tensor and $\Pi_{r}$ the generalized viscosity tensor of species $r$. The latter is by definition of zero trace and describes viscous stresses (proportional to viscosity coefficients times rate of strain tensors) but it also include terms which are quite unrelated to conventional viscosity (such as FLR terms). At large enough scales the viscous terms are negligible. The equation for the isotropic scalar pressure reads

$$
\frac{3}{2}\left(\partial_{t} p_{r}+u_{r} \cdot \nabla\right) p_{r}+\frac{5}{2} p_{r} \nabla \cdot u_{r}+\nabla \cdot q_{r}^{h}+\Pi_{r}: \nabla u_{r}=Q_{r}
$$

where $q_{r}^{h}=\frac{1}{2} m_{r} n_{r} \int\left(v-u_{r}\right)^{2}\left(v-u_{r}\right) f_{r} d^{3} v$ is the vector heat flux and $Q_{r}=\frac{1}{2} \mathbf{I}: \mathbf{H}_{r}$ the heat generated as a consequence of collisions with particles of other species.
One has $Q_{i}+Q_{e}=-R_{i}\left(u_{i}-u_{e}\right)$. It is necessary to find relations between $\Pi_{r}$, $q_{r}^{h}, R_{r}$ and $Q_{r}$ with $\tilde{n}_{r}, u_{r}$ and $T_{r}$ (where by definition $p_{r}=\tilde{n}_{r} T_{r}, \rho_{r}=\tilde{n}_{r} m_{r}$ ). These relations and the calculation of the transport coefficients in terms of microscopic quantities are presented e.g. in Braginskii.

## III. From a bi-fluid to a mono-fluid description

A mono-fluid description together with the additional approximation of neglecting electron inertia, allows the filtering of the small scales (e.g. associated with Langmuir waves).
Define the plasma velocity $u=\frac{1}{\rho} \sum_{r} \rho_{r} u_{r}$ where $\rho=\sum_{r} \rho_{r}$ and the pressure and heat flux tensors associated with each particle species in terms of the deviations from this barycentric velocity, in the form
$\mathbf{p}_{r}=m_{r} n_{r} \int(v-u) \otimes(v-u) f_{r} d^{3} v$ and
$\mathbf{q}_{r}=m_{r} n_{r} \int(v-u) \otimes(v-u) \otimes(v-u) f_{r} d^{3} v$. One has

$$
\mathbf{P}_{r}=\mathbf{p}_{r}-\rho_{r}\left(u-u_{r}\right) \otimes\left(u-u_{r}\right)
$$

and

$$
Q_{r i j k}=q_{r i j k}+p_{r i j}\left(u-u_{r}\right)_{k}+p_{r i k}\left(u-u_{r}\right)_{j}+p_{r j k}\left(u-u_{r}\right)_{i},
$$

One thus easily gets that

$$
\partial_{t} \rho+\nabla \cdot(u \rho)=0
$$

and

$$
\partial_{t}(\rho u)+\nabla \cdot(\rho u \otimes u)+\nabla \cdot \mathbf{p}-\frac{1}{c} j \times b=0,
$$

where $\mathbf{p}=\sum_{r} \mathbf{p}_{r}$ denotes the total pressure tensor and where the electric current $j=\sum_{r} q_{r} n_{r} \int v f_{r} d^{3} v=\sum_{r} \frac{q_{r}}{m_{r}} \rho_{r} u_{r}$ is given by $j=\frac{c}{4 \pi} \nabla \times b$. Note that the friction forces cancel each other and that the displacement current is here neglected.

For small amplitude MHD waves with long wave lengths, one can replace the pressure equation by

$$
\partial_{t} \mathbf{p}_{r}+\nabla \cdot\left(u \mathbf{p}_{r}+\mathbf{q}_{r}\right)+\left[\mathbf{p}_{r} \cdot \nabla u+\frac{q_{r}}{m_{r} c} b \times \mathbf{p}_{r}\right]^{\mathcal{S}}=\mathbf{H}_{r}
$$

The curent $j$ obeys

$$
\partial_{t} j+\nabla \cdot\left(\sum_{r} \frac{q_{r} \rho_{r}}{m_{r}} u_{r} \otimes u_{r}\right)+\sum_{r} \frac{q_{r}}{m_{r}} \nabla \cdot \mathbf{P}_{r}-\sum_{r} \frac{q_{r}^{2} \rho_{r}}{m_{r}^{2}}\left(e+\frac{1}{c} u_{r} \times b\right)=-\nu j .
$$

where $\nu j=-\sum_{r} \frac{q_{r}}{m_{r}} R_{r}$, with $\nu$ an average collision frequency. One has

$$
\begin{aligned}
& \partial_{t} j+\nabla \cdot\left(u \otimes j+j \otimes u-\sum_{r} \frac{q_{r} \rho_{r}}{m_{r}} u \otimes u\right)+\sum_{r} \frac{q_{r}}{m_{r}} \nabla \cdot \mathbf{p}_{r} \\
& \quad-\sum_{r} \frac{q_{r}^{2} \rho_{r}}{m_{r}^{2}} e-\frac{q^{2}}{c}\left(\frac{1}{m_{e}}+\frac{1}{m_{p}}\right) \frac{\rho u \times b}{m_{e}+m_{p}}+\frac{q}{c}\left(\frac{m_{p}}{m_{e}}-\frac{m_{e}}{m_{p}}\right) \frac{j \times b}{m_{e}+m_{p}}=-\nu j .
\end{aligned}
$$

Assuming $\frac{m_{e}}{m_{p}}$ and quasi-neutrality $\sum_{r} \frac{q_{r} \rho_{r}}{m_{r}}=0$, usual when considering slow motion of fluid elements of size greater than the Debye length, and writing $\rho_{r}=m_{r} n$ and $u \approx u_{p}$ one obtains

$$
\partial_{t} j+\nabla \cdot(u \otimes j+j \otimes u)-\frac{q^{2} n}{m_{e}}\left(e+\frac{u \times b}{c}-\frac{j \times b}{n q c}+\frac{1}{q n} \nabla \cdot \mathbf{p}_{e}\right)=-\nu j .
$$

For the weakly nonlinear dynamics of low frequency phenomena, the two first terms of the above equation are subdominant to leading order. From Maxwell equation, one then obtains the usual Hall-MHD approximation

$$
\partial_{t} b-\nabla \times(u \times b)=-\frac{c m_{p}}{q} \nabla \times\left(\frac{j \times b}{c \rho}-\frac{1}{\rho} \nabla \cdot \mathbf{p}_{e}+\frac{q}{m_{p}} \eta j\right) .
$$

The resistivity $\eta$ is related to $\nu$ by $\eta=\frac{\nu m_{e}}{n q^{2}}$. One thus has a closed system provided a closure approximation is made to express the heat fluxes. Simple closures are obtained in the isothermal or adiabatic cases where the pressure equations are replaced by $\frac{d}{d t} \frac{p}{\rho}$ or $\frac{d}{d t} \frac{p}{\rho^{\gamma}}$ respectively $\left(\frac{d}{d t}=\partial_{t}+u \cdot \nabla\right.$ denotes the convective derivative).

## IV. The pressure tensors

In the case where collisions are not enough to ensure isotropy of the pressure tensor, it is necessary to consider the equation for the full pressure tensor. This equations however involves time scales associated with the gyro-motion of the particles, making its numerical implementation difficult when larger scales are also included in the simulation.

This scale separation can in fact be used to define a reduced description where the evolution of the gyrotropic components of the pressure tensors is followed on hydrodynamic time scales, but where the non-gyrotropic ones, usually referred to as finite Larmor radius corrections, obey a slaved dynamics.

To isolate the gyrotropic components of the pressure tensor, it is convenient to write

$$
\mathbf{P}_{r} \times \widehat{b}-\widehat{b} \times \mathbf{P}_{r}=\mathbf{k}_{r}
$$

where

$$
\mathbf{k}_{r}=\frac{1}{\Omega_{r}} \frac{B_{0}}{|b|}\left[\frac{d \mathbf{P}_{r}}{d t}+\left(\nabla \cdot u_{r}\right) \mathbf{P}_{r}+\nabla \cdot \mathbf{Q}_{r}+\left(\mathbf{P}_{r} \cdot \nabla u_{r}\right)^{\mathcal{S}}\right] .
$$

In this equation, $B_{0}$ denotes the amplitude of the ambient field and $\Omega_{r}=\frac{q_{r} B_{0}}{m_{r} c}$ is the gyrotropic frequency of the particles of species $r$. The unit vector along the local magnetic field is $\widehat{b}=\frac{b}{|b|}$.

The usual procedure (Kulsrud 83) consists in performing an expansion in terms of $\Omega_{r}^{-1}$ considered as a small parameter. This approach may be conflicting with a subsequently performed weakly-nonlinear asymptotics that could mix the various orders in $1 / \Omega_{r}$.

We thus choose units such that the gyrokinetic frequency of the protons $\Omega_{p}$ is of order unity, and use a unique expansion parameter, which requires a specific relation between the small amplitudes and the long wave lengths of the perturbations.

The left hand side $\mathbf{P}_{r} \times \widehat{b}-\widehat{b} \times \mathbf{P}_{r}$ can be viewed as a self-adjoint linear operator acting on $\mathbf{P}_{r}$, whose kernel is spanned by the tensors ( $\left.\mathbf{I}-\widehat{b} \otimes \widehat{b}\right)$ and $\widehat{b} \otimes \widehat{b}$. Defining the projection on the image of this operator as

$$
\overline{\mathbf{a}}=\mathbf{a}-\frac{1}{2} \mathbf{a}:(\mathbf{I}-\widehat{b} \otimes \widehat{b})(\mathbf{I}-\widehat{b} \otimes \widehat{b})-(\mathbf{a}: \widehat{b} \otimes \widehat{b}) \widehat{b} \otimes \widehat{b}
$$

one has $\operatorname{tr} \overline{\mathbf{a}}=\mathrm{I}: \overline{\mathbf{a}}=0$ and $\overline{\mathbf{a}}: \widehat{b} \otimes \widehat{b}=0$.
The pressure tensor is written as the sum $\mathbf{P}_{r}=\mathbf{p}_{r}^{G}+\pi_{r}$ of an element of the kernel

$$
\begin{aligned}
\mathbf{p}_{r}^{G} & =\frac{1}{2} \mathbf{P}_{r}:(\mathbf{I}-\widehat{b} \otimes \widehat{b})(\mathbf{I}-\widehat{b} \otimes \widehat{b})+\left(\mathbf{P}_{r}: \widehat{b} \otimes \widehat{b}\right) \widehat{b} \otimes \widehat{b} \\
& \equiv p_{\perp r}(\mathbf{I}-\widehat{b} \otimes \widehat{b})+p_{\| r} \widehat{b} \otimes \widehat{b}
\end{aligned}
$$

and of a non-gyrotropic component $\boldsymbol{\pi}_{r}=\overline{\mathbf{P}}_{r}$ that thus satisfies $\operatorname{tr} \boldsymbol{\pi}_{r}=0$ and $\boldsymbol{\pi}_{r}: \widehat{b} \otimes \widehat{b}=0$.

## IV.a Dynamics of the gyrotropic pressures

To obtain the equations for the gyrotropic pressure components, one applies the trace and the contraction with $\widehat{b} \otimes \widehat{b}$ on both sides of the pressure tensor equation to get

$$
\mathrm{I}: \frac{d \mathbf{P}_{r}}{d t}+\left(\nabla \cdot u_{r}\right) \mathrm{I}: \mathbf{p}_{r}^{G}+\mathrm{I}:\left(\nabla \cdot \mathbf{Q}_{r}\right)+\mathrm{I}:\left(\mathbf{p}_{r}^{G} \cdot \nabla u_{r}\right)^{S}+s_{1 r}=0
$$

with $s_{1 r}=\mathrm{I}:\left(\boldsymbol{\pi}_{r} \cdot \nabla u_{r}\right)^{S}$ and

$$
\frac{d \mathbf{P}_{r}}{d t}: \widehat{b} \otimes \widehat{b}+\left(\left(\nabla \cdot u_{r}\right) \mathbf{p}_{r}^{G}+\nabla \cdot \mathbf{Q}_{r}+\left(\mathbf{p}_{r}^{G} \cdot \nabla u_{r}\right)^{S}\right): \widehat{b} \otimes \widehat{b}+s_{2 r}=0
$$

with $s_{2 r}=\left(\boldsymbol{\pi}_{r} \cdot \nabla u_{r}\right)^{S}: \widehat{b} \otimes \widehat{b}$.
One has

$$
\frac{d \mathbf{P}_{r}}{d t}: \widehat{b} \otimes \widehat{b}=\frac{d}{d t}\left(\mathbf{P}_{r}: \widehat{b} \otimes \widehat{b}\right)-\mathbf{P}_{r}: \frac{d}{d t}(\widehat{b} \otimes \widehat{b})=\frac{d p_{\| r}}{d t}-s_{3 r}
$$

with $s_{3 r}=\boldsymbol{\pi}_{r}: \frac{d}{d t}(\widehat{b} \otimes \widehat{b})$ where we used $\left(\frac{d \widehat{b}}{d t} \otimes \widehat{b}+\widehat{b} \otimes \frac{d \widehat{b}}{d t}\right):(\mathbf{I}-\widehat{b} \otimes \widehat{b})=0$ and $\left(\frac{d \widehat{b}}{d t} \otimes \widehat{b}+\widehat{b} \otimes \frac{d \widehat{b}}{d t}\right): \widehat{b} \otimes \widehat{b}=0$.

One thus gets CGL equations with heat fluxes and coupling to the nongyrotropic components of the pressure tensors,

$$
\begin{aligned}
& \partial_{t} p_{\perp r}+\nabla \cdot\left(u_{r} p_{\perp r}\right)+p_{\perp r} \nabla \cdot u_{r}-p_{\perp r} \widehat{b} \cdot \nabla u_{r} \cdot \widehat{b}+\frac{1}{2}\left(\operatorname{tr} \nabla \cdot \mathbf{Q}_{r}-\widehat{b} \cdot\left(\nabla \cdot \mathbf{Q}_{r}\right) \cdot \widehat{b}\right) \\
& \quad+\frac{1}{2}\left(s_{1 r}-s_{2 r}+s_{3 r}\right)=0 \\
& \partial_{t} p_{\| r}+\nabla \cdot\left(u_{r} p_{\| r}\right)+2 p_{\| r} \widehat{b} \cdot \nabla u_{r} \cdot \widehat{b}+\widehat{b} \cdot\left(\nabla \cdot \mathbf{Q}_{r}\right) \cdot \widehat{b}+s_{2 r}-s_{3 r}=0 .
\end{aligned}
$$

In the case of weak perturbations of an equilibrium state characterized by a uniform density, gyrotropic pressures and a uniform magnetic field, the couplings $s_{1 r}, s_{2 r}$ and $s_{3 r}$ to the non-gyrotropic pressure components, are subdominant. Furthermore, both the gyrotropic and non-gyrotropic components of the heat fluxes $\mathbf{Q}_{r}$ can a priori contribute to the gyrotropic components of $\nabla \cdot \mathbf{Q}_{r}$ entering the dynamical equations for the gyrotropic pressure components.

## IV.b Non-gyrotropic pressure contributions

Start with

$$
\boldsymbol{\pi}_{r} \times \widehat{b}-\widehat{b} \times \boldsymbol{\pi}_{r}=\overline{\mathbf{k}}_{r}
$$

A formal solution is

$$
\boldsymbol{\pi}_{r}=\frac{1}{4}\left[\widehat{b} \times \overline{\mathbf{k}}_{r} \cdot(\mathbf{I}+3 \widehat{b} \otimes \widehat{b})\right]^{\mathcal{S}}
$$

that could be used as the starting point for a perturbative computation of the solution. Instead, it is possible to perform the expansion at the level of the previous equation by writing

$$
\boldsymbol{\pi}_{r} \times \widehat{z}-\widehat{z} \times \boldsymbol{\pi}_{r}=\mathbf{k}_{r}^{\prime},
$$

where

$$
\mathbf{k}_{r}^{\prime}=\overline{\mathbf{k}}_{r}-\left(\boldsymbol{\pi}_{r} \times(\widehat{b}-\widehat{z})-(\widehat{b}-\widehat{z}) \times \boldsymbol{\pi}_{r}\right)
$$

takes into account the small deviation $\widehat{b}-\widehat{z}$ between the unit vectors $\widehat{b}$ and $\widehat{z}=(0,0,1)$ along the local and the ambient magnetic fields.

## One can formally solve

$$
\begin{aligned}
2 \pi_{r x y} & =k_{r x x}^{\prime} \\
\pi_{r y y}-\pi_{r x x} & =k_{r x y}^{\prime} \\
\pi_{r y z} & =k_{r x z}^{\prime} \\
-2 \pi_{r x y} & =k_{r y y}^{\prime} \\
-\pi_{r x z} & =k_{r y z}^{\prime}
\end{aligned}
$$

together with the conditions $\pi_{r x x}+\pi_{r y y}=-\pi_{r z z}$ and $\boldsymbol{\pi}_{r}: \widehat{b} \otimes \widehat{b}=0$ that rewrites

$$
\pi_{r z z}=-\widehat{z} \cdot \pi_{r} \cdot(\widehat{b}-\widehat{z})-(\widehat{b}-\widehat{z}) \cdot \pi_{r} \cdot \widehat{z}-(\widehat{b}-\widehat{z}) \cdot \pi_{r} \cdot(\widehat{b}-\widehat{z})
$$

implying

$$
\begin{aligned}
\pi_{r x x} & =-\frac{1}{2}\left(k_{r x y}^{\prime}+\pi_{r z z}\right) \\
\pi_{r y y} & =\frac{1}{2}\left(k_{r x y}^{\prime}-\pi_{r z z}\right)
\end{aligned}
$$

The compatibility of the above system is automatically satisfied and the non-gyrotropic components of the pressure tensor can then be computed perturbatively. For this purpose, it is convenient to split

$$
\mathbf{k}_{r}^{\prime}=\overline{\boldsymbol{\kappa}_{r}}+L\left(\boldsymbol{\pi}_{r}\right)
$$

with

$$
\boldsymbol{\kappa}_{r}=\frac{1}{\Omega_{r}} \frac{B_{0}}{|b|}\left[\frac{d \mathbf{p}_{r}^{G}}{d t}+\left(\nabla \cdot u_{r}\right) \mathbf{p}_{r}^{G}+\nabla \cdot \mathbf{Q}_{r}+\left(\mathbf{p}_{r}^{G} \cdot \nabla u_{r}\right)^{\mathcal{S}}\right]
$$

and
$L\left(\boldsymbol{\pi}_{r}\right)=\frac{1}{\Omega_{r}} \frac{B_{0}}{|b|}\left[\frac{\overline{d \boldsymbol{\pi}_{r}}}{d t}+\left(\nabla \cdot u_{r}\right) \boldsymbol{\pi}_{r}+\left(\boldsymbol{\pi}_{r} \cdot \nabla u_{r}\right)^{\mathcal{S}}\right]-\left(\boldsymbol{\pi}_{r} \times(\widehat{b}-\widehat{z})-(\widehat{b}-\widehat{z}) \times \boldsymbol{\pi}_{r}\right)$.
One then uses

$$
\frac{\overline{d \mathbf{p}_{r}^{G}}}{d t}=\left(p_{\| r}-p_{\perp r}\right) \frac{d}{d t}(\widehat{b} \otimes \widehat{b})=\left(p_{\| r}-p_{\perp r}\right) \frac{1}{|b|^{2}}\left(\frac{d b}{d t} \otimes b+b \otimes \frac{d b}{d t}-\frac{2}{|b|} \frac{d|b|}{d t} b \otimes b\right)
$$

that is computed using the generalized Ohm's law in the Hall-MHD approximation ().

In a weakly nonlinear regime, the quantity $L\left(\boldsymbol{\pi}_{r}\right)$ is of higher order than $\boldsymbol{\pi}_{r}$, which enables one to compute the non-gyrotropic pressure corrections by iterations.

Expand $\overline{\boldsymbol{\kappa}_{r}}=\boldsymbol{\chi}_{r}^{(1)}+\boldsymbol{\chi}_{r}^{(2)}+\cdots$ and $\boldsymbol{\pi}_{r}=\boldsymbol{\pi}_{r}^{(1)}+\boldsymbol{\pi}_{r}^{(2)}+\cdots$ that obey

$$
\begin{aligned}
& \boldsymbol{\pi}_{r}^{(1)} \times \widehat{z}-\widehat{z} \times \boldsymbol{\pi}_{r}^{(1)}=\boldsymbol{\chi}_{r}^{(1)} \\
& \boldsymbol{\pi}_{r}^{(2)} \times \widehat{z}-\widehat{z} \times \boldsymbol{\pi}_{r}^{(2)}=L\left(\boldsymbol{\pi}_{r}^{(1)}\right)+\boldsymbol{\chi}_{r}^{(2)}
\end{aligned}
$$

together with

$$
\begin{aligned}
& \pi_{r z z}^{(1)}=0 \\
& \pi_{r z z}^{(2)}=-\widehat{z} \cdot \boldsymbol{\pi}_{r}^{(1)} \cdot(\widehat{b}-\widehat{z})-(\widehat{b}-\widehat{z}) \cdot \boldsymbol{\pi}_{r}^{(1)} \cdot \widehat{z}-(\widehat{b}-\widehat{z}) \cdot \boldsymbol{\pi}_{r}^{(1)} \cdot(\widehat{b}-\widehat{z}) .
\end{aligned}
$$

Concentrating on the proton contribution, it is easily seen that the leading order $\boldsymbol{\pi}_{p}^{(1)}$ reproduces Yajima (1966) result

$$
\begin{aligned}
& \pi_{p x x}^{(1)}=-\pi_{p y y}^{(1)}=-\frac{p_{\perp p}}{2 \Omega_{p}}\left(\partial_{y} u_{x}+\partial_{x} u_{y}\right) \\
& \pi_{p z z}^{(1)}=0 \\
& \pi_{p x y}^{(1)}=\pi_{p y x}^{(1)}=-\frac{p_{\perp p}}{2 \Omega_{p}}\left(\partial_{y} u_{y}-\partial_{x} u_{x}\right) \\
& \pi_{p y z}^{(1)}=\pi_{p z y}^{(1)}=\frac{1}{\Omega_{p}}\left[2 p_{\| p} \partial_{z} u_{x}+p_{\perp p}\left(\partial_{x} u_{z}-\partial_{z} u_{x}\right)\right] \\
& \pi_{p x z}^{(1)}=\pi_{p z x}^{(1)}=-\frac{1}{\Omega_{p}}\left[2 p_{\| p} \partial_{z} u_{y}+p_{\perp p}\left(\partial_{y} u_{z}-\partial_{z} u_{y}\right)\right] .
\end{aligned}
$$

## V. Modeling of the heat fluxes

It is again convenient to separate $\mathbf{Q}_{r}$ in the form $\mathbf{Q}_{r}=\mathbf{q}_{r}^{G}+\mathbf{q}_{r}^{N G}$ with

$$
Q_{r i j k}^{G}=q_{\| r} \widehat{b}_{i} \widehat{b}_{j} \widehat{b}_{k}+q_{\perp r}\left(\delta_{i j} \widehat{b}_{k}+\delta_{i k} \widehat{b}_{j}+\delta_{j k} \widehat{b}_{i}-3 \widehat{b}_{i} \widehat{b}_{j} \widehat{b}_{k}\right)
$$

The equations for the gyrotropic pressure components involve

$$
\begin{aligned}
& \widehat{b} \cdot\left(\nabla \cdot \mathbf{q}_{r}^{G}\right) \cdot \widehat{b}=\nabla \cdot\left(\widehat{b} q_{\| r}\right)-2 q_{\perp r} \nabla \cdot \widehat{b} \\
& \frac{1}{2}\left(\operatorname{tr}\left(\nabla \cdot \mathbf{q}_{r}^{G}\right)-\widehat{b} \cdot\left(\nabla \cdot \mathbf{q}_{r}^{G}\right) \cdot \widehat{b}\right)=\nabla \cdot\left(\widehat{b} q_{\perp r}\right)+q_{\perp r} \nabla \cdot \widehat{b},
\end{aligned}
$$

together with the contribution of the non-gyrotropic heat fluxes to the gyrotropic part $\left(\nabla \cdot \mathbf{q}_{r}^{N G}\right)^{G}$ of $\nabla \cdot \mathbf{q}_{r}$.

## V.b The adiabatic approximation

It is valid when wave phase speeds are much larger than particles thermal velocities. The opposite case where particles travel much faster than the waves corresponds to the isothermal limit. Most of the results of the low frequency modes can be recovered by assuming isothermal electrons and adiabatic ions. For adiabatic flows one has

$$
\begin{aligned}
& \partial_{t} p_{\perp r}+\nabla \cdot\left(u_{r} p_{\perp r}\right)+p_{\perp r} \nabla \cdot u_{r}-p_{\perp r} \widehat{b} \cdot \nabla u_{r} \cdot \widehat{b}=0 \\
& \partial_{t} p_{\| r}+\nabla \cdot\left(u_{r} p_{\| r}\right)+2 p_{\| r} \widehat{b} \cdot \nabla u_{r} \cdot \widehat{b}=0
\end{aligned}
$$

that are known as the CGL equations. In the one-fluid approximation, these equations rewrite,

$$
\begin{aligned}
& \rho|b| \frac{d}{d t}\left(\frac{p_{\perp r}}{|b|}\right)=\frac{p_{\perp r}}{|b|^{2}} b \cdot\left[\nabla \times \frac{m_{p} c}{\rho q_{i}}\left(\frac{j \times b}{c}-\nabla \cdot P_{e}\right)\right] \\
& \frac{\rho^{3}}{|b|^{2}} \frac{d}{d t}\left(\frac{p_{\| r}|b|^{2}}{\rho^{3}}\right)=-2 \frac{p_{\| r}}{|b|^{2}} b \cdot\left[\nabla \times \frac{m_{p} c}{\rho q_{i}}\left(\frac{j \times b}{c}-\nabla \cdot P_{e}\right)\right] .
\end{aligned}
$$

Neglecting the Hall term and the electronic pressure gradient one gets the double adiabatic laws

$$
p_{\perp r} \propto \rho|b| \quad \text { and } \quad p_{\| r} \propto \rho^{3}|b|^{-2}
$$

The adiabatic approximation requires that the local state of the plasma be unaffected by small pertubations made at a distant point: very slow variation along magnetic lines of force. In order to extend the domain of validity of the fluid approximation, it is thus necessary to include the heat fluxes. Since the hierarchy of equations for the fluid moments is unclosed, a modelization has to be made at some point.

## Landau fluids

Relations between heat fluxes and temperature and magnetic field perturbations can be obtained within the framework of a linearized kinetic theory. Since the resulting relations involve perturbation frequencies and wavenumbers, they are not directly applicable in a general fluid theory. A method to obtain first-order in time PDE's that can be incorporated in an extended MHD model is to replace the plasma response function $W_{r}$ by its two or four-pole Padé approximant.

## VI. Plasma waves in the Vlasov theory

## VI.a Linear theory

Consider a spatially homogeneous uniformly magnetized ( $b_{0}=B_{0} \hat{z}$ ) plasma at equilibrium with a distribution function $f_{r 0}$ and no net charge or current. The Vlasov equation linearized about $f_{r 0}$ becomes (writing $f_{r}=f_{r 0}+f_{r 1}$, $\left.b=b_{0}+b_{1}, e=e_{1}\right)$

$$
\left(\frac{\partial}{\partial t}+v \cdot \nabla+\frac{q_{r}}{m_{r}} \frac{v \times b_{0}}{c} \cdot \nabla_{v}\right) f_{r 1}=-\frac{q_{r}}{m_{r}}\left(e_{1}+\frac{v \times b_{1}}{c}\right) \cdot \nabla_{v} f_{r 0}
$$

while the equilibrium distribution satisfies $\frac{q_{r}}{m_{r}} \frac{v \times b_{0}}{c} \cdot \nabla_{v} f_{r 0}=0$.
Writing

$$
v=\left(v_{x}=v_{\perp} \cos \phi, v_{y}=v_{\perp} \sin \phi, v_{z}=v_{\|}\right)
$$

and

$$
\nabla_{v}=\left(\cos \phi \partial_{v_{\perp}}-\frac{\sin \phi}{v_{\perp}} \partial_{\phi}, \sin \phi \partial_{v_{\perp}}+\frac{\cos \phi}{v_{\perp}} \partial_{\phi}, \partial_{v_{\|}}\right),
$$

one has $\frac{q_{r}}{c m_{r}}\left(v \times B_{0} \widehat{z}\right) \cdot \nabla_{v}=-\Omega_{r} \partial_{\phi}$, where $\Omega_{r}=\frac{q_{r} B_{0}}{m_{r} c}$ is the cyclotron frequency of the particles of species $r$.

The general equilibrium solution is isotropic in a plane perpendicular to $b_{0}$, i.e. satisfies $f_{r 0}=f_{r 0}\left(v_{\perp}^{2}, v_{z}\right)$. Such solutions are for example bi-Maxwellian distributions

$$
f_{r 0}=\frac{1}{(2 \pi)^{3 / 2}} \frac{m_{r}^{3 / 2}}{T_{\perp r}^{(0)} T_{\| r}^{(0) 1 / 2}} \exp \left\{-\left(\frac{m_{r}}{2 T_{\| r}^{(0)}} v_{\|}^{2}+\frac{m_{r}}{2 T_{\perp r}^{(0)}} v_{\perp}^{2}\right)\right\} .
$$

Restricting ourselves to the case of plane waves such that $e_{1}=\hat{e}_{1} \exp (i(k \cdot x-\omega t))$, $b_{1}=\hat{b}_{1} \exp (i(k \cdot x-\omega t))$ and $f_{r 1}=\hat{f}_{r 1} \exp (i(k \cdot x-\omega t))$,
and using Maxwell's equation to write $\hat{b}_{1}=\frac{c}{\omega}\left(k \times \hat{e}_{1}\right)$, one has (dropping the hats)

$$
-i(\omega-k \cdot v) f_{r 1}-\Omega_{r} \frac{\partial f_{r 1}}{\partial \phi}=-\frac{q_{r}}{m_{r}}\left(\left(e_{1}\left(1-\frac{k \cdot v}{\omega}\right)+\frac{k\left(v \cdot e_{1}\right)}{\omega}\right) \cdot \nabla_{v} f_{r 0}\right)
$$

Integrating this equations, one has

$$
\begin{aligned}
& f_{r 1}=\frac{q_{r}}{m_{r} \Omega_{r}} \exp \left(\frac{i}{\Omega_{r}} \int_{0}^{\phi}(k \cdot v-\omega) d \phi^{\prime}\right)\left[\int_{0}^{\phi} \exp \left(-\frac{i}{\Omega_{r}} \int_{0}^{\phi^{\prime}}(k \cdot v-\omega) d \phi^{\prime \prime}\right)\right. \\
& \left.\times\left(\left(e_{1}\left(1-\frac{k \cdot v}{\omega}\right)+\frac{k\left(v \cdot e_{1}\right)}{\omega}\right) \cdot \nabla_{v} f_{r 0}\right) d \phi^{\prime}+C\right]
\end{aligned}
$$

where the constant $C$ has to be determined by the constraint of periodicity. Introducing a coordinate system such that $k_{y}=0$, one has

$$
\int_{0}^{\phi}(k \cdot v-\omega) d \phi^{\prime}=\left(k_{z} v_{\|}-\omega\right) \phi+k_{x} v_{\perp} \sin \phi
$$

Using

$$
\exp (-i \lambda \sin \phi)=\sum_{l=-\infty}^{+\infty} J_{l}(\lambda) \exp (-i l \phi)
$$

where $\lambda=k_{x} v_{\perp} / \Omega_{r}$ and the $J_{l}(\lambda)$ are ordinary Bessel functions of the first kind,

One finally gets

$$
f_{r 1}=\frac{i q_{r}}{m_{r}} \exp (i \lambda \sin \phi) \sum_{l=-\infty}^{+\infty} \frac{a_{l} \cdot e_{1}}{k_{z} v_{\|}+l \Omega_{r}-\omega} \exp (-i l \phi)
$$

where

$$
\begin{aligned}
& a_{l x}=\left(\frac{k_{z} v_{\perp}}{\omega} \frac{\partial f_{r 0}}{\partial v_{\|}}+\left(1-\frac{k_{z} v_{\|}}{\omega}\right) \frac{\partial f_{r 0}}{\partial v_{\perp}}\right) \frac{1}{2}\left(J_{l+1}(\lambda)+J_{l-1}(\lambda)\right) \\
& a_{l y}=\left(\frac{k_{z} v_{\perp}}{\omega} \frac{\partial f_{r 0}}{\partial v_{\|}}+\left(1-\frac{k_{z} v_{\|}}{\omega}\right) \frac{\partial f_{r 0}}{\partial v_{\perp}}\right) \frac{1}{2 i}\left(J_{l+1}(\lambda)-J_{l-1}(\lambda)\right) \\
& a_{l z}=\frac{\partial f_{r 0}}{\partial v_{\|}} J_{l}(\lambda)+\left(\frac{k_{x} v_{\|}}{\omega} \frac{\partial f_{r 0}}{\partial v_{\perp}}-\frac{k_{x} v_{\perp}}{\omega} \frac{\partial f_{r 0}}{\partial v_{\|}}\right) \frac{1}{2}\left(J_{l+1}(\lambda)+J_{l-1}(\lambda)\right)
\end{aligned}
$$

Expressing the perturbed current $J_{1}$ in terms of $f_{r 1}$ and substituting into Maxwll's equation leads to

$$
-k \times k \times e_{1}=\frac{\omega^{2}}{c^{2}} e_{1}+\frac{i \omega}{c^{2}} 4 \pi \sum_{r} n_{r} q_{r} \int_{L} v f_{r 1} d v
$$

where the integration must be performed on a suitable contour (Landau contour) avoiding singularities.

From the latter equation it is easy to find the dispersion tensor governing the properties of all the modes propagating in this kind of plasma. An important property concerns resonant particles, moving with parallel velocities such that

$$
v_{\|}=\frac{\omega-l \Omega_{r}}{k_{z}}
$$

They yield a significant damping of its energy. For $l=0$, we recover the usual Landau damping in presence of a magnetic field, where only parallel motion is involved.

For $l \neq 0$, we have cyclotron damping which occurs when particles see a wave whose Doppler shifted frequency is some harmonic of the gyrofrequency.

Inspection of the dispersion relation (see e.g. Krall and Trievelpiece 1973) shows the complexity of the problem. If one restrict ourselves to the case where the wavelength is long compared to the ion inertial length, drastic simplification occurs: cyclotron resonance can be ignored (but Landau damping can still occur).

In the following we show that in the case where a specific wave is selected, a reductive perturbative expansion leads to a closed nonlinear wave equation. This can be seen as an example of an exact fluid closure that can serve as a benchmark for Landau fluid types of models.

## VI.b Weakly nonlinear theory in the long wavelength limit

We onsider here Alfvén waves propagating along a strong ambient magnetic field. These waves are amenable to an asymptotic expansion, directly from the Vlasov-Maxwell equation, when involving scales that are large compared to the ion Larmor radius and amplitudes small enough to keep linear dispersive effects comparable to the nonlinearities.

- The reductive perturbative expansion

Let us consider the Vlasov-Maxwell equations (we assume zero collisions). For an ambient field $B_{0} \widehat{x}$, one expands the distribution function and the electric and magnetic fields in the form $f_{r}=F_{r}^{(0)}+\epsilon\left(f_{r}^{(0)}+\epsilon f_{r}^{(1)}+\ldots\right)$, $b=B_{0} \widehat{x}+\epsilon\left(b^{(0)}+\epsilon b^{(1)}+\ldots\right)$ and $e=\epsilon\left(e^{(0)}+\epsilon e^{(1)}+\ldots\right)$, where $F_{r}^{(0)}$ denotes the equilibrium velocity distribution function, assumed rotationally symmetric around the direction of the ambient field and symmetric relatively to forward and backward velocities along this direction, thus excluding the presence of equilibrium drifts.

Denoting by $\lambda$ the Alfvén-wave propagation velocity that will be determined in the following, one defines the stretched coordinates $\xi=\epsilon^{2}(x-\lambda t), \eta=\epsilon^{3} y$ and $\zeta=\epsilon^{3} z$ and the slow time $\tau=\epsilon^{4} t$.
One writes $v=\left(v_{\|}, v_{\perp} \cos \phi, v_{\perp} \sin \phi\right)=\left(v_{\|}, v_{\perp} \vec{n}\right)=\left(v_{\|}, \vec{v}_{\perp}\right)$ and $\nabla_{v}=\left(\partial_{v_{\|}}, \vec{n} \partial_{v_{\perp}}+\vec{t} \frac{1}{v_{\perp}} \partial_{\phi}\right)$, where $\vec{n}=(\cos \phi, \sin \phi)$ and $\vec{t}=\frac{d \vec{n}}{d \phi}$.

Expanding VM equation to the successive orders, one gets

$$
\begin{aligned}
& \Omega_{r} \partial_{\phi} F_{r}^{(0)}=0 \\
& \Omega_{r} \partial_{\phi} f_{r}^{(0)}=\Sigma_{r}^{(0)} \\
& \Omega_{r} \partial_{\phi} f_{r}^{(1)}=\Sigma_{r}^{(1)} \\
& \Omega_{r} \partial_{\phi} f_{r}^{(2)}=\left(v_{\|}-\lambda\right) \partial_{\xi} f_{r}^{(0)}+\Sigma_{r}^{(2)} \\
& \Omega_{r} \partial_{\phi} f_{r}^{(3)}=\left(v_{\|}-\lambda\right) \partial_{\xi} f_{r}^{(1)}+\left(\vec{v}_{\perp} \cdot \nabla_{\perp}\right) f_{r}^{(0)}+\Sigma_{r}^{(3)} \\
& \Omega_{r} \partial_{\phi} f_{r}^{(4)}=\left(v_{\|}-\lambda\right) \partial_{\xi} f_{r}^{(2)}+\left(\vec{v}_{\perp} \cdot \nabla_{\perp}\right) f_{r}^{(1)}+\partial_{\tau} f_{r}^{(0)}+\Sigma_{r}^{(4)}
\end{aligned}
$$

where

$$
\Sigma_{r}^{(s)}=\frac{q_{r}}{m_{r}}\left(e^{(s)}+\frac{1}{c} v \times b^{(s)}\right) \cdot \nabla_{v} F_{r}^{(0)}+\sum_{p+q=s-1} \frac{q_{r}}{m_{r}}\left(e^{(p)}+\frac{1}{c} v \times b^{(p)}\right) \cdot \nabla_{v} f_{r}^{(q)} .
$$

One derives easily

$$
\begin{aligned}
& \widehat{x} \times \partial_{\xi} e^{(0)}=\frac{\lambda}{c} \partial_{\xi} b^{(0)} \\
& \widehat{x} \times \partial_{\xi} e^{(1)}+\nabla_{\perp} \times e^{(0)}=\frac{\lambda}{c} \partial_{\xi} b^{(1)} \\
& \widehat{x} \times \partial_{\xi} e^{(2)}+\nabla_{\perp} \times e^{(1)}=\frac{\lambda}{c} \partial_{\xi} b^{(2)}-\frac{1}{c} \partial_{\tau} b^{(0)}
\end{aligned}
$$

and more generally for any $p \geq 0$,

$$
\widehat{x} \times \partial_{\xi} e^{(2+p)}+\nabla_{\perp} \times e^{(1+p)}=\frac{\lambda}{c} \partial_{\xi} b^{(2+p)}-\frac{1}{c} \partial_{\tau} b^{(p)} .
$$

It immediately follows that $b_{\|}^{(0)}=0$ and $e_{\perp}^{(0)}=-\frac{\lambda}{c} \widehat{x} \times b_{\perp}^{(0)}$. No mean value is assumed for the various fields when averaged over the $\xi$ variable.

- When averaged on the angle $\phi$, the eq. for $f_{r}^{(0)}$ gives $e_{\|}^{(0)}=0$, and

$$
\partial_{\phi} f_{r}^{(0)}=\frac{1}{B_{0}}\left(\partial_{\phi}\left(b_{\perp}^{(0)} \cdot \vec{v}_{\perp}\right) \partial_{v_{\|}} F_{r}^{(0)}-\left(v_{\|}-\lambda\right) b_{\perp}^{(0)} \cdot \partial_{\phi} \nabla_{v_{\perp}} F_{r}^{(0)}\right)
$$

Defining for any positive integer $j$ the operator

$$
\mathcal{D}_{j}^{ \pm}=\left(\lambda-v_{\|}\right)\left(\frac{\partial}{\partial v_{\perp}} \pm \frac{j}{v_{\perp}}\right)+v_{\perp} \frac{\partial}{\partial v_{\|}},
$$

and also $D \equiv \mathcal{D}_{0}^{ \pm}$, one writes

$$
f_{r}^{(0)}=\frac{1}{B_{0}}\left(b_{\perp}^{(0)} \cdot \vec{n}\right) \mathcal{D} F_{r}^{(0)}+\bar{f}_{r}^{(0)}
$$

where the last term of the RHS refers to a possible contribution independent of the $\phi$ variable, that will be later shown to be identically zero.

- The eq. for $f_{r}^{(1)}$ gives:
$e_{\|}^{(1)}=0, e_{\perp}^{(1)}=-\frac{\lambda}{c} \widehat{x} \times b_{\perp}^{(1)}$, and $\partial_{\xi} b_{\|}^{(1)}+\nabla_{\perp} \cdot b_{\perp}^{(0)}=0$. One can prescribe $e_{\perp}^{(1)}=0$ and $b_{\perp}^{(1)}=0$. The modulus of the local magnetic field is given by $|b|=B_{0}\left(1+\epsilon^{2} A\right)+O\left(\epsilon^{3}\right)$ with $A=\frac{b_{\|}^{(1)}}{B_{0}}+\frac{\left|b_{\perp}^{(0)}\right|^{2}}{2 B_{0}^{2}}$.

One has

$$
\partial_{\phi} f_{r}^{(1)}=\frac{1}{B_{0}}\left[\left(v_{\|}-\lambda\right)\left(\widehat{x} \times b_{\perp}^{(0)}\right) \cdot \nabla_{v_{\perp}} f_{r}^{(0)}+\left(\vec{v}_{\perp} \times b_{\perp}^{(0)}\right) \partial_{v_{\|}} f_{r}^{(0)}\right]
$$

- Using solvability at next order:
one easily checks that $e_{\perp}^{(2)}=-\frac{\lambda}{c} \widehat{x} \times b_{\perp}^{(2)}+\frac{1}{c} \widehat{x} \times \partial_{\xi}^{-1} \partial_{\tau} b_{\perp}^{(0)}$ where $\partial_{\xi}^{-1}$ denotes the anti-derivative, $e_{\|}^{(2)}=b_{\|}^{(2)}=0$ and, as announced, $\bar{f}_{r}^{(0)}=0$.
- Propagation velocity of the wave

Applying the vectorial operator $\sum_{r} m_{r} n_{r} \int d^{3} v \vec{v}_{\perp}$ on the two sides of the eq. for $f_{r}^{(2)}$ allows one to determine the propagation velocity $\lambda$ of the Alfvén wave.
In the LHS of the resulting equation, one gets

$$
\sum_{r} n_{r} m_{r} \Omega_{r} \int \vec{v}_{\perp} \partial_{\phi} f_{r}^{(2)} d^{3} v=\frac{B_{0}}{4 \pi} \partial_{\xi} b_{\perp}^{(0)}
$$

while in the RHS, one has

$$
\sum_{r} m_{r} n_{r} \int\left(v_{\|}-\lambda\right) \vec{v}_{\perp} \partial_{\xi} f_{r}^{(0)} d^{3} v=\frac{1}{B_{0}}\left(p_{\|}^{(0)}-p_{\perp}^{(0)}+\lambda^{2} \rho^{(0)}\right) \partial_{\xi} b_{\perp}^{(0)}
$$

with the equilibrium parallel and transverse pressures $p_{\|}^{(0)}=\sum_{r} m_{r} n_{r} \int v_{\|}^{2} F_{r}^{(0)} d^{3} v$ and $p_{\perp}^{(0)}=\sum_{r} m_{r} n_{r} \int \frac{v_{\perp}^{2}}{2} F_{r}^{(0)} d^{3} v$, and the density $\rho^{(0)}=\sum_{r} m_{r} n_{r} \int F_{r}^{(0)} d^{3} v$.

One then concludes that the propagation velocity $\lambda$ is given by

$$
\lambda^{2} \rho^{(0)}=\frac{\left|B_{0}\right|^{2}}{4 \pi}+p_{\perp}^{(0)}-p_{\|}^{(0)}
$$

where the usual Alfvén velocity is affected by the anisotropy of the equilibrium pressure tensor. In order to prevent the system to be firehose unstable, the RHS is assumed positive.

- Wave-particle resonance

The solvability condition on the eq. for $f_{r}^{(3)}$, together with the quasi-neutrality constraint $\sum_{r} q_{r} n_{r} \int \bar{f}_{r}^{(1)} d^{3} v=0$ prescribe $\bar{f}_{r}^{(1)}$ and $e_{\|}^{(3)}$. One has
$\left(v_{\|}-\lambda\right) \partial_{\xi} \bar{f}_{r}^{(1)}=-\left\langle\left(\vec{v}_{\perp} \cdot \nabla_{\perp}\right) f_{r}^{(0)}\right\rangle-\frac{q_{r}}{m_{r}} \frac{\partial F_{r}^{(0)}}{\partial v_{\|}} e_{\|}^{(3)}-\frac{q_{r}}{m_{r}} \sum_{p+q=2}\left\langle\left(e^{(p)}+\frac{v \times b^{(p)}}{c}\right) \cdot \nabla_{v} f_{r}^{(q)}\right\rangle$
that displays a singularity when the longitudinal velocity $v_{\|}$of the particles equals the propagation velocity $\lambda$ of the wave.

After some algebra, one gets

$$
\left(v_{\|}-\lambda\right) \partial_{\xi} \bar{f}_{r}^{(1)}=\left(v_{\|}-\lambda\right) R_{r}+S_{r} \frac{\partial F_{r}^{(0)}}{\partial v_{\|}}
$$

with

$$
\begin{aligned}
& R_{r}=-\frac{v_{\perp}}{2} \frac{\partial F_{r}^{(0)}}{\partial v_{\perp}} \partial_{\xi} A+\frac{1}{2} \mathcal{D}_{1}^{+} \mathcal{D} F_{r}^{(0)} \partial_{\xi} \frac{\left|b_{\perp}^{(0)}\right|^{2}}{2 B_{0}^{2}} \\
& S_{r}=\frac{v_{\perp}^{2}}{2} \partial_{\xi} A+\frac{q_{r}}{m_{r}} \partial_{\xi} \varphi
\end{aligned}
$$

where

$$
-\partial_{\xi} \varphi=e_{\|}^{(3)}+\frac{1}{c B_{0}} \partial_{\xi}^{-1} \partial_{\tau} b_{\perp}^{(0)} \times b_{\perp}^{(0)}
$$

identifies with the electric field along the local magnetic field.
One thus gets $\partial_{\xi} \bar{f}_{r}^{(1)}=R_{r}+\chi_{r}$, where $\chi_{r}$ corresponds to the singular contribution driven by $S_{r} \frac{\partial F_{r}^{(0)}}{\partial v_{\|}}$.

This singularity results from the assumption that in the frame traveling at the propagation velocity $\lambda$ of the wave, all the dynamics takes place on a time scale $O\left(\epsilon^{-4}\right)$, a condition that is broken near the resonance. One should thus define (Mjølhus, and Wyller, 1988) $\chi_{r}=\lim _{\epsilon \rightarrow 0} \chi_{r}^{\{\epsilon\}}$ where $\chi_{r}^{\{\epsilon\}}$ obeys

$$
\epsilon^{2} \partial_{\tau} \chi_{r}^{\{\epsilon\}}+\left(v_{\|}-\lambda\right) \partial_{\xi} \chi_{r}^{\{\epsilon\}}=\partial_{\xi} S_{r} \frac{\partial F_{r}^{(0)}}{\partial v_{\|}}
$$

Since $S_{r}$ does not depend on $v_{\|}$, one can explicitly solve in the form

$$
\chi_{r}^{\{\epsilon\}}(\xi, \tau)=\chi_{r}^{\{\epsilon\}}\left(\xi-\left(v_{\|}-\lambda\right) \frac{\tau}{\epsilon^{2}}, 0\right)+\frac{1}{v_{\|}-\lambda} \frac{\partial F_{r}^{(0)}}{\partial v_{\|}}\left(S_{r}(\xi)-S_{r}\left(\xi-\left(v_{\|}-\lambda\right) \frac{\tau}{\epsilon^{2}}\right)\right) .
$$

One then computes

$$
\begin{aligned}
\int \chi_{r} d v_{\|} & =\lim _{\epsilon \rightarrow 0} \int \chi_{r}^{\{\epsilon\}} d v_{\|}=\mathrm{P} \int \frac{1}{v_{\|}-\lambda} \frac{\partial F_{r}^{(0)}}{\partial v_{\|}} d v_{\|} S_{r}(\xi)+\left.\pi \frac{\partial F_{r}^{(0)}}{\partial v_{\|}}\right|_{v_{\|}=\lambda} \mathcal{H}_{\xi}\left\{S_{r}\right\} \\
& =\mathcal{G}_{r} S_{r}
\end{aligned}
$$

where P holds for principal value, $\mathcal{H}_{\xi}\left\{S_{r}\right\}=\frac{1}{\pi} \mathrm{P} \int \frac{S_{r}(\zeta)}{\zeta-\xi} d \zeta$ denotes the Hilbert transform with respect to the $\xi$ variable, and

$$
\mathcal{G}_{r}=\mathrm{P} \int \frac{1}{v_{\|}-\lambda} \frac{\partial F_{r}^{(0)}}{\partial v_{\|}} d v_{\|}+\left.\pi \frac{\partial F_{r}^{(0)}}{\partial v_{\|}}\right|_{v_{\|}=\lambda} \mathcal{H}_{\xi} .
$$

As a consequence of the electric neutrality of the equilibrium state, $\sum_{r} q_{r} n_{r} \int R_{r} d^{3} v=0$. The condition $\sum_{r} q_{r} n_{r} \int \bar{f}_{r}^{(1)} d^{3} v=0$ then implies

$$
2 \pi \sum_{r} q_{r} n_{r}\left[\int d\left(\frac{v_{\perp}^{2}}{2}\right) \frac{v_{\perp}^{2}}{2} \mathcal{G}_{r} \partial_{\xi} A+\frac{q_{r}}{m_{r}} \int d\left(\frac{v_{\perp}^{2}}{2}\right) \mathcal{G}_{r} \partial_{\xi} \varphi\right]=0
$$

or equivalently

$$
-\partial_{\xi} \varphi=\mathcal{L}^{-1} \mathcal{M} \partial_{\xi} A
$$

where one has defined the operators

$$
\mathcal{L}=2 \pi \sum_{r} \frac{q_{r}^{2} n_{r}}{m_{r}} \int_{0}^{\infty} d\left(\frac{v_{\perp}^{2}}{2}\right) \mathcal{G}_{r} \quad, \quad \mathcal{M}=2 \pi \sum_{r} q_{r} n_{r} \int_{0}^{\infty} d\left(\frac{v_{\perp}^{2}}{2}\right) \frac{v_{\perp}^{2}}{2} \mathcal{G}_{r} .
$$

Note that the inversion of the operator $\mathcal{L}$ is easily performed due to the property $\mathcal{H}_{\xi}=-\mathcal{H}_{\xi}^{-1}$ of the Hilbert transform and thus It follows that

$$
\begin{aligned}
\int \partial_{\xi} \bar{f}_{r}^{(1)} d v_{\|}=\int R_{r} d v_{\|}+\mathcal{G}_{r} S_{r}= & \int \frac{1}{2} \mathcal{D}_{1}^{+} \mathcal{D} F_{r}^{(0)} d v_{\|} \partial_{\xi} \frac{\left|b_{\perp}^{(0)}\right|^{2}}{2 B_{0}^{2}}-\int \frac{v_{\perp}}{2} \frac{\partial F_{r}^{(0)}}{\partial v_{\perp}} d v_{\|} \partial_{\xi} A \\
& +\mathcal{G}_{r}\left(\frac{v_{\perp}^{2}}{2}-\frac{q_{r}}{m_{r}} \mathcal{L}^{-1} \mathcal{M}\right) \partial_{\xi} A
\end{aligned}
$$

- The Kinetic DNLS equation

One now applies the operator $\sum_{r} m_{r} n_{r} \int d^{3} v \vec{v}_{\perp}$. on the eq. for $f_{r}^{(4)}$.
Using Faraday's equation, one gets

$$
\sum_{r} m_{r} n_{r} \Omega_{r} \int \vec{v}_{\perp} \partial_{\phi} f_{r}^{(4)} d^{3} v=\frac{B_{0}}{4 \pi}\left(\partial_{\xi} b_{\perp}^{(2)}-\nabla_{\perp} b_{\|}^{(1)}\right)
$$

One also easily obtains $\sum_{r} m_{r} n_{r} \int \vec{v}_{\perp} \partial_{\tau} f_{r}^{(0)} d^{3} v=-\frac{\lambda \rho^{(0)}}{B_{0}} \partial_{\tau} b_{\perp}^{(0)}$ and

$$
\begin{aligned}
& \sum_{r} m_{r} n_{r} \int \vec{v}_{\perp}\left(\vec{v}_{\perp} \cdot \nabla_{\perp}\right) f_{r}^{(1)} d^{3} v= \\
& \nabla_{\perp} \sum_{r} m_{r} n_{r} \int \frac{v_{\perp}^{2}}{2} \bar{f}_{r}^{(1)} d^{3} v-\frac{1}{4} \sum_{r} m_{r} n_{r} \int v_{\perp}^{2}\binom{\sin 2 \phi \partial_{y}-\cos 2 \phi \partial_{z}}{-\cos 2 \phi \partial_{y}-\sin 2 \phi \partial_{z}} \partial_{\phi} f_{r}^{(1)} d^{3} v
\end{aligned}
$$

where the second term $T_{2}$ (including the sign) of the RHS is given by

$$
T_{2}=\frac{1}{8 \pi}\binom{\partial_{y}\left(b_{y}^{(0) 2}-b_{z}^{(0) 2}\right)+2 \partial_{z}\left(b_{y}^{(0)} b_{z}^{(0)}\right)}{-\partial_{z}\left(b_{y}^{(0) 2}-b_{z}^{(0) 2}\right)+2 \partial_{y}\left(b_{y}^{(0)} b_{z}^{(0)}\right)} .
$$

Furthermore, as a consequence of the neutrality condition and the expansion of the electric current one has

$$
\sum_{r} m_{r} n_{r} \frac{q_{r}}{m_{r}} \int \vec{v}_{\perp}\left(e^{(4)}+\frac{1}{c} v \times b^{(4)}\right) \cdot \nabla_{v} F_{r}^{(0)} d^{3} v=0
$$

and

$$
\begin{gathered}
\sum_{p+q=3} \sum_{r} m_{r} n_{r} \frac{q_{r}}{m_{r}} \int \vec{v}_{\perp}\left(e^{(p)}+\frac{1}{c} v \times b^{(p)}\right) \cdot \nabla_{v} f_{r}^{(q)} d^{3} v \\
\quad=-\frac{1}{4 \pi}\left[\left(\partial_{y} b_{z}^{(0)}-\partial_{z} b_{y}^{(0)}\right) \widehat{x} \times b_{\perp}^{(0)}+b_{\|}^{(1)} \partial_{\xi} b_{\perp}^{(0)}\right]
\end{gathered}
$$

This contribution is easily added to $T_{2}$. Using the divergenceless condition, the sum reduces to $-\frac{1}{4 \pi} \partial_{\xi}\left(b_{\|}^{(1)} b_{\perp}^{(0)}\right)$. Finally,

$$
\begin{aligned}
& \sum_{r} m_{r} n_{r} \int \vec{v}_{\perp}\left(v_{\|}-\lambda\right) \partial_{\xi} f_{r}^{(2)} d^{3} v=\sum_{r} m_{r} n_{r} \int v_{\perp}\left(v_{\|}-\lambda\right) \partial_{\xi}\left(f_{r}^{(2)} \vec{t}\right) d^{3} v \\
& =+\frac{\lambda}{B_{0} \Omega_{i}}\left(3 p_{\| i}^{(0)}+\lambda^{2} \rho_{i}^{(0)}-2 p_{\perp i}^{(0)}\right) \partial_{\xi \xi}\left(\widehat{x} \times b_{\perp}^{(0)}\right)+\frac{B_{0}}{4 \pi} \partial_{\xi} b_{\perp}^{(2)}-\frac{1}{4 \pi} \partial_{\xi}\left(b_{\|}^{(1)} b_{\perp}^{(0)}\right)-\frac{\lambda \rho^{(0)}}{B_{0}} \partial_{\tau} b_{\perp}^{(0)} \\
& \quad+\frac{1}{8 \pi B_{0}} \partial_{\xi}\left(\left|b_{\perp}^{(0)}\right|^{2} b_{\perp}^{(0)}\right)+\frac{1}{B_{0}} \partial_{\xi}\left(\sum_{r} m_{r} n_{r} \int\left[\left(v_{\|}-\lambda\right)^{2}-\frac{v_{\perp}^{2}}{2}\right] \bar{f}_{r}^{(1)} d^{3} v b_{\perp}^{(0)}\right),
\end{aligned}
$$

where only the contribution of the ions was retained in the dispersive term.

Summing up the various contributions computed above, one notes that the terms involving $b_{\perp}^{(2)}$ cancel out. Writing $\bar{f}_{r}^{(1)}=\left\langle\bar{f}_{r}^{(1)}\right\rangle_{\xi}+\tilde{\bar{f}}_{r}^{(1)}$ where the brackets $\langle\cdot\rangle_{\xi}=\lim _{L \rightarrow \infty} \frac{1}{2 L} \int_{-L}^{+L} \cdot d \xi$ indicate averaging along the direction of the ambient field and the tilded quantities are the fluctuations about this mean value, one sees that $\left\langle\bar{f}_{r}^{(1)}\right\rangle_{\xi}$ contributes.

One gets the dynamical equation

$$
\partial_{\tau} b_{\perp}^{(0)}+\delta \partial_{\xi \xi}\left(\widehat{x} \times b_{\perp}^{(0)}\right)-B_{0} \nabla_{\perp}\left(\frac{\widetilde{P}}{2 \lambda \rho^{(0)}}\right)+\frac{\partial}{\partial \xi}\left[\left(\frac{\widetilde{P}}{2 \lambda \rho^{(0)}}+\langle U\rangle_{\xi}\right) b_{\perp}^{(0)}\right]=0
$$

with a dispersion coefficient $\delta$ given by

$$
\delta=\frac{1}{2 \Omega_{i}}\left(\lambda^{2}+3 \frac{p_{\| i}^{(0)}}{\rho^{(0)}}-2 \frac{p_{\perp i}^{(0)}}{\rho^{(0)}}\right)=\frac{1}{2 \Omega_{i} \rho^{(0)}}\left(\frac{B_{0}^{2}}{4 \pi}+2 p_{\| i}^{(0)}-p_{\perp i}^{(0)}-p_{\| e}^{(0)}-p_{\perp e}^{(0)}\right) .
$$

Here the subscripts $e$ and $i$ refer to electrons and ions (assumed of a unique species) respectively.

The ion density at equilibrium has furthermore been replaced by the total plama density. One also has defined

$$
\begin{equation*}
\widetilde{P}=\left(\frac{B_{0}^{2}}{4 \pi}+2 p_{\perp}^{(0)}+\mathcal{N}-\mathcal{M}^{2} \mathcal{L}^{-1}\right) \widetilde{A} \tag{1}
\end{equation*}
$$

where the magnetic fluctuations $\widetilde{b}_{\| \|}^{(1)}$ along the ambient field is prescribed in terms of the transverse component $b_{\perp}^{(0)}$ by the divergenceless condition. For pulses whose extension is comparable to the longitudinal scale $\epsilon^{-2}$, mean fields are zero and the system is closed. The resulting system was derived by Rogister (1971) using a Fourier space formalism.
Note that the action of the Landau damping on the Alfvén waves is mediated by the coupling with ion-acoustic waves that are directly affected.
In 1D and for situations where $\beta=\frac{8 \pi p^{(0)}}{B_{0}^{2}} \ll \frac{T_{e}}{T_{i}}$, i.e. when kinetic effects are negligible, KDNLS reduces to the usual DNLS equation

$$
\begin{equation*}
\partial_{\tau} b+\frac{i}{2 R_{i}} \partial_{\xi \xi} b+\frac{1}{4(1-\beta)} \partial_{\xi}\left(|b|^{2} b\right)=0 \tag{2}
\end{equation*}
$$

which is integrable by inverse scattering (Kaup and Newell, J. Math. Physics, 19, 798 (1978)).
(i) Modulational stability of a circularly polarized wave

The DNLS equation admits an exact solution in the form of circularly polarized Alfvén waves $b=b_{0} e^{-i \sigma(k \xi-\omega \tau)}$.

Linearization of the DNLS equation about the above solution leads to the dispersion relation

$$
\Omega^{2}-2 K\left(\frac{\sigma k}{R_{i}}+\frac{b_{0}^{2}}{2(1-\beta)}\right) \Omega+\left(\frac{k^{2}}{R_{i}^{2}}-\frac{K^{2}}{4 R_{i}^{2}}+\frac{3 \sigma k b_{0}^{2}}{4 R_{i}(1-\beta)}+\frac{3 b_{0}^{4}}{16(1-\beta)^{2}}\right)=0
$$

which predicts a modulational instability for right-hand (left-hand) polarized waves when $\beta>1+\frac{b_{0}^{2} R_{i}}{4 k}$ (respectively $\beta<1-\frac{b_{0}^{2} R_{i}}{4 k}$ ).
(ii) Two-parameter solitons

They form in the nonlinear stage of the modulation instability of the circularly polarized solutions. Rescale $\xi$ so that $R_{i}=1$. Mjølhus ${ }^{1}$ found solitary wave packet solutions: $b=a \exp i \theta$ with real amplitude $a=a(x-v t)$ and phase $\theta$. The local wavenumber $\kappa=\theta_{x}$ and frequency $\nu=-\theta_{t}$ obey, in a simple case:

$$
\begin{aligned}
& a^{2}=\frac{4\left(\nu_{0}+\kappa_{0}\right)}{\left(\nu_{0}+2 \kappa_{0}\right)^{1 / 2} \cosh \left[\left(x-v t-x_{0}\right) / \delta\right]+\kappa_{0}} \\
& \kappa=\kappa_{0}+(3 / 4) a^{2} \quad \delta^{-1}=2\left(\kappa_{0}^{2}+\nu_{0}\right)^{1 / 2} \\
& \nu=\nu_{0}+(3 / 4) a^{2} \quad v=-2 \kappa_{0}
\end{aligned}
$$



Figure 1: Envelope (solid) and real part (dotted) of the above soliton for $\kappa_{0}=0.5, \delta=1, \nu_{0}=0$.

[^0]
## Landau damping of circularly polarized Alfvén wave trains

In one dimension the KDNLS equation writes

$$
\partial_{\tau} b+\frac{\partial}{\partial \xi}\left(b(\alpha+\gamma \mathcal{H})|b|^{2}\right)+i \delta \frac{\partial^{2}}{\partial \xi^{2}} b=0
$$

When $|b|^{2}$ is constant, there is no Landau damping: the $\mathcal{H}$ term vanishes. However it is non zero in presence of modulation, hence the misleading denomination of nonlinear Landau damping.

Two main points (Fla, Mjølhus and Wyller, Physica Scripta 40, 219 (1989)):

- Instability of the circularly polarized wave for all parameter values (even RH polarization, $\beta<1$, although it is weak in that case).
- For typical parameters instability of the RH case is suppressed for $\beta>1$. A quasi-resonance survives only in the special case when the temperature of the electrons strongly exceeds that of the ions.

Also: along with damping, wavenumber decreases, a consequence of the conservation of helicity $K=\frac{1}{2 i} \int_{-\infty}^{+\infty}\left[b^{\star} \int_{-\infty}^{x} b d x^{\prime}-b \int_{-\infty}^{x} b^{\star} d x^{\prime}\right] d x=-2 \pi \mathcal{P} \int_{-\infty}^{+\infty} \frac{\left|b_{k}\right|^{2}}{k} d k$.

Numerical simulations show the formation of stationary S-type and arc-polarized directional and rotational discontinuities (Medvedev et al. PRL 78, 4934 (1997)). Linear waves evolve to rotational discontinuities while circular ones do not evolve, a phenomenon due to the fact that Landau damping is NOT restricted to small scales.

When dealing with Alfvén wave trains that include a large number of pump wavelengths of order $\epsilon^{-2}$ the mean fields are no longer negligible. It is given by
$\langle U\rangle_{\xi}=-\frac{B_{0}^{2}}{8 \pi \lambda \rho^{(0)}}\left(\left\langle\frac{\left|b_{\perp}^{(0)}\right|^{2}}{2 B_{0}^{2}}\right\rangle_{\xi}-2\left\langle\frac{b_{\|}^{(1)}}{B_{0}}\right\rangle_{\xi}\right)-\frac{1}{2 \lambda \rho^{(0)}} \sum_{r} m_{r} n_{r} \int\left[\left(v_{\|}-\lambda\right)^{2}-\frac{v_{\perp}^{2}}{2}\right]\left\langle\bar{f}_{r}^{(1)}\right\rangle_{\xi} d^{3} v$
that can be viewed as a convective velocity which locally corrects the Alfvén wave speed. Its computation requires the estimate of $\left\langle\bar{f}_{r}^{(1)}\right\rangle_{\xi}$ by pushing to higher order the expansion of the Vlasov-Maxwell system.

Furthermore, the system requires the solvability condition

$$
\frac{B_{0}^{2}}{4 \pi}\left\langle\frac{b_{\|}^{(1)}}{B_{0}}\right\rangle_{\xi}+\sum_{r} m_{r} n_{r} \int \frac{v_{\perp}^{2}}{2}\left\langle\bar{f}_{r}^{(1)}\right\rangle_{\xi} d^{3} v=\Gamma(\tau)
$$

where $\Gamma(\tau)$ denotes a function of $\tau$. For solutions decaying at large transverse distance or obeying periodic boundary conditions, no mean field $\left\langle b_{\perp}^{(0)}\right\rangle_{\xi}$ is thus driven.

## The 3D-KDNLS equation for parallel Alfvén wavetrains

Defining the operator $\mathcal{K}=\mathcal{N}-\mathcal{M}^{2} \mathcal{L}^{-1}$,

$$
\begin{aligned}
& \left(\partial_{\tau}+\langle U\rangle_{\xi} \partial_{\xi}\right) b_{\perp}^{(0)}+\frac{\partial}{\partial \xi}\left(\frac{\widetilde{P} b_{\perp}^{(0)}}{2 \lambda \rho^{(0)}}\right)-\frac{B_{0}}{2 \lambda \rho^{(0)}} \nabla_{\perp} \widetilde{P}+\delta \partial_{\xi \xi}\left(\widehat{x} \times b_{\perp}^{(0)}\right)=0 \\
& \rho^{(0)} \partial_{\tau}\langle U\rangle_{\xi}=c_{1}\left(\nabla_{\perp} \cdot\left\langle\widetilde{P} \frac{b_{\perp}^{(0)}}{B_{0}}\right\rangle_{\xi}-\left\langle\widetilde{A} \mathcal{K} \partial_{\xi} \widetilde{A}\right\rangle_{\xi}\right)-c_{2}\left\langle\widetilde{A} \mathcal{K} \partial_{\xi} \widetilde{A}\right\rangle_{\xi, \eta, \zeta} \\
& \partial_{\xi} \widetilde{b}_{\|}^{(1)}+\nabla_{\perp} \cdot b_{\perp}^{(0)}=0
\end{aligned}
$$

with $\widetilde{A}=\frac{\widetilde{b}_{\|}^{(1)}}{B_{0}}+\frac{\left.\widetilde{b_{\perp}^{(0)}}\right|^{2}}{2 B_{0}^{2}}, \widetilde{P}=\left(\frac{B_{0}^{2}}{4 \pi}+2 p_{\perp}^{(0)}+\mathcal{K}\right) \widetilde{A}$.
VI.c Calculation of the heat fluxes: towards a Landau-fluid closure

The transverse heat flux is defined as $q_{\perp}=\sum_{r} m_{r} n_{r} \int \frac{1}{2}\left(V_{\perp}-U_{\perp}\right)^{2}\left(V_{\|}-U_{\|}\right) f_{r} d^{3} v$, where the velocities are calculated in the local frame of reference. To leading order, $q_{\perp}=\epsilon^{2} q_{\perp}^{(1)}+O\left(\epsilon^{3}\right)$. A straightforward expansion leads to

$$
\widetilde{q}_{\perp}^{(1)}=\lambda p_{\perp}^{(0)}\left(\frac{\widetilde{p}_{\perp}^{(1)}}{p_{\perp}^{(0)}}-\frac{\widetilde{\rho}^{(1)}}{\rho^{(0)}}-\widetilde{A}\right)
$$

The parallel heat flux is $q_{\|}=\sum_{r} m_{r} n_{r} \int\left(V_{\|}-U_{\|}\right)^{3} f_{r} d^{3} v$. Again to leading order, $q_{\|}=\epsilon^{2} q_{\|}^{(1)}+O\left(\epsilon^{3}\right)$, and one has

$$
\widetilde{q}_{\|}^{(1)}=\lambda p_{\|}^{(0)}\left(\frac{\widetilde{p}_{\|}^{(1)}}{p_{\|}^{(0)}}-3 \frac{\widetilde{\rho}^{(1)}}{\rho^{(0)}}+2 \widetilde{A}\right)
$$

The heat fluxes also read

$$
\begin{gathered}
\frac{q_{\perp r}^{(1)}}{v_{t h, r} p_{\perp r}^{(0)}}=-\frac{T_{\perp r}^{(0)}}{T_{\| r}^{(0)}} \frac{c_{r} \mathcal{W}_{r}}{1-\frac{T_{\perp r}^{(0)}}{T_{\| r}^{(0)}} \mathcal{W}_{r}} \frac{T_{\perp r}^{(1)}}{T_{\perp r}^{(0)}} \\
\frac{q_{\| r}^{(1)}}{v_{t h, r} p_{\| r}^{(0)}}=\frac{c_{r}\left(c_{r}^{2}-3+\mathcal{W}_{r}^{-1}\right)}{c_{r}^{2}-1+\mathcal{W}_{r}^{-1}} \frac{T_{\| r}^{(1)}}{T_{\| r}^{(0)}}
\end{gathered}
$$

where $c_{r}=\lambda / v_{t h, r}$ and

$$
\mathcal{W}_{r} \equiv \mathcal{W}\left(c_{r}\right)=\frac{1}{\sqrt{2 \pi}} \mathrm{P} \int \frac{\zeta e^{-\zeta^{2} / 2}}{\zeta-c_{r}} d \zeta+\sqrt{\frac{\bar{\pi}}{2}} c_{r} e^{-c_{r}^{2} / 2} \mathcal{H}_{\xi} .
$$

Using suitable Padé approximants for $\mathcal{W}_{r}$, one easily obtains first order partial differential equations for the heat fluxes.

As an example, the parallel heat flux can be modeled by the following equation

$$
\left(\frac{d}{d t}+\frac{v_{t h, r}}{\sqrt{\frac{8}{\pi}}\left(1-\frac{3 \pi}{8}\right)} \mathcal{H} \partial_{x}\right) \frac{q_{\| r}^{(1)}}{v_{t h, r} p_{\|}^{(0)}}=\frac{1}{1-\frac{3 \pi}{8}} v_{t h, r} \partial_{x} \frac{T_{\| r}^{(1)}}{T_{\| r}^{(0)}}
$$

that was also obtained by Snyder, Hammett and Dorland (1997). With a similar, although slighltly more complicated (as it involves magnetic field disturbances) equation for $q_{\perp r}$, one has a fluid model that reproduces the correct weakly nonlinear dynamics of long parallel Alfvén waves, up to the replacement of the plasma response function by suitable Padé approximants. Note that such a model needs to be modified for more general waves propagating at a finite angle to the magnetic field, to account for the coupling with non-gyrotropic contributions (T.P. and Sulem 2004).

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