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Magnetic field line reconnection in dissipationless regimes and mixing of the Lagrangian invariants in strongly magnetized, two-dimensional, plasma configurations

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Magnetic Connections and Outline

Magnetic topology plays an important role in the global dynamics of high temperature plasmas.

Within the ideal MHD plasma description, where

$$\vec{E} + \vec{u} \times \vec{B}/c = 0, \tag{1}$$

which leads to

$$\partial_t \vec{B} = \nabla \times (\vec{u} \times \vec{B}), \tag{2}$$

with $\vec{E}(\vec{x}, t)$, $\vec{B}(\vec{x}, t)$ the plasma electric and magnetic field and $\vec{u}(\vec{x}, t)$ the fluid plasma velocity, *two plasma elements*, separated by the vector $\delta\vec{l}(\vec{x}, t)$ *that are initially connected by a magnetic field line*

$$\delta\vec{l} \times \vec{B} = 0, \qquad \text{at} \quad t = 0$$

remain connected by a field line at any subsequent time, since $\delta\vec{l} \times \vec{B}$ is transported with the plasma (i.e., its Lie derivative vanishes)

$$\partial_t \, (\delta\vec{l} \times \vec{B}) + (\vec{u} \cdot \nabla) \, (\delta\vec{l} \times \vec{B}) + (\nabla \vec{u}) \cdot (\delta\vec{l} \times \vec{B}) = 0. \tag{3}$$

This condition introduces a topological linking (magnetic connection) between plasma elements that is preserved during the ideal MHD plasma evolution.

Magnetic linking constraints the plasma dynamics by making configurations with lower magnetic energy, but different topological linking, inaccessible.

Magnetic reconnection partially removes these constrains by allowing the field lines to decouple locally (i.e., around critical points) from the plasma motion and to reknit in a different net of connections. This localized breaking of the connections arises from physical effects neglected in Eq.(1) that are weak all over the plasma, but are locally enhanced by the formation of small spatial scales around critical points.

In collisionless magnetic field line reconnection the decoupling between the magnetic field and the plasma motion occurs because of finite electron inertia (in the fluid limit) or thermal effects (in the kinetic plasma description).

However, in the absence of dissipation, the plasma response both in the fluid and in the kinetic electron treatment admits *generalized linking conditions that are preserved during the process of magnetic reconnection* and that in a two-dimensional configuration take the simple form of Lagrangian invariants.

In this presentation I will focus on

• the role of these generalized linking conditions in collisionless two-dimensional magnetic field line reconnection and on

• the analytical and numerical results obtained in Refs. [1,2] in the study of the nonlinear development of magnetic reconnection in the fluid and in the drift-kinetic limits.

[1] E. Cafaro, *et al.*, Phys. Rev. Lett., **80**, 4430 (1998);
D. Grasso, *et al.*, Phys. Rev. Lett., **86**, 5051 (2001);
D. Del Sarto, *et al.*, Phys. Rev. Lett., **91**, 235001 (2003).
[2] T. Liseikina, *et al.*, submitted to Phys. Plasmas, (2003).

Generalized Magnetic Connections: an Example

In a collisionless cold plasma model the effect of electron inertia and of the Hall term in Ohm.s law

$$\vec{E} + \frac{\vec{u}}{c} \times \vec{B} = -\frac{m_e}{e}\frac{d\vec{u}_e}{dt} + \frac{1}{nec}\vec{J} \times \vec{B}, \tag{4}$$

can be accounted for by introducing the vector fields

$$\vec{B}_e \equiv \vec{B} - (m_e c/e)\nabla \times \vec{u}_e = \nabla \times \vec{A}_e \tag{5}$$

(subscripts e denote electron quantities and $\vec{u}_e$ is the electron fluid velocity) and

$$\vec{E}_e \equiv \vec{E} + m_e \nabla u_e^2/(2e) + m_e \partial_t \vec{u}_e/e = -\nabla\varphi_e - \partial_t \vec{A}_e/c, \tag{6}$$

where the generalized vector potential $\vec{A}_e$ is proportional to the fluid electron canonical momentum and φ_e to the total electron energy and reduce to vector potential $\vec{A}$ and to the electrostatic potential φ in the limit of massless electrons.

The vector fields $\vec{B}_e(\vec{x},t)$ and $\vec{E}_e(\vec{x},t)$ satisfy the homogeneous Mawxell's equations and the ideal Ohm's law in the form

$$\vec{E}_e + \frac{\vec{u}_e}{c} \times \vec{B}_e = 0, \tag{7}$$

which leads to the generalized liking condition

$$\partial_t\ (\delta\vec{l} \times \vec{B}_e) + (\vec{u}_e \cdot \nabla)\ (\delta\vec{l} \times \vec{B}_e) + (\nabla\vec{u}_e) \cdot (\delta\vec{l} \times \vec{B}_e) = 0. \tag{8}$$

Similarly, all the ideal MHD theorems (magnetic flux conservation, magnetic helicity conservation, linking number etc,) are recovered by substituting $\vec{B}_e$ for $\vec{B}$ and $\vec{u}_e$ for $\vec{u}$.

Energy conservation and transitions between magnetic equilibria

The breaking of the magnetic connections allows the system to access configurations with lower magnetic energy.

The possibility of a transition between two magnetic *equilibria* with different magnetic energies can be easily conceived in the case of dissipative reconnection, when the local decoupling between the magnetic field and the plasma motion is due to electric resistivity,

$$\vec{E} + \frac{\vec{u}}{c} \times \vec{B} = \eta \vec{J}, \tag{9}$$

since the excess magnetic energy that is released in the transition can be transformed into heat.
The possibility of such a transition between two equilibrium states is less obvious in the nondissipative case where energy can only be transferred into mechanical or (reversible) internal energy so that one could expect that the system cannot be "stopped" in a new stationary equilibrium with a lower magnetic energy.

Indeed this apparent difficulty is not very different from the one that occurs in the treatment of Landau damping in Vlasov's equation for the distribution function $f(\vec{x}, \vec{v}, t)$. In Vlasov's equation no energy is dissipated and particle-points in phase space that lie initially on an $f = const$ hypersurface and that move along the characteristics of the single-particle Hamiltonian $H(\vec{x}, \vec{v}, t)$ lie at all times on an $f = const$ hypersurface (with the same value of the constant). This amounts to say that, in the absence of collisions, f-connections are preserved.

Two-dimensional configurations: connections and Lagrangian invariants

The concept of magnetic connections simplifies in the case of two-dimensional (2-D) configurations where all quantities depend on x, y and on time t only.

The magnetic configurations of interest here are characterized by a strong, externally imposed, B_z field which is taken to be fixed and does not play the role of a dynamical variable and by an inhomogeneous shear field in the x-y plane associated with a current density $J(x, y, t)$ along the z-axis. The field B_z plays a very important physical role in determining the model that is appropriate to represent the plasma dynamics in the x-y plane. Plasma configurations where B_z is absent display a different behaviour both in the fluid and in the kinetic description.

In such a 2-D configuration, the magnetic and the electric field can be expressed as

$$\vec{B} = B_0\vec{e}_z + \nabla\psi(x, y, t) \times \vec{e}_z, \tag{10}$$

$$\vec{E} = -\nabla\varphi(x, y, t) + \vec{e}_z\partial_t\psi(x, y, t)/c, \tag{11}$$

where the flux function $\psi(x, y, t)$ is the z-component of the vector potential of the shear magnetic field and φ is the electrostatic potential

Then, the conserved connections between plasma elements moving in the x-y plane take the form of Lagrangian invariants i.e., can be expressed in term of scalar quantities that are advected by the plasma motion and are constant along characteristics.

In the ideal MHD limit this Lagrangian invariant corresponds to the z component A_z of the magnetic vector potential i.e. to the flux function ψ. Plasma elements that lie initially on an $\psi = const$ curve in the x-y plane and that move along the characteristics of the stream function φ remain at all times on the same $\psi = const$ curve, i.e., ψ-connections are preserved.

Cold fluid finite-mass electrons

If the effect of electron inertia is included (in a cold electron fluid), the Lagrangian invariant corresponds to the z component $A_z - (m_e c/e) u_{e,z}$ of the " vector potential" of the field $\vec{B}_e$, which is proportional to the z component of the electron canonical fluid momentum.

In most cases of interest for magnetic reconnection in a configuration with a strong B_z field, the density perturbations can be taken to be small. Then, the term proportional to the z component $u_{e,z}$ of the electron velocity can be rewritten in terms of the z component of the electron current density J (we disregard the ion motion along field lines). Within this approximation, denoting as customary $A_z - (m_e c/e) u_{e,z}$ by F, we have

$$F(x,y,t) = \psi(x,y,t) - d_e^2 \nabla^2 \psi(x,y,t) = \psi(x,y,t) + d_e^2 J(x,y,t), \tag{12}$$

with

$$J \equiv -\nabla^2 \psi$$

the z component of the current density and

$$d_e \equiv c/\omega_{pe}$$

the collisionless electron skin depth.

The Lagrangian invariant F is advected by the stream function φ of the electron motion in the x-y plane which is proportional to the electrostatic potential according to

$$\frac{\partial F}{\partial t} + [\varphi, F] = 0 \tag{13}$$

with the Poisson brackets $[f, g]$ defined by

$$[f, g] = \mathbf{e}_z \cdot \nabla f \times \nabla g. \tag{14}$$

This equation arises from the parallel component of Ohm's equation (4). The stream function φ obeys the equation

$$\frac{\partial U}{\partial t} + [\varphi, U] = [J, \psi], \tag{15}$$

where $U = \nabla^2 \varphi$ is proportional to the plasma fluid vorticity. This equation arises from the electron continuity equation and quasi-neutrality, after expressing the ion perturbed density in terms of the divergence of the ion polarization drift.

Warm fluid finite-mass electrons

When the effects of electron temperature are included, electron parallel compressibility leads to a modification of the conserved connections and introduces a new microscopic scale-length

$$\rho_s \equiv (m_e/m_i)^{1/2} v_{the}/\Omega_i$$

the so called ion-sound gyro-radius.

When this contribution is included, an anisotropic electron pressure tensor appears in Ohm's law[1] and modifies the structure of the conserved connections.

In this warm fluid finite-mass electron regime, two generalized connections are conserved which are expressed by the Lagrangian invariants $G_{\pm}(x, y, t)$ defined by

$$G_{\pm} = \psi - d_e^2 \nabla^2 \psi \pm d_e \rho_s \nabla^2 \varphi, \tag{16}$$

that are advected by the generalized stream functions

$$\varphi_{\pm} = \varphi \pm \rho_s / d_e \psi \tag{17}$$

The new advection equations are

$$\frac{\partial G_{\pm}}{\partial t} + [\varphi_{\pm}, G_{\pm}] = 0. \tag{18}$$

Note the formal analogy with the standard 1-D Vlasov-Poisson problem for electrostatic Langmuir waves: the set of Eqs. (18) has the form of two coupled 1D Vlasov equations, with x and y playing the role of the coordinate and of the conjugate momentum for the "distribution functions" $G_{\pm}$ of two "particle" species with opposite charges in the Poisson-type equation for φ and equal charges in the Yukawa-type equation for ψ

$$d_e \rho_s \nabla^2 \varphi = (G_{\pm} - G_{\pm})/2, \tag{19}$$

$$\psi - d_e^2 \nabla^2 \psi = (G_{\pm} + G_{\pm})/2.$$

The stream functions $\varphi_{\pm}$ play the role of the single particle Hamiltonians.

[1]The effect of this term is to cancel a drift term in the electron inertia contribution, Then Eq.(13) becomes $\partial_t F + [\varphi, F] = \varrho_s^2 [U, \psi]$.

Geometry: 3-Forms

The set of Eqs. (18) can be conveniently rewritten in terms of the 3-Forms equalities

$$dG_{\pm} \wedge dx \wedge dy = dG_{\pm} \wedge d\varphi_{\pm} \wedge dt. \tag{20}$$

From Eqs.(20) it is immediate to see that, on each of the two manifolds $G_{\pm} = const$, we recover Hamilton's equations for x and y with Hamiltonians $\varphi_{\pm}$

$$dx \wedge dy = d\varphi_{\pm} \wedge dt, \tag{21}$$

which give the characteristics $x = x_{\pm}(t)$ and $y = y_{\pm}(t)$ of the Lagrangian advection of $G_{\pm}$.

Equivalently, by interpreting x and y as conjugate canonical variables ($[x, y] = 1$), we can consider (separately) $G_{\pm}$ as canonical variables and define appropriate conjugate variables $K_{\pm}(x, y, t)$ (i.e., such that $[G_{\pm}, K_{\pm}] = 1$) which are also Lagrangian invariants[2].

In this case we can see that the plasma evolution can be described geometrically as the area-preserving evolution of two Lagrangian meshes, G_+, K_+ and G_-, K_- respectively, that are advected along the characteristics described by the stream functions $\varphi_{\pm}(x, y, t)$. These stream functions determine the dynamics of the system through Eqs.(16,19).

[2] $K_{\pm}$ obey the equations $(\partial_t K_{\pm})\nabla G_{\pm} - (\partial_t G)_{\pm}\nabla K_{\pm} = \nabla\varphi_{\pm}$.

Dynamics: Energy functional

The dynamics of the plasma configuration is governed by the conserved energy functional $\mathcal{H}(\psi, \varphi)$ [3]

$$\mathcal{H}(\psi, \varphi) \equiv \int d^2x \left(|\nabla\psi|^2 + |\nabla\varphi|^2 + d_e^2 J^2 + \varrho_s^2 U^2\right)/2. \tag{22}$$

The first term $\int d^2x |\nabla\psi|^2/2$ represents the magnetic energy in the shear field, $\int d^2x |\nabla\varphi|^2/2$ the plasma fluid kinetic motion, $\int d^2x d_e^2 J^2/2$ the energy of the ordered electron kinetic energy along field lines and $\int d^2x \varrho_s^2 U^2/2$ the work done by the parallel electron compression. This last term disappears in the limit of a cold electron fluid ($\varrho_s \to 0$) [4].

Note that the magnetic energy and the plasma fluid kinetic motion, contain first derivatives or ψ and φ while the ordered electron kinetic energy and the work done by the parallel electron compression contain second derivatives, i.e., one can expect that the former two dominate at large scales and the latter two at small spatial scales.

Note in addition that, in the absence of dissipative effects, there are no characteristic dissipative scalelengths in Eqs. (18) that can limit the nonlinear formation of small spatial scales.

[3] This energy functional is related to the possibility of describing Eqs. (18) in the form of Hamiltonian *field* equations with non-canonical variables and degenerate non-canonical Poisson brackets. The kernel of these Poisson brackets is given by an infinite set of Casimirs defined as $\int d^2x \mathcal{C}_\pm(G_\pm)$ with $\mathcal{C}_\pm$ arbitrary smooth functions.

P.J. Morrison, *Phys. Fluids*, 27, 886 (1984);

T.J., Schep, *et al.*, Phys. Plasmas, 1, 2843 (1994);

B.N., Kuvshinov, *et al.*, Phys. Letters A, 191, 296 (1994).

[4] By redefining the energy functional $\mathcal{H}$ by adding the appropriate combination of Casimirs, we can rewrite $\mathcal{H}$ in the form $\mathcal{H}(G_\pm, \varphi_\pm) = -\int d^2x \; (G_+\varphi_+ + G_-\varphi_-)/2$.

B.N., Kuvshinov, *et al.*, J. Plasma Physics, 59, 727 (1998)

1 Nonlinear Reconnection Regimes

We are interested here in the nonlinear evolution of collisionless reconnection instabilities which arise because of the initial inhomogeneous current distribution in the x-y plane. This procedure is different from the one where forced magnetic reconnection is studied in configurations where magnetic flux is forced from their boundaries (in our case reconnection is forced by the initial conditions).

As mentioned above, the decoupling between the plasma motion and the magnetic field occurs around critical points that correspond, in the 2-D configurations under examination, to the zeros of the shear field i.e., to the zeros of $\nabla\psi(x, y, t)$.

As is customary for magnetic configurations of interest for laboratory plasmas, we consider initial configurations where critical points have degenerated into a critical line, i.e., initial configurations that depend only on one coordinate (say x) and where the shear field vanishes along a line (the null line):

$$\psi_0 = \psi_0(x), \qquad \text{with} \qquad \partial\psi_0(x)/\partial x = 0|_{x=0}.$$

The early development (linear phase) of the reconnection instabilities in such configurations has been thoroughly examined in the literature in terms of threshold conditions for the onset of the instabilities, growth rates and role of the boundary layer at the null line. In this layer a large current density cumulates and the topology of the shear field starts to be changed with the formation of magnetic islands bounded by magnetic separatrices.

The interest here is to examine the nonlinear phase of a collisionless reconnection instability and the eventual saturation of the island growth.

As a more technical remark I recall here that the threshold condition of the reconnection instabilities in null line configurations is controlled by the value of a parameter, usually denoted by Δ', which "measures" both the total current that the instability drives in the boundary layer and the magnetic energy flux that is convected by the instability flows towards the null line.
In the case of collisionless reconnection instabilities, where the decoupling between the magnetic field and the plasma occurs because of electron inertia, the regime that is of greatest interest is the so called large Δ'-regime ($d_e\Delta'$ of order unity).

The early nonlinear phase of the development of the reconnection instability in the cold electron limit was examined numerically and analytically in
M. Ottaviani and F. Porcelli, Phys. Rev. Lett., 71, 3802 (1993)
and shown to lead to a narrow current layer along the initial null line and to a super-exponential phase with a reconnection rate, as measured by the reconnected flux $\delta\psi(t)$ at the island X-point, larger than in the linear phase.

In order to investigate the long term nonlinear evolution of a fast growing (large $d_e\Delta'$) reconnection instability produced by electron inertia in a sheared magnetic equilibrium configuration with a null line Eqs.(18) were integrated numerically in

E. Cafaro, *et al.*, Phys. Rev. Lett., **80**, 4430 (1998);
D. Grasso, *et al.*, Phys. Rev. Lett., **86**, 5051 (2001);
D. Del Sarto, *et al.*, Phys. Rev. Lett., **91**, 235001 (2003).

Numerical Results

In this series of simulations periodic conditions were taken along y and the configuration parameters were chosen such that only one mode can be linearly unstable.

For reference we mention that for the results that will be presented here, the typical mesh sizes are $N_x = 2048$ and $N_y = 512$.
Random perturbations were imposed on the equilibrium configuration $\psi_0(x) = -L/[2\cosh^2(x/L)]$ in a simulation box with $L_x = 2L_y = 4\pi L$, taking $d_e = 3/10L$ and ϱ_s/d_e in the range 0-1.5.
The accuracy of the integration was verified by testing the effects of numerical dissipation on the conservation of the energy and of the Lagrangian invariants.

Formation of small spatial scales in the nonlinear phase

The Lagrangian invariants $G_\pm$ differ from the flux function ψ by the term $d_e^2 J \pm d_e \varrho_s U$ which has small coefficients but involves higher spatial derivatives. As shown in Cafaro et al., magnetic reconnection proceeds unimpeded in the nonlinear phase because of the development near the X point of the magnetic island of increasingly small spatial scales that *effectively decouple ψ from $G_\pm$*.

In Hamiltonian regimes the formation of such scales does not stop at some finite resistive scalelength.
This corresponds to the formation of increasingly narrow current and vorticity layers.
Because of the conserved $G_\pm$ connections, the spatial localization and structure of these layers depends on the value of ϱ_s/d_e.

Mixing of the Lagrangian invariants and island growth saturation

As mentioned above, in the reconnection model adopted, magnetic energy $\int d^2x|\nabla\psi|^2$ is transformed, in principle reversibly, into two forms of kinetic energy, one, $\int d^2x|\nabla\varphi|^2$, related to the plasma motion in the x-y plane and one, $\int d^2x d_e^2 J^2$, to the electron current along z and, for $\varrho_s \neq 0$, into electron parallel compression $\int d^2x \varrho_s^2 U^2$.
The last two energies involve quantities with higher derivatives.

Being the system Hamiltonian, it is not a priori clear whether a reconnection instability can induce a transition between two *stationary plasma configurations* with different magnetic energies, as is the case for resistive plasma regimes where the excess energy is dissipated into heat.

Taking $\varrho_s/d_e \sim 1$, in Grasso et al. it was shown that, in spite of energy conservation, this transition is possible at a "macroscopic" level.

A new coarse-grained stationary magnetic configuration can be reached because, as the instability develops, the *released magnetic energy is removed at an increasingly fast rate from the large spatial scales towards the small scales that act a perfect sink.*

This allows the saturation of the island growth.

Similarly, the constraints imposed by the conservation of the $G_\pm$ connections cease to matter at a macroscopic level.

The advection of the two Lagrangian invariants $G_\pm$ is determined by the stream functions $\varphi_\pm$. The winding, caused by this differential rotation type of advection. makes $G_\pm$ increasingly filamented inside the magnetic island, leading to a mixing process.

These filamentary structures of $G_\pm$ do not influence the spatial structure of ψ which remains regular.

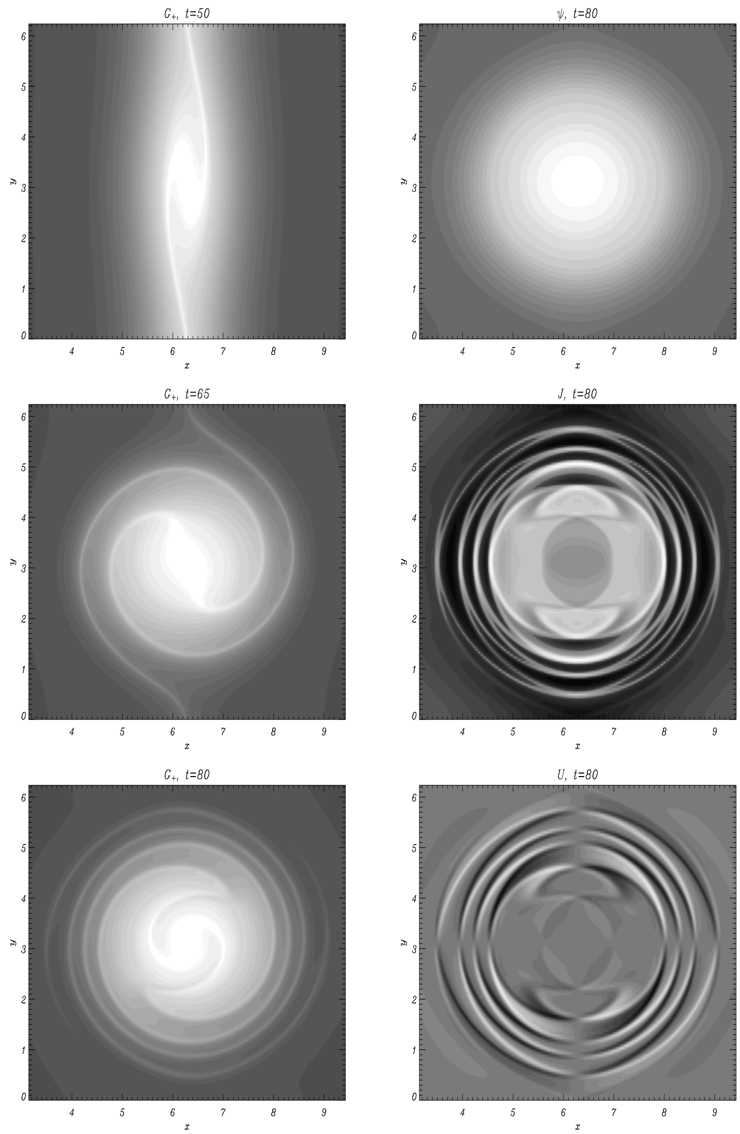

Fig.1 Laminar mixing: contour of G_+ at $t = 50, 65, 80$ (right column) and of Ψ, J and U (left column from top to bottom) at $t = 80$ for $\rho_s/d_e = 1.5$.

The analogy with the Bernstein-Greene-Kruskal (BGK) solutions of the Vlasov equation obtained in

C. Lancellotti and J.J. Dorning, *Phys. Rev. Lett.* **81**, 5137 (1998).
G. Manfredi, *Phys. Rev. Lett.* **79**, 2815 (1997).
M. Brunetti, F. Califano, F. Pegoraro, *Phys. Rev.* **E62**, 4109 (2000).

for the nonlinear Landau damping of Langmuir waves was discussed in Grasso et al.

Onset of a secondary Kelvin Helmoltz instability: turbulent versus laminar mixing

The advection, and consequently the mixing, of the Lagrangian invariants can be either laminar or turbulent depending on the value of ϱ_s/d_e.

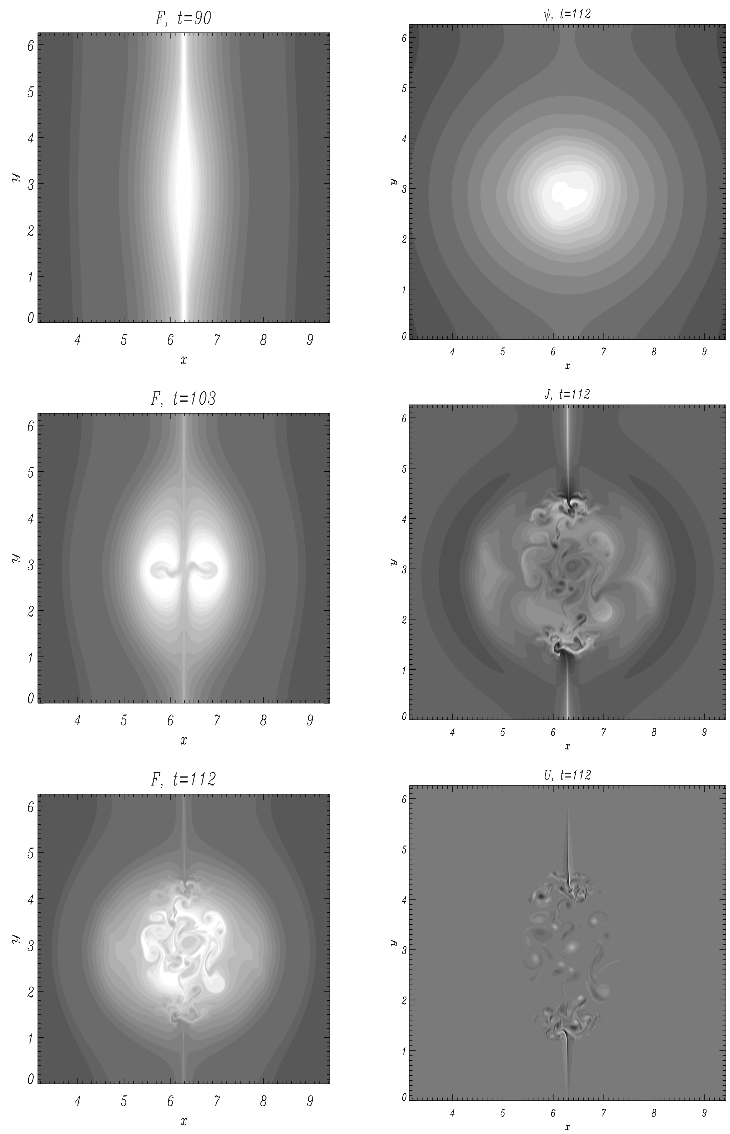

Fig.2 Turbulent mixing: contour of F at $t = 90, 103, 112$ (left column) and of Ψ, J and U (right column from top to bottom) at $t = 112$ for $\rho_s/d_e = 0$.

The transition between these two regimes was shown in Del Sarto et al. to be related to the onset of a secondary Kelvin Helmoltz-type (K-H) instability driven by the velocity shear of the plasma motions that form because of the development of the reconnection instability.

Whether or not the K-H instability becomes active before the island growth saturates, determines whether a (macroscopically) stationary reconnected configuration is reached and affects the redistribution of the magnetic energy.

Cold electrons

In the cold electron limit, $\varrho_s/d_e = 0$, the system of Eqs. (18) becomes degenerate and the generalized connections are determined by a single Lagrangian invariant F.

Initially, F is advected along a hyperbolic pattern given by the stream function φ which has a stagnation point at the O-point of the magnetic island.
This motion leads to the stretching of the contour lines of F towards the stagnation point and to the formation of a bar-shaped current layer along the equilibrium null line, which differs from the cross shaped structure found in the initial phase of the reconnection instability for $\varrho_s/d_e \neq 0$.

Subsequently, F contours are advected outwards in the x-direction. At this stage F starts to be affected by a K-H instability that causes a full redistribution of F.
In this phase the spatial structure of F is dominated by the twisted filaments of the current density which spread through the central part of the magnetic island.

The contours of the vorticity U exhibit a well developed turbulent distribution of monopolar and dipolar vortices, while those of ψ remain regular although they pulsate in time.

The energy balance shows that part of the released magnetic energy remains in the form of plasma kinetic energy corresponding to the fluid vortices in the magnetic island and that an oscillatory exchange of energy persists

(see J. Bergmans, et al., Phys. Rev. Lett., 87, 195002 (2001).)

between the plasma kinetic energy and the electron kinetic energy corresponding to the pulsations of the island shape.

This turbulent evolution of the nonlinear reconnection process persists in the non degenerate, finite electron temperature, case where the two Lagrangian invariants $G_\pm$ determine the generalized linking conditions

However, as the ratio ϱ_s/d_e is increased, i.e. as the electron temperature effects become more important, the onset of the K-H instability occurs later during the island growth and its effect on the current layer distribution becomes weaker.

For $\varrho_s/d_e \sim 1$, no sign of a secondary instability is detectable during the time the island takes to saturate its growth.

In the transitional regime, the advection pattern and the current layer structures exhibit an intermediate behaviour. Initially, $G_\pm$ are advected in opposite directions with a differential rotation, as is the case for $\varrho_s/d_e = 1$. At later times they acquire features characteristic of the evolution of F in the degenerate $\varrho_s = 0$ case and their advection becomes K-H unstable leading to an almost turbulent distribution.

1.1 Need for a kinetic electron description

The above results show that the conservation of the generalized connections in the reconnection process leads to the formation of current and vorticity layers with spatial scales that, in the absence of dissipation, becomes increasingly small with time.

In this nonlinear phase of the development of the reconnection instability, the fluid approximation may become inconsistent inside the layers. The generalized connections and the constraints that they exert on the plasma dynamics apply to the case of a fluid plasma, where fluid elements can be defined and the linking property between fluid plasma elements can be formulated.

It thus becomes important to understand what is the role of the topological invariants in a kinetic electron description where e.g., the canonical momenta of the single electrons do not simply add up to give the fluid conserved Lagrangian invariant F discussed above.

The role of a finite electron temperature on the topological properties of the plasma is already evident from the above results, since the contribution of the parallel electron compressibility introduces two new Lagrangian invariants $G_\pm$ and two different streaming functions $\varphi_\pm$ instead of F and φ, and consequently changes the nonlinear evolution of reconnection in a significant way.

Drift kinetic formulation

Let $\mathcal{F}(x, y, v_{||}, t)$ be the drift-kinetic electron distribution function, where $v_{||}$ is the electron velocity along field lines. It is convenient to adopt as kinetic variable the electron canonical momentum, divided by the electron mass, $p_{||}$, defined by [5]

$$p_{||} \equiv v_{||} - \psi, \tag{23}$$

where we recall that $v_{||}$ is the coordinate in velocity space.
Since we consider two dimensional (z independent) fields and perturbations, $p_{||}$ is a particle constant of the motion.

In the $x, y, p_{||}, t$ variables the drift kinetic equation for the distribution function

$$f(x, y, p_{||}, t) \equiv \mathcal{F}(x, y, v_{||}, t)$$

reads

$$\frac{\partial f}{\partial t} + [\varphi - \psi p_{||} - \psi^2/2, f] = 0 - [\varphi_{kin}, f], \tag{24}$$

with $\varphi_{kin} = \varphi - \psi p_{||}/c - \psi^2/2$.

H.J. de Blank, Phys. Plasmas, **8**, 3927 (2001);

G. Valori, *Fluid and kinetic aspects of collisionless magnetic reconnection*, ISBN 90-9015313-6, Print Partners Ipskamp, Enschede, the Netherlands (2001).

Note that in Eq.(24) the spatial derivatives are taken at constant $p_{||}$ and not at constant $v_{||}$.
For each fixed value of $p_{||}$, the time evolution of f corresponds to that of a Lagrangian invariant "density" advected by the velocity field obtained from the generalized stream function φ_{kin}.
The advection velocity field is different on each $p_{||} = const$ foil.
In this formulation the drift distribution function f obeys a advection equation that, at fixed $p_{||}$, has the same algebraic structure as those adopted in the fluid approach.

$$df \wedge dx \wedge dy = df \wedge d\varphi_{kin} \wedge dt. \tag{25}$$

Thus f consists of an infinite number of Lagrangian invariants, each of them advected with a different velocity, that take the place of the two fluid invariants $G_\pm$.

[5] We adopt the following normalizations $\varphi = e\varphi/m_e v_{the}^2$, $\psi = e\psi/m_e c v_{the}$, $x, y = x/L, y/L$, $t = t m_e v_{the}^2 c/L^2 e B_0$, $p_{||} = p_{||}/v_{the}$, where L is a characteristic length and the other symbols are standard

The fluid quantities are defined in terms of distribution function f as follows

$$\int dp_{||} f(x,y,p_{||},t) \ = \ n(x,y,t), \tag{26}$$

$$\int dp_{||} p_{||} f(x,y,p_{||},t) \ = \ [u(x,y,t) - \psi(x,y,t)] \ n(x,y,t),$$

$$\int dp_{||} [p_{||} - u(x,y,t) + \psi(x,y,t)]^2 f(p_{||},x,y,t) \ = \ \Pi_{||||},$$

where $n(x,y,t)$ and $u(x,y,t)$ are the normalized electron density and fluid velocity and $\Pi_{||||}(x,y,t)$ is the (z,z) component of the pressure tensor. Then Ampere's equation reads

$$d_e^2 \nabla^2 \psi = nu, \tag{27}$$

and, as in the fluid case, the ion equation of motion together with quasineutrality give

$$(n - n_0) = \rho_s^2 \nabla^2 \varphi, \tag{28}$$

where $n_0 = n_0(x)$ is the initial normalized density and the density variations are supposed to remain small.

The above system of equations admits a conserved energy functional

$$\int dxdy \ [d_e^2 (\nabla \psi)^2 + \rho_s^2 (\nabla \varphi)^2 + nu^2 + \Pi_{||||}]/2 = const \tag{29}$$

Aside for the normalization, the main difference between these energy terms and the corresponding ones derived in the fluid case is in the expression of the electron compression work, as natural in a kinetic theory, the pressure tensor $\Pi_{||||}$ cannot be expressed in terms of the lower order moments of the distribution function.

Electron equilibrium distribution function

The stationary solutions of Eq.(24) are of the form $f = f(p_{||}, \varphi_{kin})$. Using the identity for the single particle energy

$$v_{||}^2/2 - \varphi = p_{||}^2/2 - \varphi_{kin}, \tag{30}$$

we can write a stationary distribution function that depends only on the particle energy as $f = f(p_{||}^2/2 - \varphi_{kin})$, while the well known static ($\varphi_0 = 0$) Harris pinch equilibrium distribution is given by[6]

$$f = f_0 \exp\left[-(p_{||}^2 - 2\varphi_{kin}) - 2v^* p_{||}\right] \tag{31}$$

In order to have a less inhomogeneous plasma configuration we can add a pedestal (Maxwellian) distribution function of the form

$$f_{ped} = f_{00} \exp\left[-(p_{||}^2 - 2\varphi_{kin})\right].$$

The corresponding self consistent vector potential $\psi_0(x)$ is given by $\psi_0(x) = (1/v^*)\ \ln(\cosh x)$ and the shear magnetic field has the standard hyperbolic tangent distribution.

[6]In velocity variable $v_{||}$ this distribution corresponds to $\mathcal{F}_0 \exp\left[-(v_{||} - v^*)^2 - 2v^*\psi\right]$ and leads to a particle and current density of the form $n = n_0 \exp(-2v^*\psi)$ and $j = -n_0 v^* \exp(-2v^*\psi)$, where j is normalized on $n_o e v_{the}$ and v^* is the standard parameter related to the diamagnetic fluid motion.

Evolution of the $p_{||} = const$ foils

We write the distribution function with $f(x, y, t, p_{||})$ as

$$f(x, y, t, p_{||}) = \int d\bar{p}_{||}\delta(\bar{p}_{||} - p_{||})f(x, y, t, \bar{p}_{||}).$$

This is a foliation of the electron distribution function in terms of the infinite number of Lagrangian invariants obtained by taking the distribution function f at fixed electron canonical momentum.
Within the drift-kinetic equation each $\bar{p}_{||}$-foil evolves independently, while all foils are coupled through Maxwell's equations.

The total number of particles in each foil is constant in time.

In the initial configuration, the spatial dependence of each $\bar{p}_{||}$-foil is given for the case of the Harris distribution by

$$\exp{(2\bar{\varphi}_{kin})} = \exp{(-2\psi\bar{p}_{||} - \psi^2)} = \exp{[\bar{p}_{||}^2 - \hat{v}_{||}(x)^2]}, \tag{32}$$

where $\hat{v}_{||}(x) \equiv v_{||}(\psi, \bar{p}_{||}) = \bar{p}_{||} + (1/v^*)\ \ln{(\cosh x)}$. For negative values of $\bar{p}_{||}$ the maximum of the argument of the exponent in Eq.(32) is located at

$$x = \pm \text{arccosh}[\exp{(-v^*\bar{p}_{||})}]$$

i.e. the foil is localized in space within two symmetric bands, respectively to the right and to the left of the neutral line of the magnetic configuration. For positive values of $\bar{p}_{||}$ all the foils are centered around $x = 0$.

Nonlinear twist dynamics of the foils

In the adopted drift kinetic framework the $\bar{p}_{||}$-foils take the role of the Lagrange invariants $G_{\pm}$ of the fluid plasma description.

In this perspective, the dynamics of the foils can be predicted by looking at the form of stream function φ_{kin} inside each foil.

The advection velocity can be written as

$$\vec{e}_z \times \nabla(\varphi - \psi p_{||} - \psi^2/2) = \tag{33}$$

$$\vec{e}_z \times \nabla\varphi - (p_{||} + \psi)\ \nabla\psi \times \vec{e}_z$$

which represents the particle $E \times B$ drift and their free motion along field lines. At fixed $p_{||} = \bar{p}_{||}$ we see that depending on the sign of $\psi + \bar{p}_{||} = \hat{v}_{||}(x)$, the advection velocity field takes two counter oriented rotation patterns, reminiscent of those that advect G_- (G_+) in fluid theory.

In the equilibrium configuration where all quantities are function of $\psi = \psi(x)$ and $\varphi = 0$, this advection corresponds to the free particle motion along $\psi = const$ surfaces inside each foil.

However, when the instability starts to move the plasma along the x axis and $\partial\varphi/\partial y \neq 0$, the portions of the foil where $\hat{v}_{||} > 0$ or where $\hat{v}_{||} < 0$ will bend in opposite directions. This will lead to a distortion and twist of the foils and to their eventual spatial mixing, analogously to the mixing of $G_{\pm}$ in the fluid theory.

Numerical results

For a Harris equilibrium the evolution of the reconnection instability is characterized by three dimensionless parameters that can be expressed as the dimensionless ion sound ion sound gyro-radius ρ_s and the electron skin depth d_e from Poisson's and Ampere's equations respectively, and n_0. In fact, when d_e and n_0 are given, v^*, and thus ψ_0, are determined implicitly by the choice of L.
The size of the simulation box along y has been chosen equal to 4π such that the parameter Δ' is positive only for the lowest order mode corresponding to $k_y = 1/2$ so that only the $k_y = 1/2$ mode can be linearly unstable. The simulation box is 40 long in the x direction, with periodic boundary conditions in y and first type boundary conditions in x.

We have taken fixed $\rho_s = 1$ and
$d_e = 1, v^* = 4$, corresponding to $\psi_0 = 1/4, n_0 = 1/16$,
$d_e = 1, v^* = 2 \;\; (=> \psi_0 = 1/2, n_0 = 1/4)$,
$d_e = 0.5, v^* = 2 \;\; (=> \psi_0 = 1/2, n_0 = 1/16)$.

Smaller values of v^* correspond to larger instability growth rates i.e., to a faster evolving instabilities where the saturation of the island growth is reached sooner.

The growth rate increases with d_e faster than linearly.
The instability saturation is shown for the case with $d_e = 1$ and $v^* = 4$.

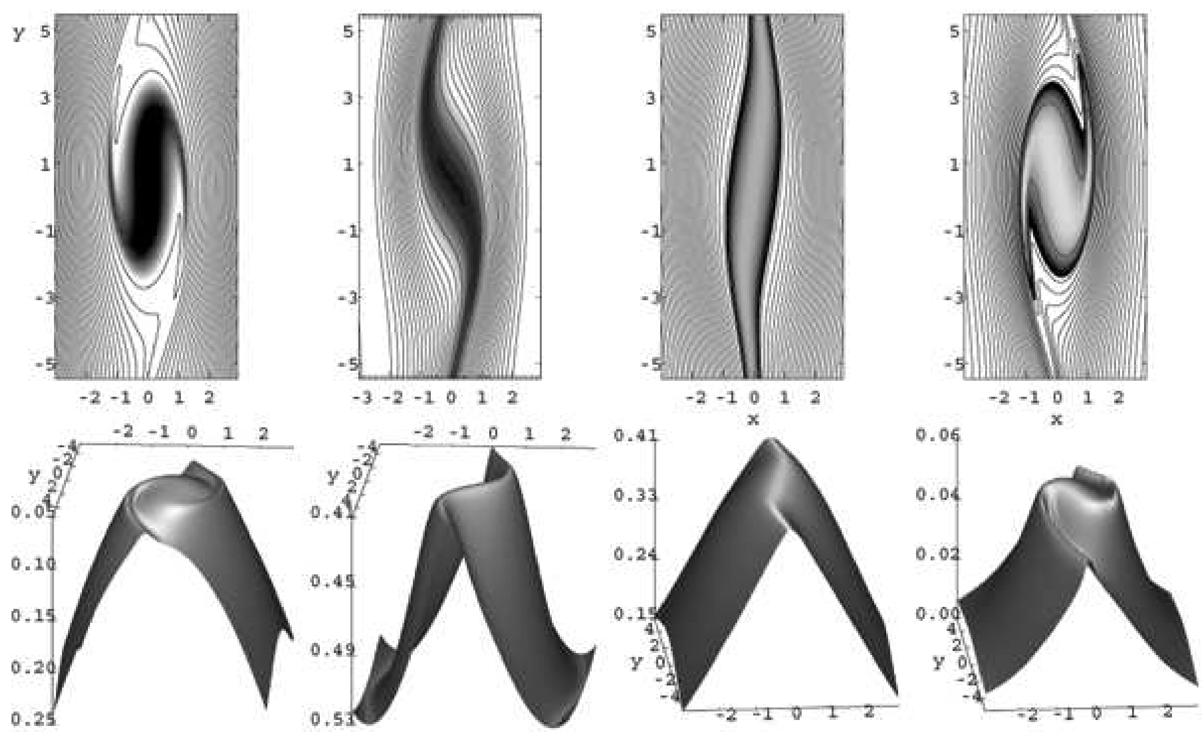

Fig.3 Contour plots (top) and 3D plots (bottom) of the $p_{||}$ = constant foils of electron distribution function at $t = 100$ for $p_{||} = -1.5, -0.5, 0.5, 1.5$ from left to right in the interval $-3 < x < 3$ around the neutral line. Note the different scales in the vertical axes of the 3D plots.

The evolution of the $\bar{p}_{||}$-foils $f(x, y, \bar{p}_{||}, t)$, restricted to the interval $-3 < x < 3$ around the neutral line, is shown at $t = 100$ for $\bar{p}_{||} = -1.5, -0.5, 0, 0.5, 1.5$, together with the contour plots of the stream function φ_{kin} in x-y for the same values of $\bar{p}_{||}$ and the same interval in x.
Foils corresponding to negative values of $\bar{p}_{||}$ were initially localized in two symmetric bands to the left and to the right of the neutral line and are thus modified by the onset of the reconnection instability only in their portion that extend into the reconnection region.
On the contrary foils corresponding to positive values of $\bar{p}_{||}$ were initially localized around $x = 0$ and are thus twisted by the development of the reconnection instability. The contour plots of the stream function φ_{kin} corresponds to a differential rotation in the x-y plane. The sign of the rotation is opposite for positive and for negative values of $\bar{p}_{||}$. The mixing caused by this differential rotation of the $p_{||}$-foils is evident.

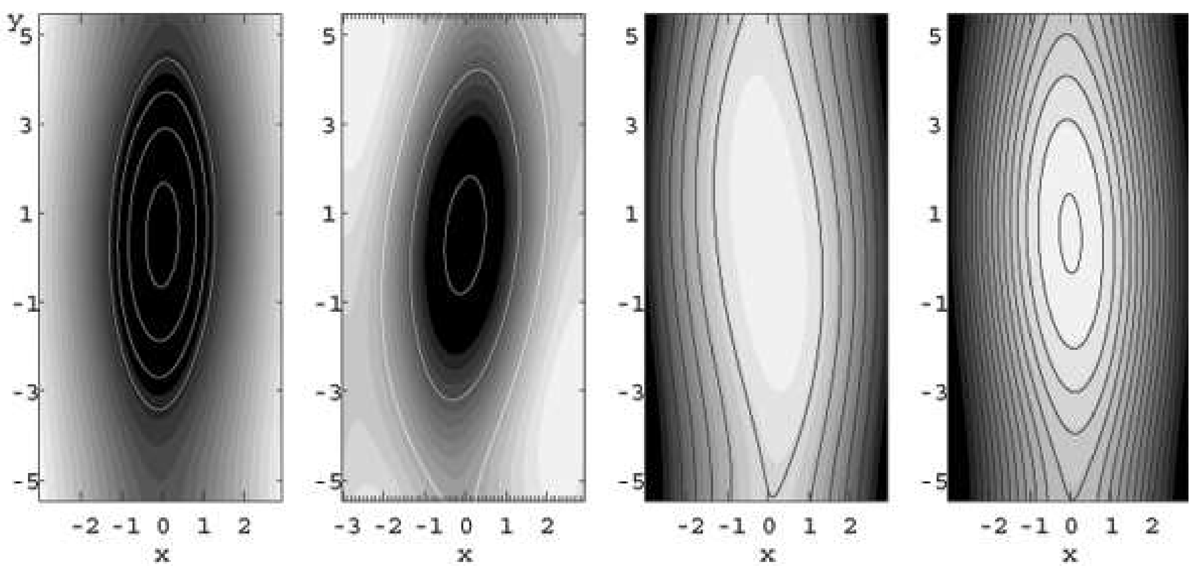

Fig.4 Contour plots of the kinetic stream function φ_{kin} for the same values of the parameters and of $p_{||}$ as in Fig. 3.

2 Conclusions

We have analyzed [7] the role of the Lagrangian invariants in the fluid and in the drift kinetic case.
This has allowed us to establish a clear link between the fluid and the kinetic regimes of the reconnection instability since the (two) fluid invariants and the (infinite) drift-kinetic invariants evolve in time in an analogous fashion.

In particular we have shown that the mixing of the Lagrangian invariants in x-y space leads to the formation of smaller and smaller spatial scales both in the fluid and in the kinetic regimes. The corresponding energy transfer towards increasingly smaller scales allows in both cases for the saturation of the magnetic island growth.

Finally, within the range of parameters explored in the simulations discussed in the present paper, we have not evidenced any onset of a secondary instability.

This result is fully consistent with the fluid simulations that show that the onset of the Kelvin-Helmoltz instability is impeded by increasing the electron temperature.

[7] "Foliation and mixing of the electron drift-kinetic distribution function in non-linear two-dimensional magnetic reconnection" by T. Liseikina, F. Pegoraro and E. Yu. Echkina, submitted for publication in Phys. of Plasmas.