

Non-Local Fluid Closure Approximations to Model Long Mean-Free-Path Dynamics.

Greg Hammett

w3.ppppl.gov/~hammett

Princeton Plasma Physics Lab

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Outline

* Motivation: long λ_{mfp} important in fusion & many other cases.

Chapman-Enskog breaks down

* Phase-mixing paradigm problem

fluid closure fit to $\nu \sim |k|^{-1} \nu_t$ damping

Fourier transform \Rightarrow non-local heat flux

* Alternate derivation: renormalization

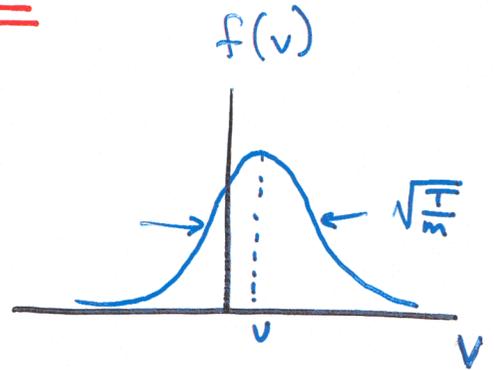
* Caveats

Classical Fluids are Highly Collisional

$$\lambda_{mfp} \sim 10^{-4} \text{ cm in air}$$

Particles move together in LTE

Local Maxwellian specified by n, v, T

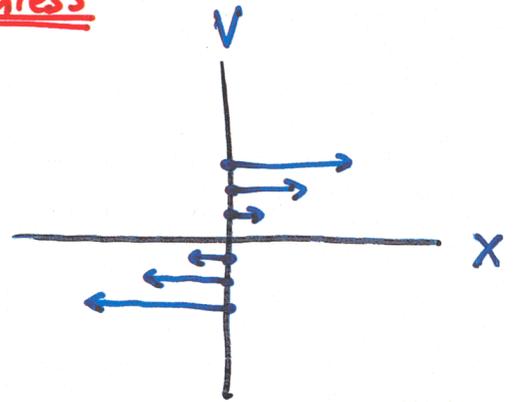


Particles slowly diffuse apart.

Fusion Plasmas are often Highly Collisionless

$$\lambda_{mfp} \sim \text{kilometers} \gg \text{tokamak size}$$

Particles quickly separate + "phase-mix"
 \Rightarrow collisionless Landau-damping



Other Low-Collisionality Systems:

* Often occur in space + astrophysical plasmas.

* The "~~fluid~~ gas" of stars evolving as a galaxy.

* Computer chips getting so small that feature sizes are approaching λ_{mfp} .

Widely thought that fluid simulations of plasma turbulence were inherently unable to model important kinetic effects.

Classical Chapman-Enskog closure procedure used by Braginskii fails for collisionless plasmas and misses kinetic effects such as Landau damping. (Improved by Chang & Callen)

"When collisions are infrequent, ... (the) heat flow depends on the detailed nature of the velocity distribution function, and cannot be determined in any simple way from the macroscopic (fluid) equations."

Spitzer, Physics of Fully Ionized Gases, 1962, p. 25 (see also p. 159).

"A property of Langmuir waves that is predicted by the Vlasov theory but which is completely outside the scope of fluid theory is the collisionless damping of electrostatic potentials..."

Krall and Trivelpiece, Principles of Plasma Physics, 1973, p. 386.

"What has happened to the Landau damping? One cannot expect the Landau damping to manifest itself in such a procedure, a power series expansion in $(k v_{th} / \omega)$, for in the Landau problem, in this limit, the damping goes as

$$\text{Im}(Z) \sim \exp\left(-\frac{1}{(k v_{th} / \omega)^2}\right) = e^{-\frac{1}{\epsilon^2}}$$

i.e., the damping goes to zero faster than any power of $(k v_{th} / \omega)$."

Oberman, Matt-57, Project Matterhorn, Princeton, 1960, p.8.

Paradigm Problem: Approximating e^{-1/ϵ^2} (the Universal Language of Mathematics)

Taylor-series expansion:

$$f(x) \approx f(0) + x f'(0) + \frac{1}{2} x^2 f''(0) + \dots$$

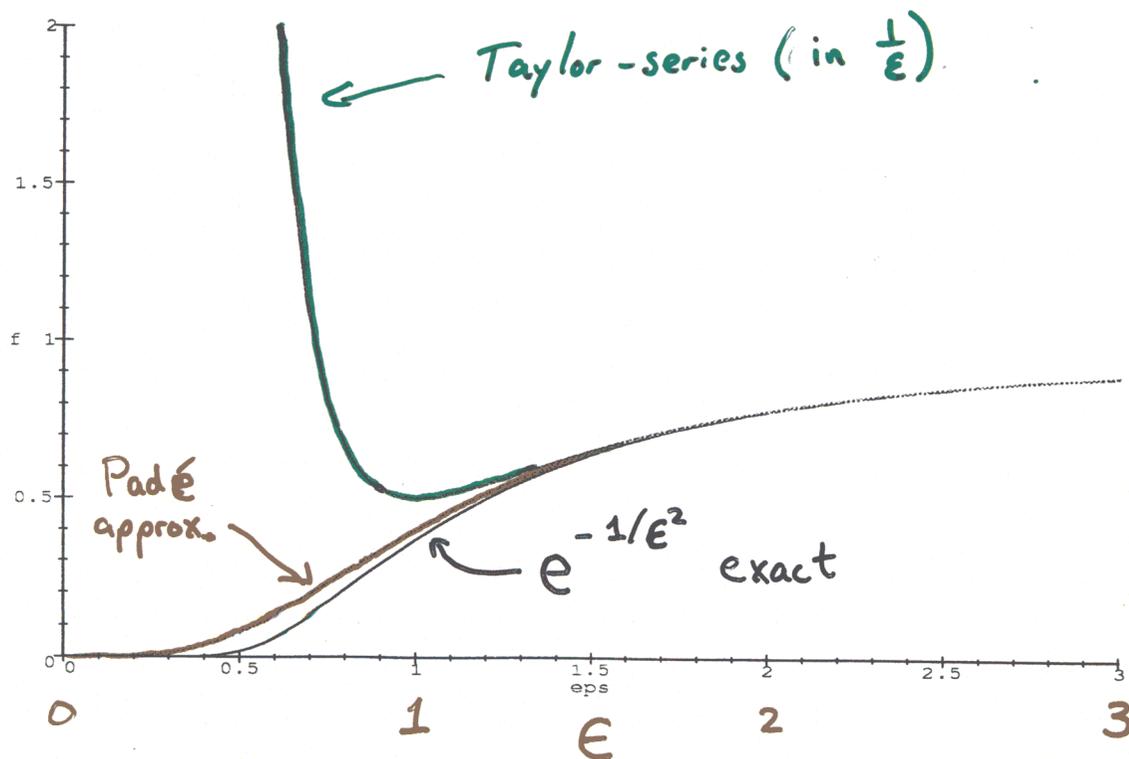
$$e^{-1/\epsilon^2} \approx 0 + \epsilon 0 + \frac{1}{2} \epsilon^2 0 + \dots$$

or substitute $x = 1/\epsilon^2$, and try expanding around $x = 0$:

$$e^{-x} \approx 1 - x + \frac{1}{2} x^2 = 1 - \frac{1}{\epsilon^2} + \frac{1}{2} \frac{1}{\epsilon^4} + \dots$$

Diverges as $\epsilon \rightarrow 0$!!!

(Reminiscent of “renormalization” problems in quantum mechanics, etc., where perturbation expansions must be summed to all orders...)



Relevant to plasmas:

$$\text{Im}(Z) \propto e^{-\left(\frac{\omega}{k v_t}\right)^2} = e^{-\frac{1}{\epsilon^2}}$$

$\epsilon \rightarrow 0$ in cold plasma fluid limit

Padé Approximations of e^{-1/ϵ^2}

Padé Approximations are Ratio of Polynomials.
(Taylor series are a simple polynomial)

Example:

$$e^{-1/\epsilon^2} = \frac{1}{e^{+1/\epsilon^2}} \approx \frac{1}{1 + \frac{1}{\epsilon^2} + \frac{1}{2}\frac{1}{\epsilon^4} \dots} = \frac{2\epsilon^4}{1 + 2\epsilon^2 + 2\epsilon^4}$$

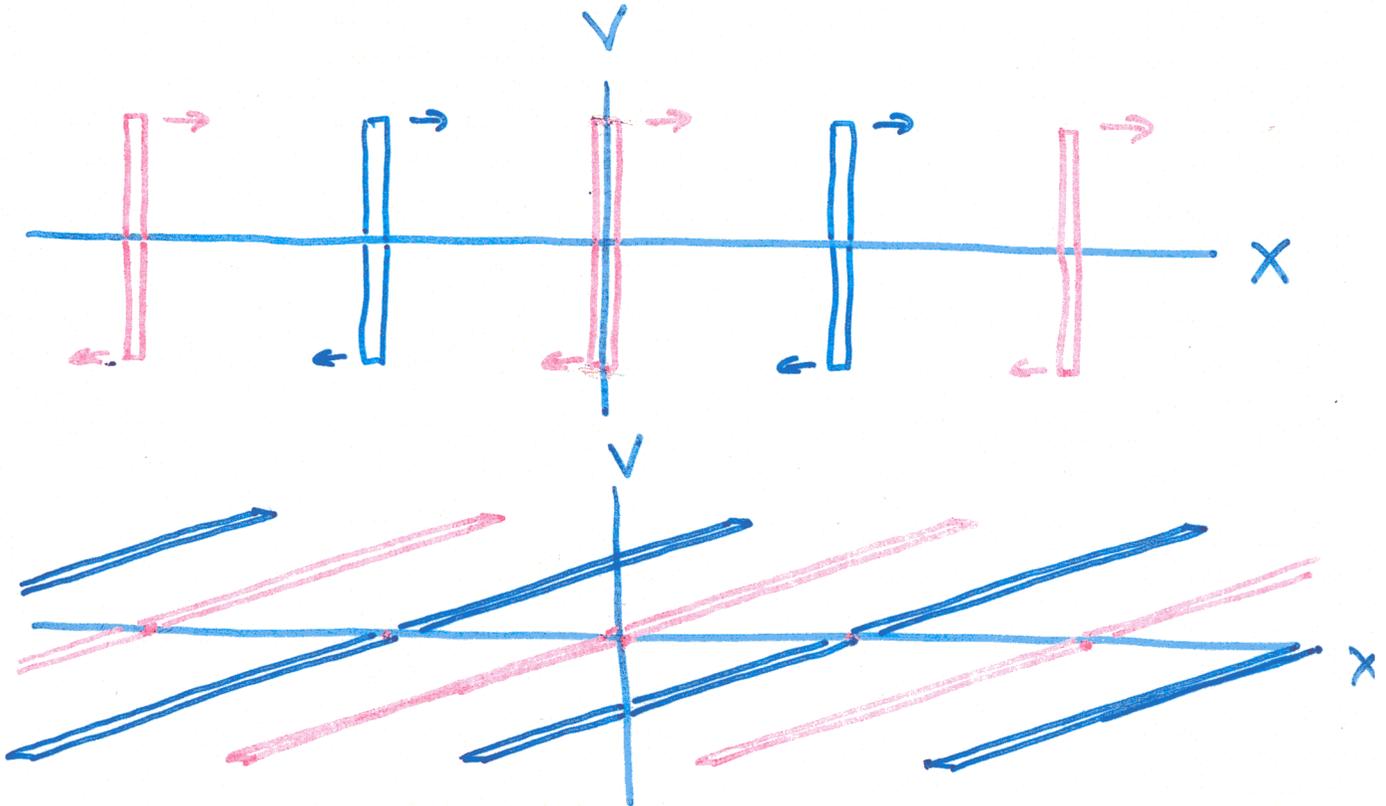
“4-pole approximation”

Padé approximations are often much more robust
(they have bounded errors) than Taylor series.

The trick for more complicated equations is finding
the equivalent of a Padé approximation.

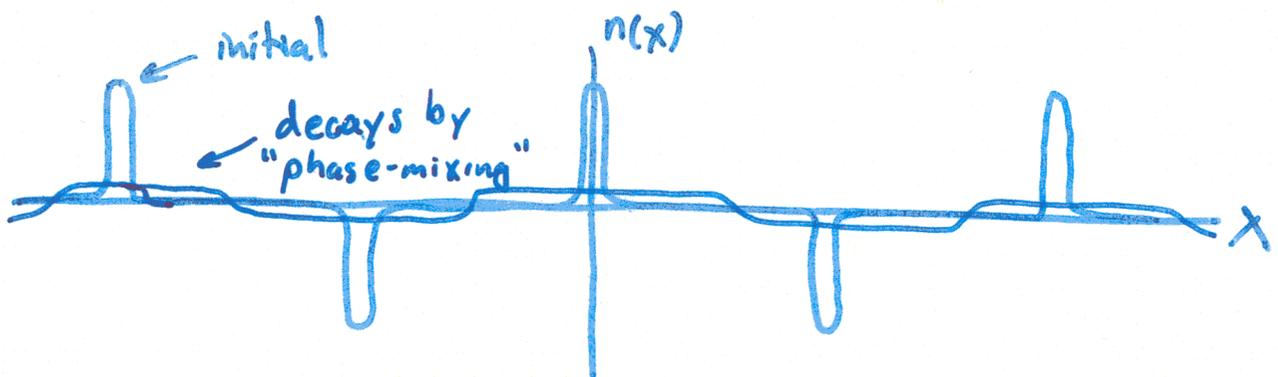
$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} = 0$$

Contours of $f(x, v)$:



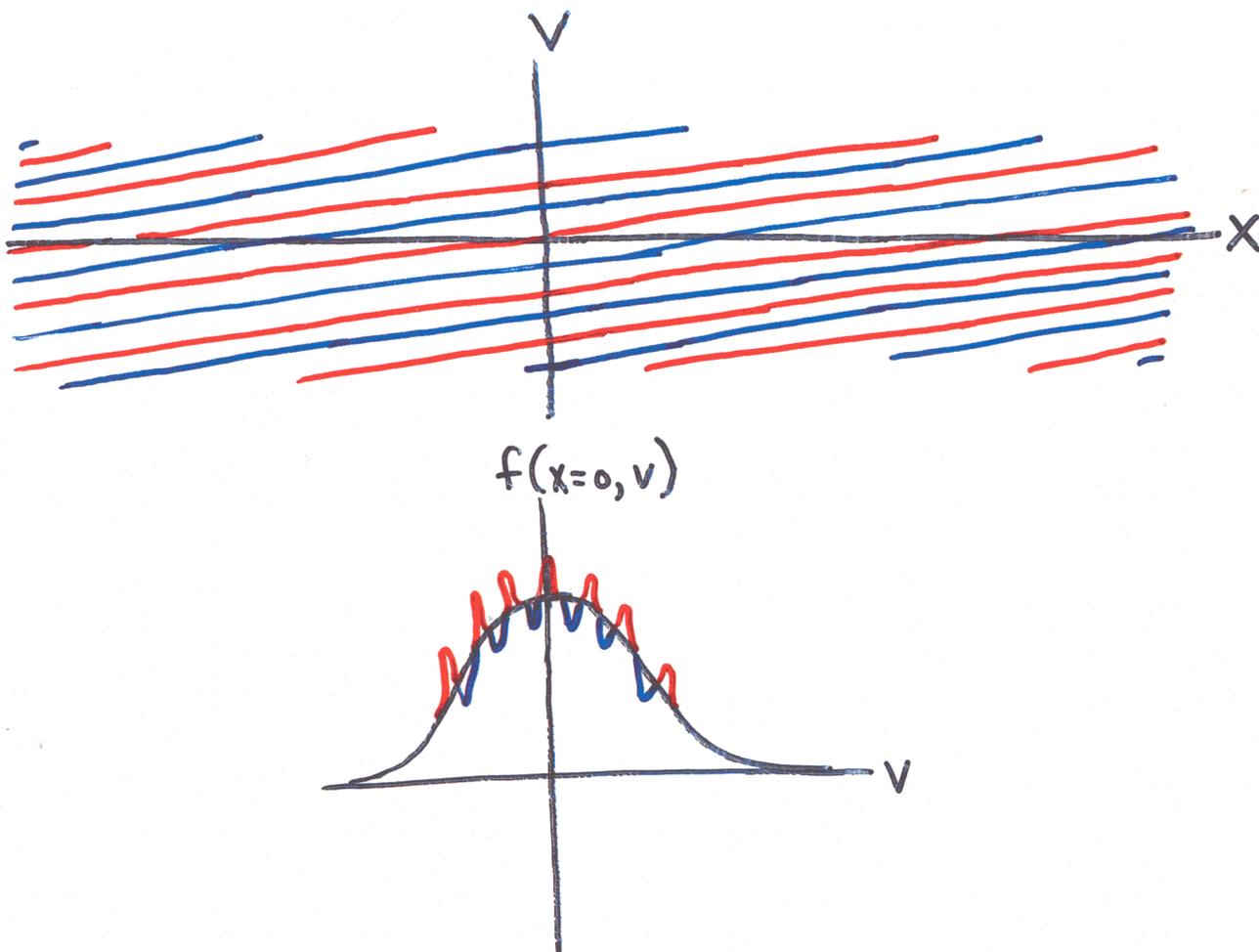
As time ↑, goes to finer velocity scales...

$$n(x) = \int dv f$$



Phase-mixing damping rate $\sim |k| v_{\pm}$, roughly analogous to shearing rate $|\nabla u|$ or eddy-turnover time $|k|/U$ in Navier-Stok

Long time limit:



Even weak collisions will wipe out such small scale velocity features

$$G(f) = \nu_{\text{col}} v^2 \frac{\partial^2 f}{\partial v^2}$$

$$f \propto e^{-ikvt}$$

$$\frac{\partial^2 f}{\partial v^2} \approx -k^2 t^2 f$$

Collisions dominate at $\tau \propto \nu_{\text{col}}^{-1/3}$

A Simple Phase-Mixing Paradigm

(Carl Oberman reminded me of this view of Landau damping.)

Consider a 1-D kinetic Eq. for $f(z, v, t)$, with no \underline{E} field:

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial z} = 0$$

The exact solution is $f(z, v, t) = f_0(z - vt, v)$

Consider a single Fourier mode in z with a Maxwellian distribution in v :

$$f_0 = n_0 e^{ikz} f_M(v)$$

$$f = n_0 e^{ik(z-vt)} \frac{1}{\sqrt{2\pi v_t^2}} e^{-v^2/(2v_t^2)}$$

At any fixed v , f oscillates in time with $\omega = kv$ & no damping. However, any v -moment of f will exponentially decay in time:

$$n(z, t) = \int dv f = n_0 \frac{e^{ikz}}{\sqrt{2\pi v_t^2}} \underbrace{\int dv}_{\text{mixing}} \underbrace{e^{-ikvt}}_{\text{phases}} e^{-v^2/(2v_t^2)}$$

$$n(z, t) = n_0 e^{ikz} e^{-k^2 v_t^2 t^2 / 2}$$

The Closure Problem in the Fluid Moment Hierarchy

$$\int dv v^l * \quad \frac{\partial f}{\partial t} + v \frac{\partial f}{\partial z} + \frac{e}{m} E_{||} \frac{\partial f}{\partial v} = 0$$

$$\frac{\partial}{\partial t} \int dv v^l f + \frac{\partial}{\partial z} \int dv v^{l+1} f - \frac{e}{m} E_{||} \int dv l v^{l-1} f = 0$$

$$l=0 \Rightarrow \quad \frac{\partial n}{\partial t} + \frac{\partial}{\partial z} (n v) = 0$$

$$l=1 \Rightarrow \quad \frac{\partial}{\partial t} (m n v) + \frac{\partial}{\partial z} (v m n v) = - \frac{\partial p}{\partial z} + e n E_{||}$$

$$l=2 \Rightarrow \quad \frac{\partial p}{\partial t} + \frac{\partial}{\partial z} (v p) = -2 p \frac{\partial v}{\partial z} - \frac{\partial q}{\partial z}$$

etc.

Exact, nonlinear, conservation laws (particles, momentum, energy, ...)

But each $\langle v^l \rangle$ Eq. requires knowledge of $\langle v^{l+1} \rangle$

= Infinite Hierarchy \Rightarrow Closure problem

Usual approximations: neglect a higher moment

$p=0$ cold plasma approximation, No phase-mixing.

$q=0$, or $q=\infty$ (isothermal $\nabla T=0$) still no phase-mixing.

Seemingly "better" approximation: $\langle v^{10} \rangle = \alpha \langle v^8 \rangle \frac{\langle v^8 \rangle}{\langle v^6 \rangle}$

still fails to reproduce phase-mixing.

Traditional fluid eqs. fail to reproduce phase-mixing
 (even if higher-order moments are kept.)

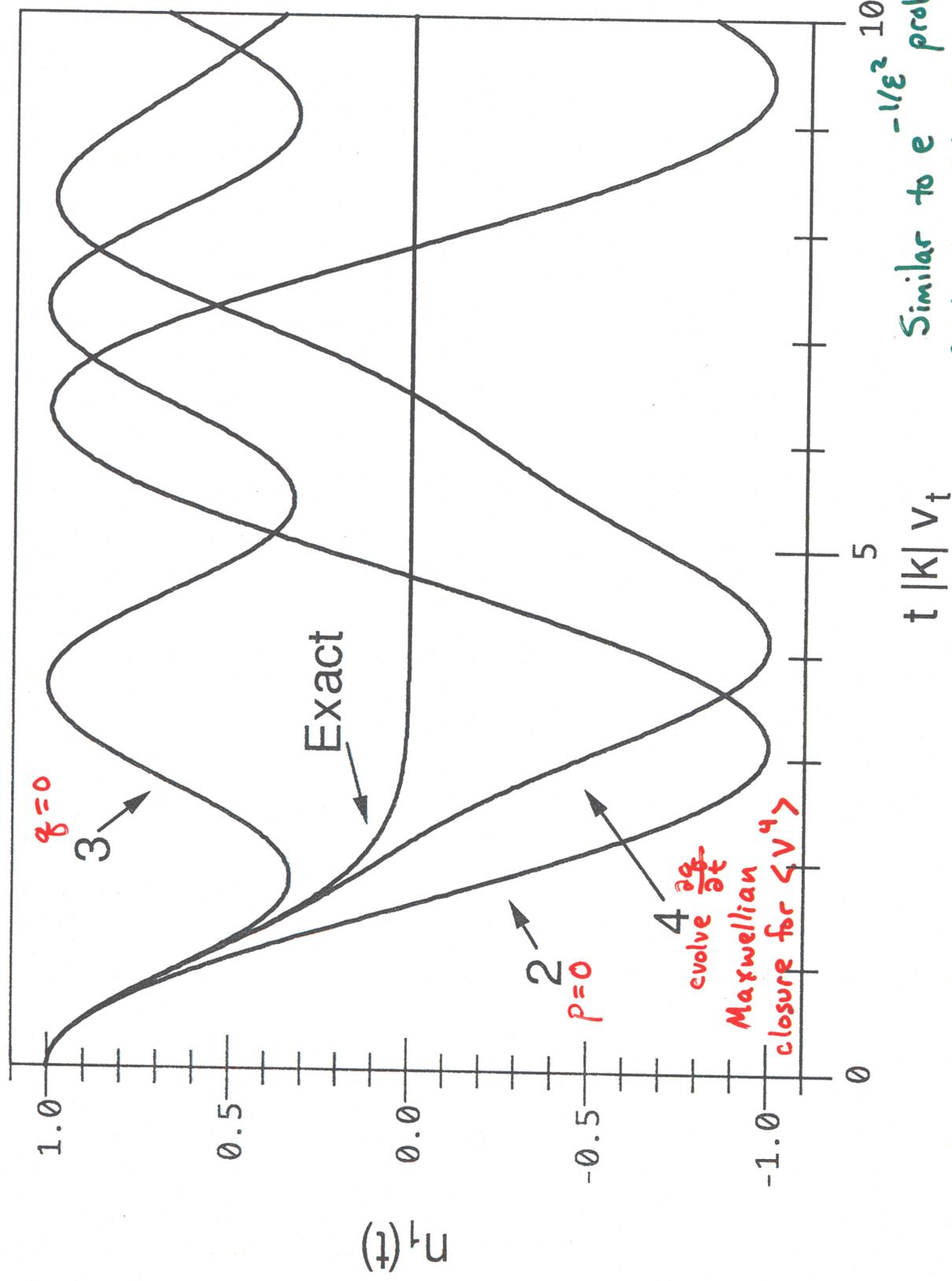


Fig. 1

Similar to e^{-1/ϵ^2} problem!
 Good as a Taylor-series for short time,
 breaks down for long time...

Diffusive-type Closure needed to model

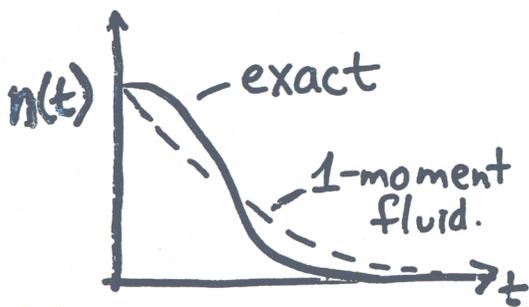
Collisionless Phase-Mixing.

Simplest possible 1-moment fluid model (too simple for most purposes). Start with exact density conservation Eq.:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial z}(nu) = 0$$

Approximate higher moment (nu) in terms of lower moments (n), with a diffusive-type term to model phase-mixing damping:

$$nu \approx -D \frac{\partial n}{\partial z} \Rightarrow \frac{\partial n_k}{\partial t} + Dk^2 n_k = 0$$



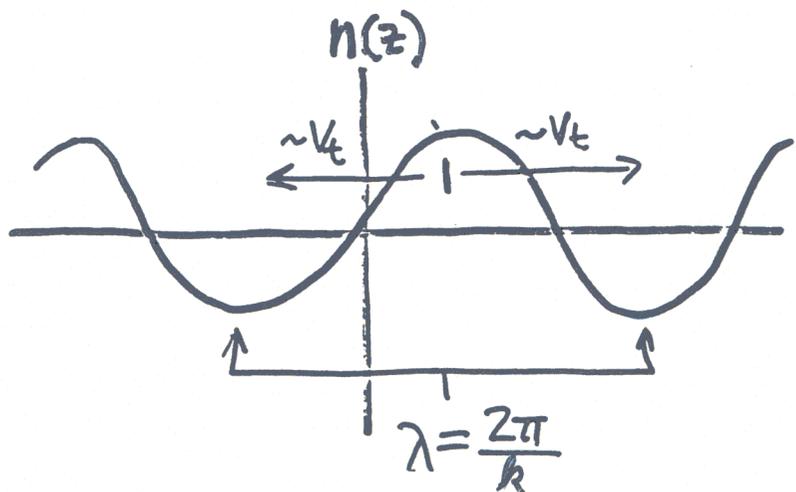
$$\text{Fluid Model: } n_k(t) \propto e^{-Dk^2 t}$$

$$\text{Exact Kinetic: } \propto e^{-k^2 v_t^2 t^2 / 2}$$

$$D = \sqrt{\frac{2}{\pi}} \frac{v_t}{|k|}$$

$$\text{Damping rate } \nu \sim |k| v_t \sim Dk^2$$

(D is an integral operator in real space.)



New closure approximation for fluid moments
successfully models long-time-scale phase-mixing.

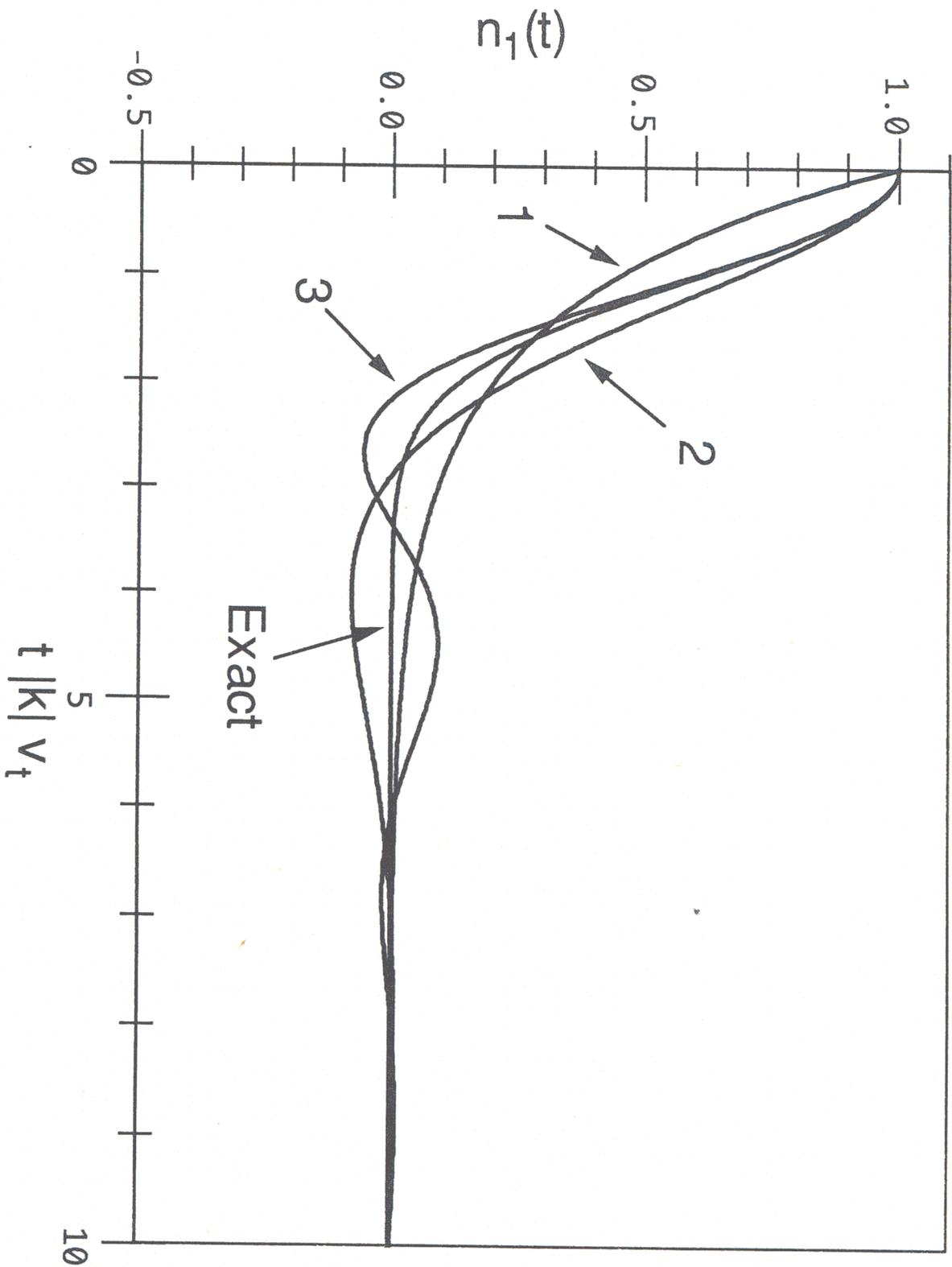


Fig. 2

New Landau-fluid closure approximation
Converges rapidly as more moments are kept.

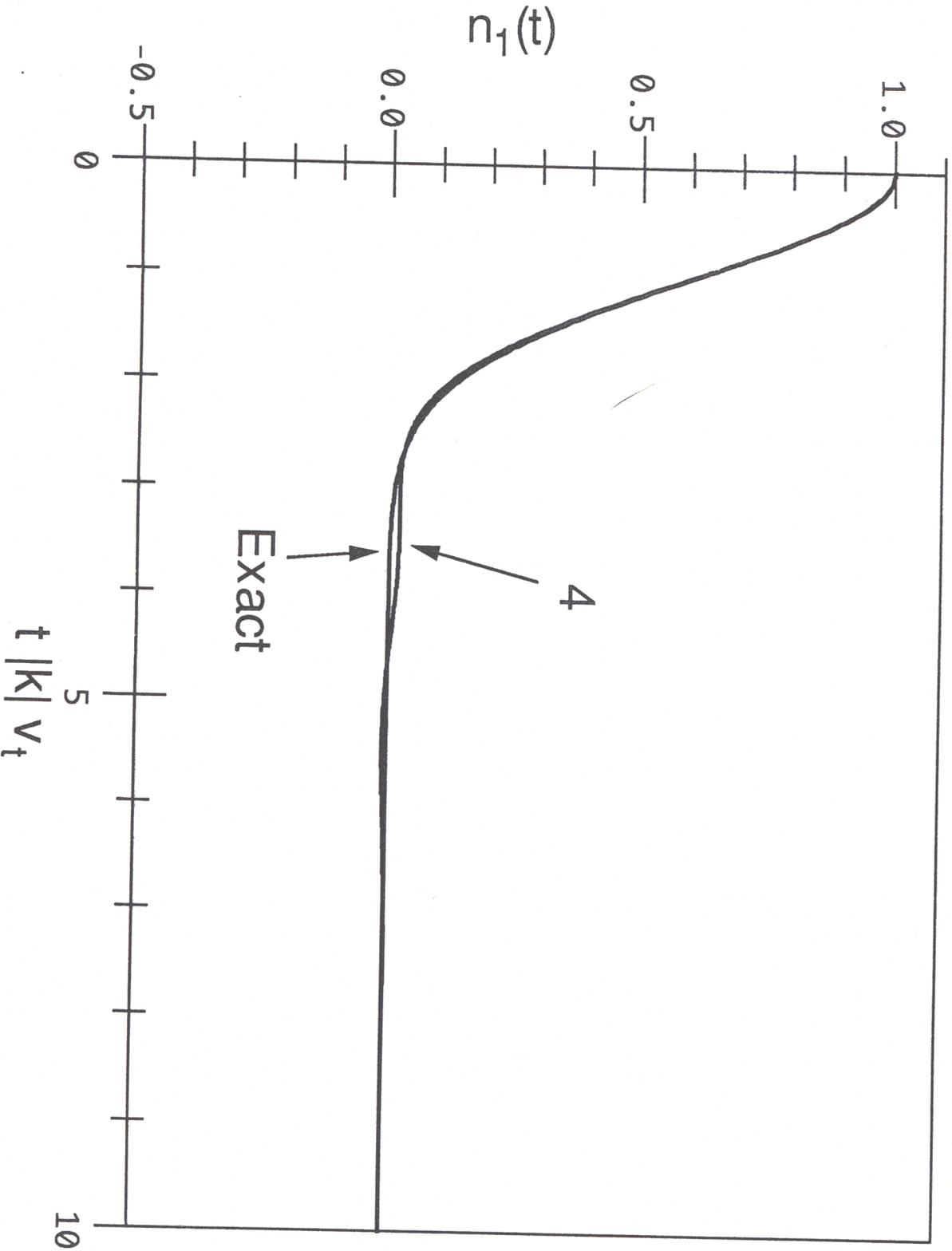


Fig. 3

$$q = -n\chi \nabla T$$

Thermal Diffusivity
(Conductivity)

$$\chi \cong \frac{v_t^2}{\nu_{\text{coll}} + |k|v_t}$$

Collisional limit
Braginskii
Chapman-Enskog

Landau-damping
Phase-mixing limit
on heat flux.

related to "flux limiters"
in laser-plasma problems.

$$\chi \approx \frac{v_t}{|k|} \Rightarrow \text{non-local operator in real space, not simple diffusion.}$$

We developed these Landau-closure approximations for tokamak drift-wave turbulence. Other applications:

- * α -particle driven TAE instability (Spong, Hedrick, et al.)
- * Resistive-wall MHD (Bondeson & Ward)
- * Laser-plasma filamentation (Kaiser et al.)
- * Extend Zakharov Eqs. of Langmuir turbulence, ionospheric applications (Goldman & Newman, H. Rose et al.)

Probably other useful applications.

Tokamak core microturbulence is
small scale & small amplitude

$$n = n_0 + \tilde{n}(z, t)$$

$$T = T_0 + \tilde{T}(z, t)$$

↑
Small

How to generalize non-local heat flux
to
 $n(z) \neq n_0$?

Constraints? Have to satisfy entropy ↑
Test problems?

Luciani, Mora, & Virmont 1983

nonlocal heat flux semi-empirical

orig. for Laser-plasma interactions

compared w/ expts. & codes with large variations in n & T

still used: S. Nayakshin, "Feeding the black hole w/ condensing accretion flows..."
astro-ph/0402469

Small fluctuation limit:

$$q_{\text{LMV}}(z) = \int \frac{dz'}{2\lambda} \underbrace{q_{\text{ce}}(z')}_{-n \frac{V_t^2}{v_{\text{coll}}} \frac{\partial T}{\partial z'}} e^{-\frac{|z-z'|}{\lambda}}$$

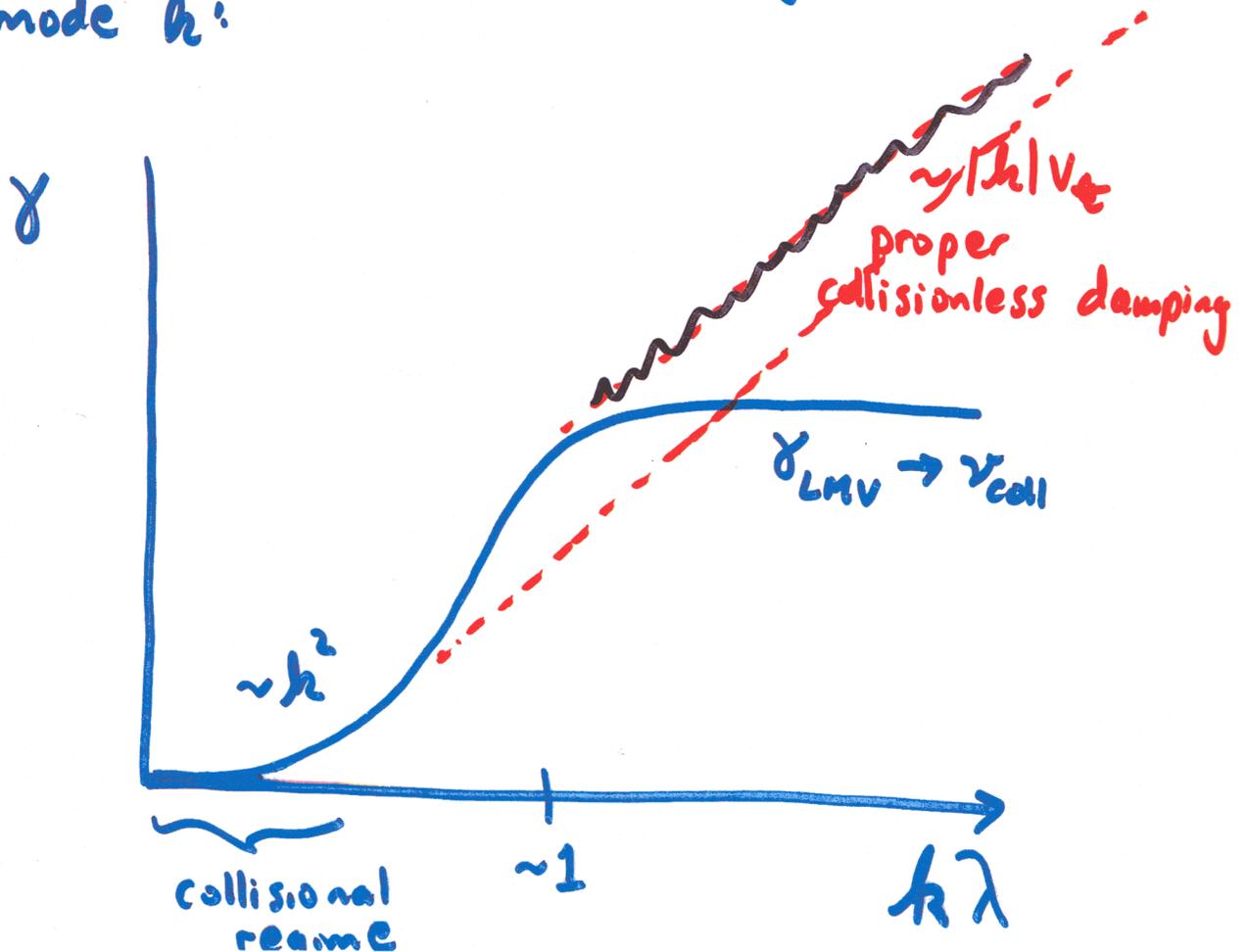
Integrate by parts:

$$q_{LMV} \propto - \int dz' \frac{T(z+z') \text{Sgn}(z')}{\lambda} e^{-\frac{|z'|}{\lambda}}$$

differs from our

$$q_{LF} \propto - \int dz' \frac{T(z+z')}{z'} = g\left(\frac{z'}{\lambda}\right)$$

Fourier transform to find damping rate γ for mode h :



$$q = -n \chi \nabla T$$

$$q_k = -n \frac{v_t}{|k|} i k T_k$$

Fourier-Transform $q_k \rightarrow q(z)$

$$q(z) = -n_0 v_{t0} \text{P.V.} \int_{-\infty}^{\infty} dz' \frac{T(z+z')}{z'}$$



Hilbert Transform

incl. collisions (Snyder, Hammett, Dorland 97)

$$q(z) = -n_0 v_{t0} \int_0^{\infty} dz' \frac{T(z+z') - T(z-z')}{z'} \frac{1}{1 + \frac{z'^2}{\lambda^2}}$$



Non-local derivative



Cutoff
 $\sim \lambda_{mp}$

Renormalized Dissipation Approach

Parker & Carati, PRL (1995)

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} = \frac{\partial}{\partial v} \left(\nu_0 v f + \nu_0 v_t^2 \frac{\partial f}{\partial v} \right)$$

↑
= 1 norm.

Chapman - Enskog : $\nu_0 \rightarrow \infty$

Here : $\nu_0 \rightarrow 0$

$$f(x, v, t) = \int dk \int d\omega e^{ikx - i\omega t} \hat{f}(k, \omega, v)$$

Hermite polynomial basis useful

[Hammett et al 93, Armstrong et al 70,

Grant & Feix 67, Sadowski 67]

$$\hat{f}(k, \omega, v) = e^{-\frac{v^2}{2}} \sum_{\ell} \alpha_{\ell}(k, \omega) h_{\ell}(v)$$

Orthonormal
Hermite Polynomials

$$\alpha_{\ell} = \int dv h_{\ell}(v) \hat{f}(k, \omega, v) \quad \text{same info as} \\ \int dv v^{\ell} f$$

$$(-i\omega + l\nu_0)\alpha_l + ik\{\sqrt{l}\alpha_{l-1} + \sqrt{l+1}\alpha_{l+1}\} = 0$$

↑ dominates for sufficiently large l
 $\alpha_{l+1} \approx 0$

$\omega \rightarrow 0$ "long time behavior in renormalization group theories"

(Forster, Nelson, Stephen 77)

$$l\nu_0\alpha_l + ik\sqrt{l}\alpha_{l-1} = 0$$

Solve for α_l & insert into next lower Eq:

$$\left[(l-1)\nu_0 + \frac{\hbar^2}{\nu_0} \right] \alpha_{l-1} + ik\sqrt{l-1}\alpha_{l-1} = 0$$

$$\equiv (l-1)\tilde{\nu}[l-1]$$

↑ renormalized collision rate

$$\tilde{\nu}[l-1] = \nu_0 + \frac{\hbar^2}{(l-1)\tilde{\nu}[l]}$$

recursion formula

Fixed point for $l \ll \frac{h^2}{v_0^2}$

$$\tilde{\nu}[l] \approx \frac{|h|}{\sqrt{l}}$$

putting in dimensional factors gives same scaling as original approach

$$\tilde{\nu}[l] \approx \frac{|h| v_t}{\sqrt{l}}$$

various extensions possible...

Caveats

Landau-fluid closures are approximations
won't always work

Conservation of trouble

2 main challenges:

① If small ~~and~~ scale features in $f(v)$ become important

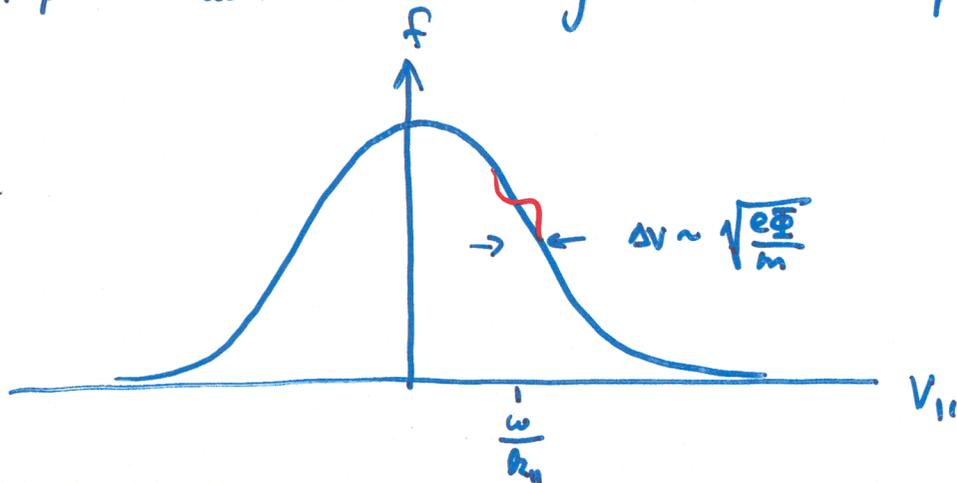
- weak turbulence

- particle trapping in wave. ζ in $v_{||}$ direction
Mattor et al.

② Multidimensional extensions
in the presence of
recurring orbits...

More complete discussion in paper....

Example: Quasilinear flattening turns off damping:



To resolve Δv scale requires $N \sim \left(\frac{v_t}{\Delta v}\right)^2$ Hermite polynomials or fluid moments.

For some problems (plasma echoes, ITG weak-turbulence very near marginal stability - Mator) the Landau-Fluid approach would require a huge # of moments \Rightarrow inefficient.

But for tokamak turbulence in most parameter regimes:

* Many modes at various $\frac{\omega}{k_{||}}$, Can't turn off one mode without driving another... Probably gives rise to turbulent diffusion in $v_{||}$?
 \leftarrow Landau-Fluid approx. okay...

* Even weak collisions can be important at small v scales:

$$\frac{\partial f}{\partial t} = \nu v_t^2 \frac{\partial^2 f}{\partial v^2} \sim \nu \left(\frac{v_t}{\Delta v}\right)^2 \tilde{f}$$

* Tokamak turbulence driven primarily by spatial gradients of density & temperature, not sensitive to velocity space details. Basic picture of Nonlinear EXB coupling between unstable modes & Landau-damped modes...

Caveats: (Hammett et al., Plasma Physics Contr. Fus. 35,
973 (1993))

There are some cases where the Landau-fluid approximations don't work well (or require many moments to converge), where it is necessary to follow full details of full $f(x, v, t)$. (plasma echoes, quasilinear flattening near narrow resonances). (Mattor, Phys. Fluids B 4, 3952 (1992)).

But works well as a model of rate at which f phase-mixes to small velocity scales which can then be ignored.

(Philosophically similar to subgrid turbulence models, or to the EDQNM simplified vs. of the DIA)

Usually there are many processes which quickly wipe out small velocity scales (collisions) or make them incoherent (turbulence).

Should be well-suited to typical strong-turbulence tokamak regimes.

Provide accurate linear growth rates for instabilities which drive the turbulence, & for damped modes.

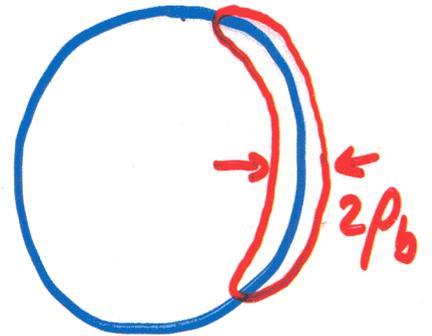
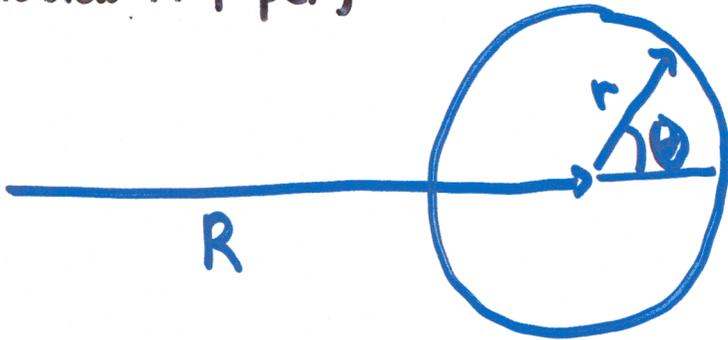
$E \times B$ nonlinearities in fluid Eqs. couple these together.

Fluid moment Eqs. express important conservation laws (particles, momentum, magnetic moment, parallel energy, ...) which constrain the nonlinear dynamics of the turbulence.

Multi-dimensional Challenge

(for illustration only, more precise problem in paper)

banana orbits..



$$2\rho_b \sim \frac{r}{\sqrt{\epsilon}} \ll r$$

$$\frac{\partial f}{\partial t} + \frac{v_{||}}{R} \frac{\partial f}{\partial \theta} + \frac{E\rho}{R} \sin\theta \frac{\partial f}{\partial r} = 0$$

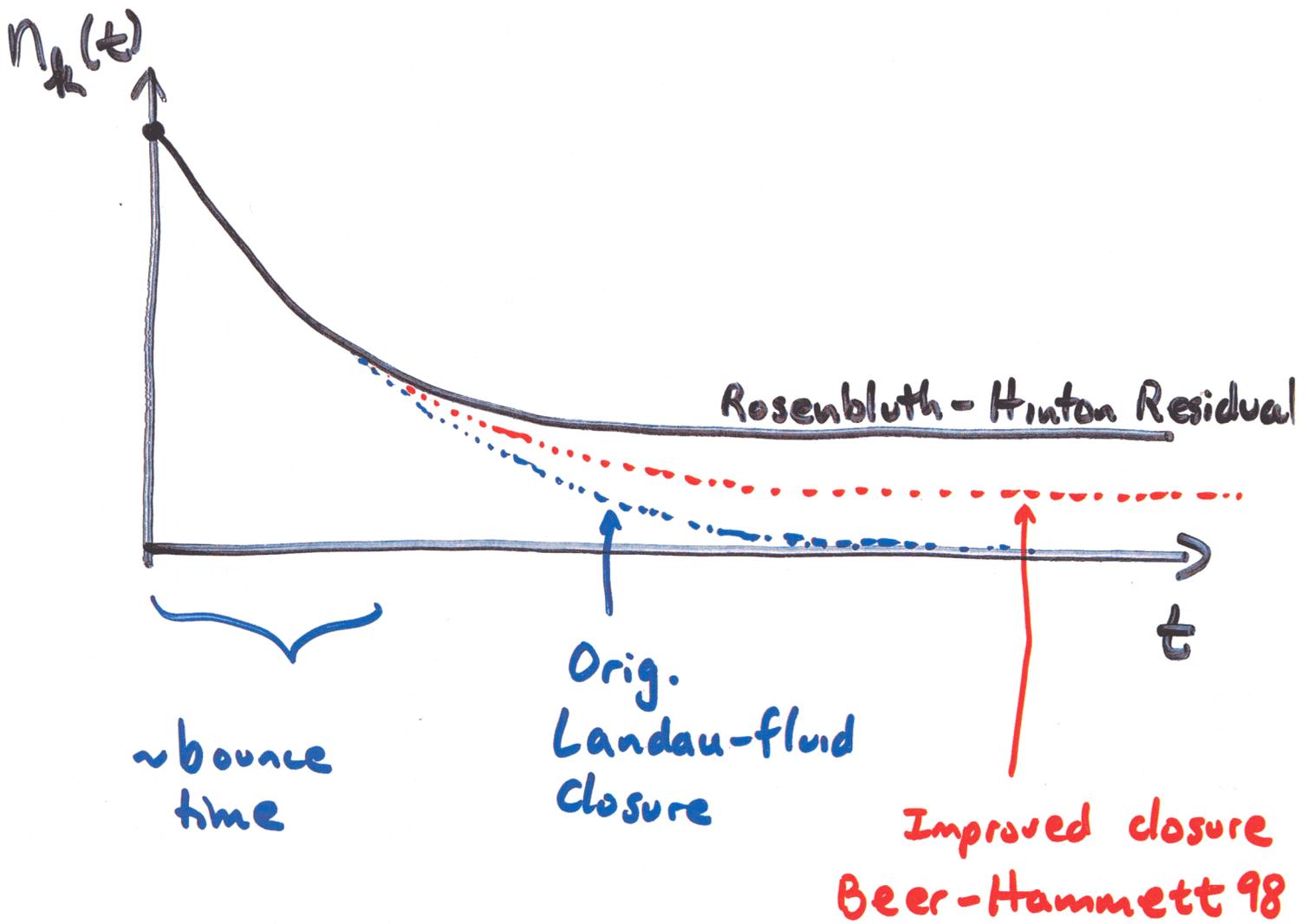
$$\text{I.C. } f(r, \theta, E, \lambda, t=0) \propto e^{-E} e^{i\lambda r}$$

$$v_{||} = \pm \sqrt{2E} \sqrt{1 - \lambda B(\theta)}$$

$$B(\theta) = B_0 (1 - \epsilon \cos\theta)$$

$$n_\lambda(t) = \int d^3v f$$

$$\propto B(\theta) \int_0^\infty dE \sqrt{E} \int_{-\lambda_{\max}}^{\lambda_{\max}} \frac{d\lambda}{\sqrt{1 - \lambda B(\theta)}} f$$



Would like to do better.

Direct Continuum/Eulerian codes

for solving $f(\vec{x}, E, \lambda, t)$ (5D + time)

very successful in recent years.

- advanced algorithms: pseudo-spectral
implicit linear dynamics,
high order Gauss-Legendre integration

- parallelize well

- Less resolution in ~~space~~ velocity space
needed than in \vec{x}

~ 10 E

~ 30 λ

$\sim \overset{64}{\cancel{52}} * 96 * 32$ \vec{x}

Class of interesting problems
where resolution needed is modest

Not as extreme as high-Reynolds #
Navier-Stokes. But some ^{important} problems
need more resolution.

References

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