

Resonance Problems in
Photonics - II:

Scattering resonances of
Microstructures and
Homogenization

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Outline

I Photonic microstructures and their optical properties

II The scattering resonance problem for Schrödinger equation (scalar Helmholtz approx)

III Multiple scale analysis for class of microstructures

numerical method, comparison of asymptotics and DNS (Higher order homogenization)

IV Rigorous theory for high contrast microstructures

(large amplitude and oscillatory potentials)

preconditioned Lippman-Schwinger equation

→ estimates on error in homogenization
depending on regularity of
microstructure

Microstructure enables control of optical properties of medium



Photonic bandgap (PBG) effect, Bragg resonance,
Dispersive v. nonlinear effects

3D PBG

Bulk Bragg grating

Photonic crystal

fiber

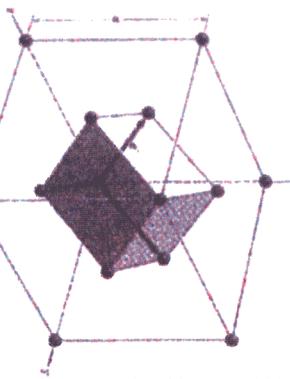


Figure 15. The rhombohedral primitive cell of the face-centered cubic crystal. The primitive translation vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ connect the lattice point at the origin with lattice points at the face centers. As shown, the primitive vectors are:

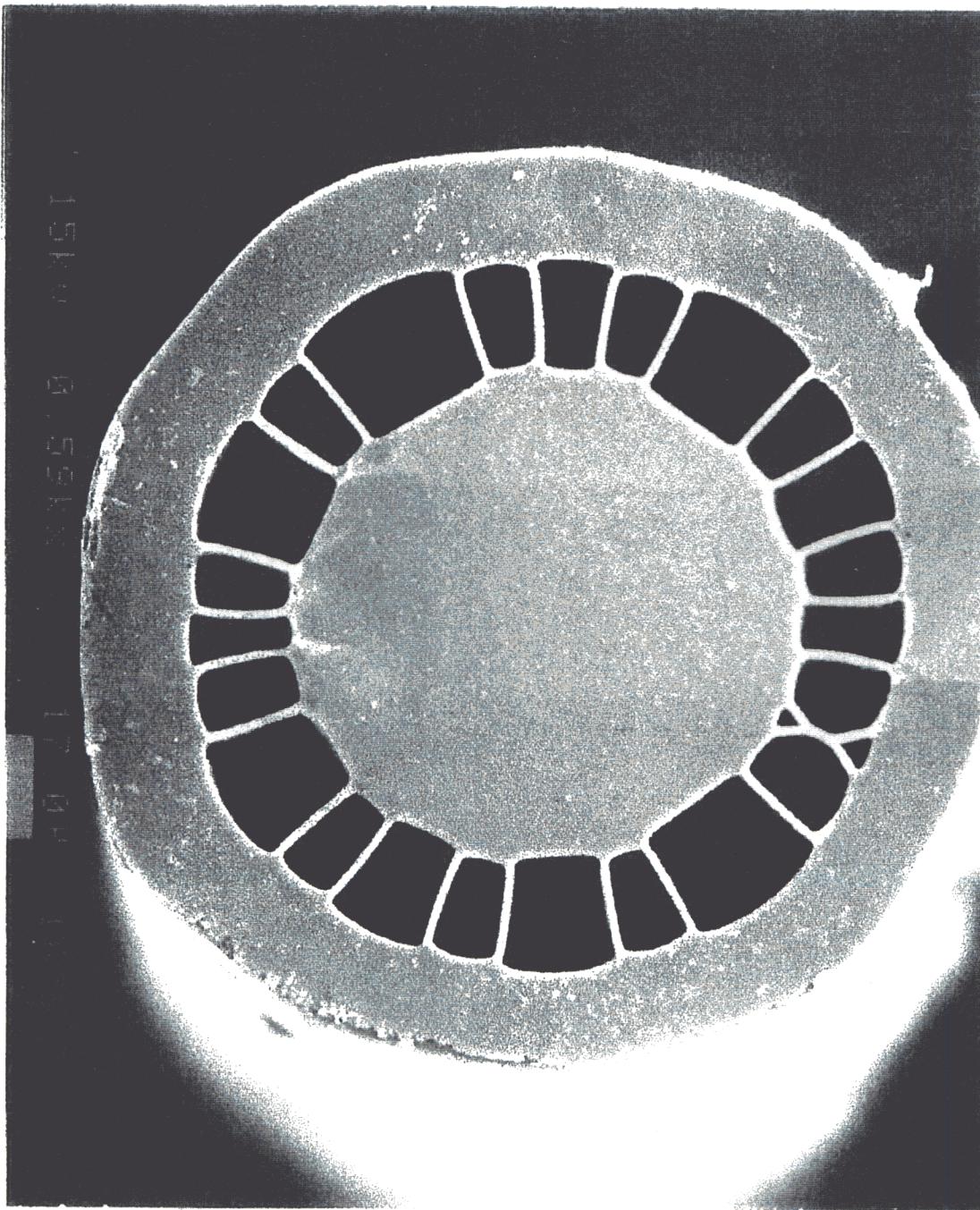
$$\mathbf{a} = \frac{\sqrt{3}}{2} (\mathbf{i} + \mathbf{j}) ; \quad \mathbf{b} = \frac{\sqrt{3}}{2} (\mathbf{j} + \mathbf{k}) ; \quad \mathbf{c} = \frac{\sqrt{3}}{2} (\mathbf{i} + \mathbf{k}) .$$

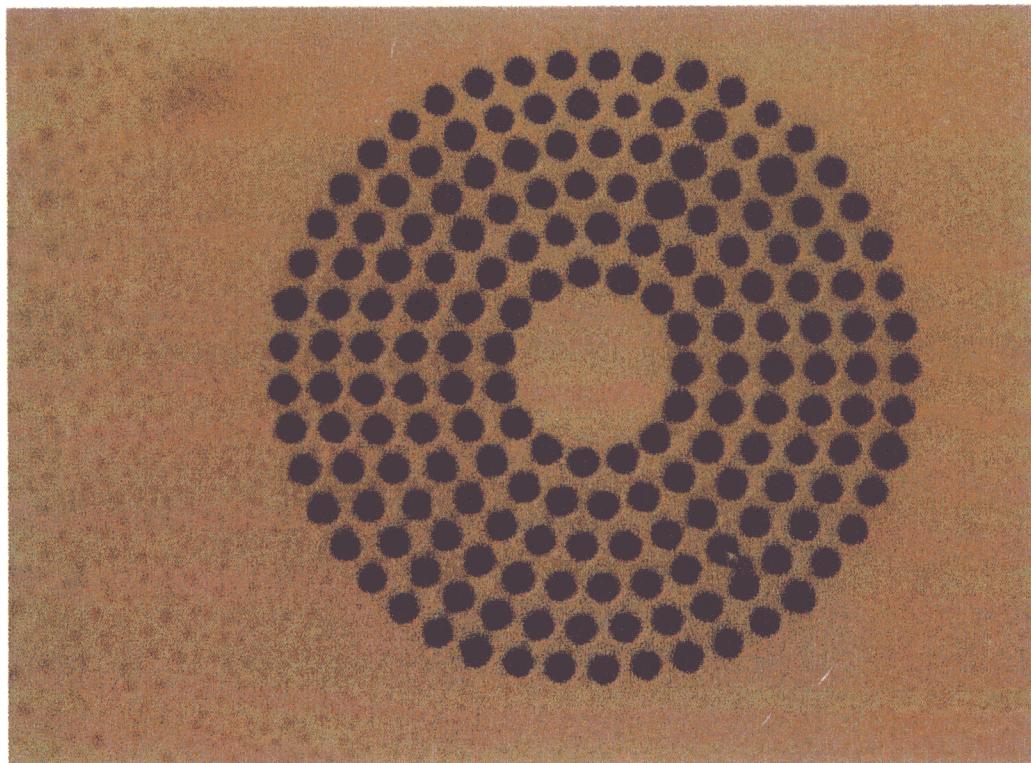
The angles between the axes are 60°.

Transverse and/or Longitudinal
Microstructure

Fig. 1(c)

Eggleton et al.





Ryan Bise, OFS

Photonic microstructures - composite media whose *optical properties* can be tuned through:

- variations of distribution of microfeatures
- variations of geometry of individual microfeatures
- variations in refractive index contrasts
e.g. air-glass - 50%

Application potential:

Communications – tunable devices, transmission media

Quantum information science – Cavity QED

Optical properties

Modes: $E_\omega(\mathbf{x})e^{i(\beta(\omega)z-\omega t)}$, $\beta = \beta(\omega)$

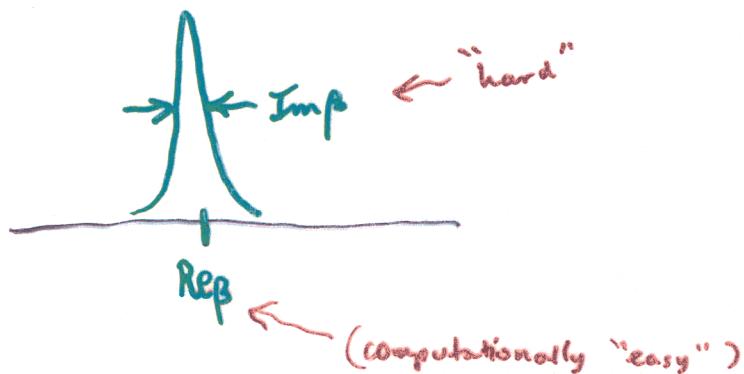
$$\begin{aligned} E(\mathbf{x}_\perp, z, t) &\sim \int e^{i(\beta(\omega)z-\omega t)} E_\omega(\mathbf{x}_\perp) d\omega \\ &\sim e^{i(\beta(\omega_0)z-\omega_0 t)} \\ &\times \underbrace{\int e^{i(\omega-\omega_0)[\beta'(\omega_0)z-t]} e^{i\frac{1}{2}(\omega-\omega_0)^2\beta''(\omega_0)z} d\omega}_{\text{energy prop.}} \underbrace{E_{\omega_0}(\mathbf{x}_\perp)}_{\text{spreading}} d\omega \end{aligned}$$

- Group velocity $\sim \beta'(\omega_0)^{-1}$

- Dispersion $\sim \beta''(\omega_0)$

(WRS - zero disp. pt.
shift)

- Leakage rates (scattering resonances)
 $\sim \Im \beta(\omega_0) > 0$



III. Model Problem

("W-W" 1930)

$$\frac{\partial u}{\partial t} - c \frac{\partial u}{\partial x} = 0 \quad \text{"waves"} \\ c, \text{ speed}$$

$$i\dot{a} - \omega a = 0 \quad \text{"oscillator"}$$

$$u(x,t) = f(x+ct)$$

$$a(t) = a_0 e^{-i\omega t}$$

Couple discrete oscillator to wave supporting medium

$$(1) \frac{\partial u}{\partial t} - c \frac{\partial u}{\partial x} = \varepsilon \chi(x) a(t)$$

$$(2) i\dot{a}(t) - \omega a(t) = -i \varepsilon \int \chi(x) u(x,t) dx$$

$$\text{Total energy} = \int |u(x,t)|^2 dx + |a(t)|^2 \\ \text{conserved} .$$

Solve IVP : $\partial_t u - c \partial_x u = \varepsilon X a$
 $i \dot{a} - \omega a = -i \varepsilon \int X u dx$

$a(0) = a_0 , \quad u(x, 0) = u_0(x) \equiv 0$

for simplicity

Also, take $X(x) = \frac{1}{\sigma} \chi\left(\frac{x}{\sigma}\right) \rightarrow \delta_0$

as $\sigma \downarrow 0$ Dirac mass at 0

Solution : $a(t) \sim a_0 e^{-i\omega t} e^{-\frac{2\varepsilon^2}{|c|} t}$
 as $t \rightarrow +\infty$

1) Under perturbation, frequency ω becomes a complex frequency

$a(t) \sim e^{-i\omega_* t}$

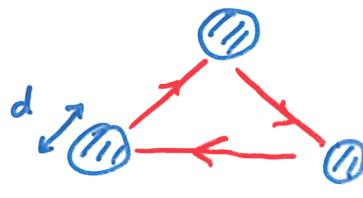
$\omega_*(\varepsilon) = \omega - \frac{2i}{|c|} \varepsilon^2$

2) Derivation of damping or friction in a conservative system - by coupling a lower dim. dyn. system to an ∞ -dim dyn. syst

How does $\text{Im } \beta$ (scattering resonance
imag. part)

vary with details of microstructure?

- For $\lambda \ll d$ microstructure will have a large impact on $\text{Im } \beta$

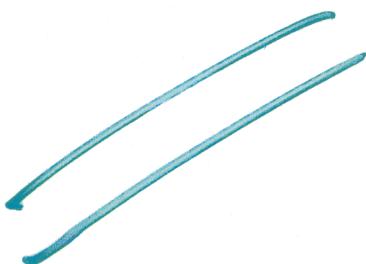


- geometric optics
- semiclassical analysis

* We focus here on $\lambda > d$ -

averaging effects, homogenization

and also find large effects due to
microstructure



“Modes” propagating in \hat{z} direction

$$\vec{E}(\mathbf{x}, t) = [\vec{e}_\perp(\mathbf{x}_\perp) + e_{\text{lon}}(\mathbf{x}_\perp)\hat{z}] e^{i(\beta \mathbf{x} \cdot \hat{\mathbf{z}} - \omega t)}, \quad \hat{z} \cdot \vec{e}_\perp = 0$$

Transverse \vec{E} field components (Maxwell's eqns):

$$(\Delta_\perp + k^2 n^2(x_\perp) - \beta^2) \vec{e}_\perp = \mathcal{V}_1 \vec{e}_\perp$$

$$\mathcal{V}_1 \vec{e}_\perp \equiv -\vec{\nabla}_\perp \left(\vec{e}_\perp \cdot \vec{\nabla}_\perp \ln n^2 \right)$$

$$\Delta_{\perp} = \nabla_{\perp}^2$$

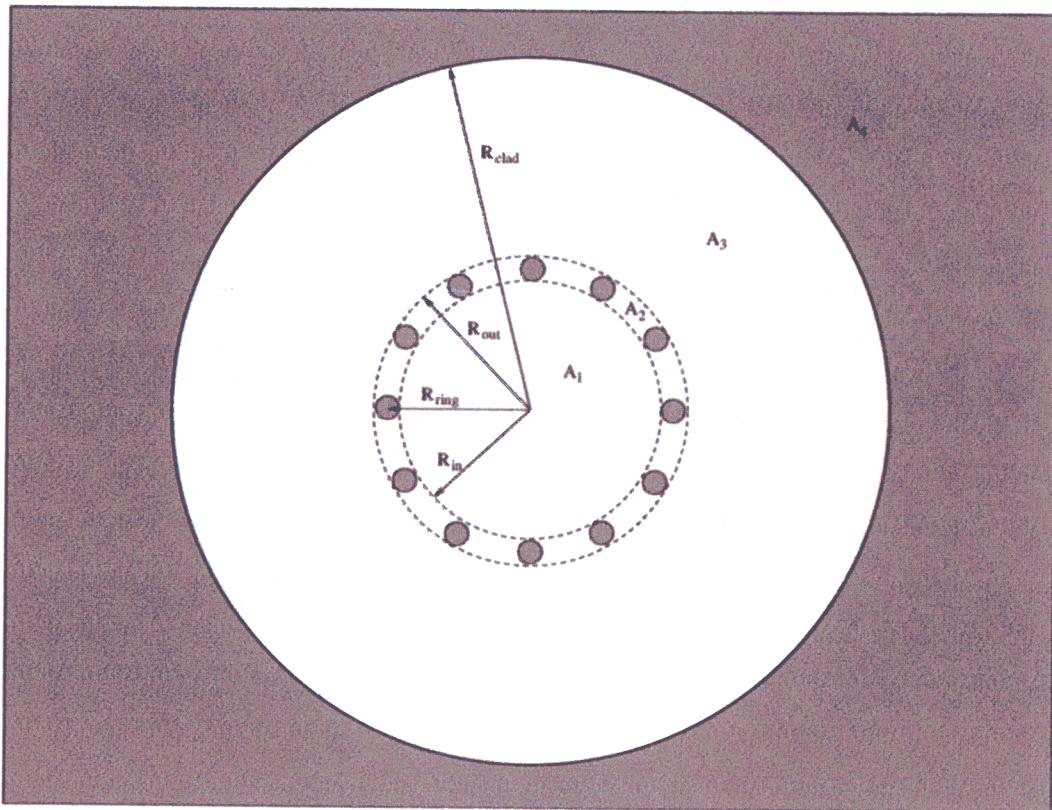
Scalar Helmholtz approximation

$$(\Delta_{\perp} + k^2 n^2(\mathbf{x}_{\perp})) \varphi = \beta^2 \varphi$$

$$k = \frac{2\pi}{\lambda}, \quad \omega = ck$$

Microstructure waveguide

$$n(\mathbf{x}) = n(\mathbf{x}_\perp) = n(x_1, x_2), \text{ref. ind.}$$
$$\mathbf{x} = \mathbf{x}_\perp + \hat{z}x_3, \hat{z} = (0, 0, 1)$$



e.g. glass ($n_g \sim 1.5$)
air ($n_h \sim 1.0$)

$$\text{---} \quad n = n_g$$

Formulation as a Schrödinger equation

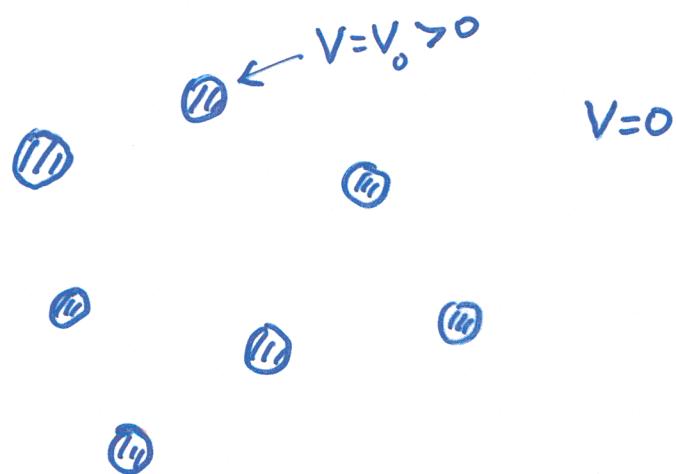
$$(\Delta + k^2 n^2(x)) \varphi(x) = \beta^2 \varphi(x) \quad (x = x_\perp)$$

Define *potential*, $V(x)$, and *energy*, E

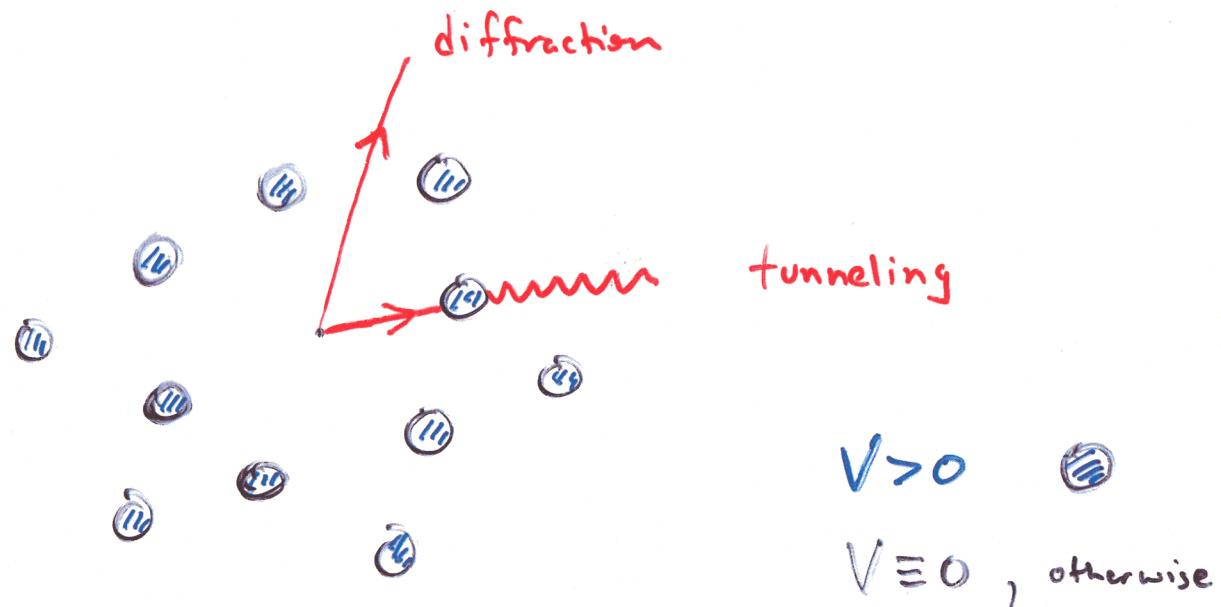
$$V(x) = k^2 (n_g^2 - n^2(x)), \quad E = k^2 n_g^2 - \beta^2,$$

$$n(x) \equiv n_g \text{ for } |x| \geq R_*$$

$$\rightarrow H \varphi \equiv (-\Delta + V)\varphi = E \varphi$$



Loss mechanisms - diffraction vs. tunneling radiation condition at infinity



Scattering Resonance Problem (SRP)

$$H \varphi \equiv (-\Delta + V)\varphi = E \varphi$$

φ outgoing as $|x| \rightarrow \infty$

Boundary condition at ∞ is non-self-adjoint

$\Rightarrow E$ complex

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Outgoing radiation condition

$$-\Delta\varphi = E \varphi, \quad |x| \geq R_*$$

$$\varphi = \sum_{l=-\infty}^{+\infty} \left(c_l^{(1)} \underbrace{e^{il\theta} H_l^{(1)}(\sqrt{E} r)}_{\text{outgoing}} + c_l^{(2)} \underbrace{e^{-il\theta} H_l^{(2)}(\sqrt{E} r)}_{\text{incoming}} \right)$$

φ outgoing at infinity $\iff c_l^{(2)} = 0$ for all l

Alternative characterization:

$$\begin{aligned}\varphi &= (-\Delta - E)_+^{-1} \rho, \quad \rho \text{ localized} \\ (-\Delta - E)_+^{-1} &= \text{outgoing Green's fn}\end{aligned}$$

Scattering resonance problem (SRP) - general remarks

$$H \varphi \equiv (-\Delta + V)\varphi = E \varphi$$

φ outgoing as $|x| \rightarrow \infty$

- Outgoing boundary condition at infinity is **non-selfadjoint**-eigenvalues, E are complex
 - $\Im E = -\Im \beta^2$ determines the leakage rate: $e^{i(\beta z - \omega t)} E_\omega$
 - Resonance modes, φ , **grow exponentially as $x \rightarrow \infty$**

Multiple scales / Homogenization

Expansion

$$H\varphi = (-\Delta + V)\varphi = E\varphi$$

φ outgoing as $|x| \rightarrow \infty$

Multiscale potentials : $V(r, \theta, N\theta) = V(r, \theta, \Theta)$

View φ as depending on slow (r, θ) and fast ($N\theta$) variables

$$\varphi(r, \theta; N) = \Phi^{(N)}(r, \theta, \Theta), \quad \Theta = N\theta$$

$$\left(\Delta_{\text{rad}} + \frac{1}{r^2} \partial_\theta^2 \right) \varphi + V \varphi = E \varphi$$

\iff

$$\begin{aligned} & \left(\Delta_{\text{rad}} + \frac{1}{r^2} (\partial_\theta + N \partial_\Theta)^2 \right) \Phi^{(N)} + V(r, \theta, \Theta) \Phi^{(N)} \\ &= E \Phi^{(N)} \end{aligned}$$

Seek an asymptotic N^{-1} expansion of solutions of the SRP

$$\begin{aligned}\Phi^{(N)} &= \Phi_0 + \frac{1}{N}\Phi_1 + \frac{1}{N^2}\Phi_2 + \frac{1}{N^3}\Phi_3 + \frac{1}{N^4}\Phi_4 + \dots \\ E^{(N)} &= E_0 + \frac{1}{N}E_1 + \frac{1}{N^2}E_2 + \dots\end{aligned}$$

Hierarchy of equations of the form:

$$\begin{aligned}\frac{1}{r^2}\partial_\Theta^2\Phi_j &= G_j(r, \theta, \Theta), \quad \Phi_j \text{ outgoing} \\ G_j(r, \theta, \Theta; \Phi_0, \dots, \Phi_{j-1}, E_0, \dots, E_{j-1}) &\quad (1)\end{aligned}$$

Solvability condition:

$$\int_0^{2\pi} G(r, \theta, \Theta) d\Theta = 0$$

\iff

Nonstiff Inhomogeneous PDE with outgoing BC at ∞

\Rightarrow efficient numerical method

Homogenization Expansion for N-fold symmetric structures:

$$\begin{pmatrix} \Phi^{(N)} \\ E^{(N)} \end{pmatrix} = \underbrace{\begin{pmatrix} \Phi_{\text{av}}(r, \theta) \\ E_{\text{av}} \end{pmatrix}}_{\text{average theory}} + \underbrace{\frac{1}{N^2} \begin{pmatrix} \Phi_2(r, \theta, N\theta) \\ E_2 \end{pmatrix}}_{\text{microstructure correction}} + \mathcal{O}\left(\frac{1}{N^3}\right)$$

- $\Phi_{\text{av}}(r, \theta), E_{\text{av}}$ is a scattering resonance of the averaged Hamiltonian: (homogenization)

$$H_{\text{av}} = -\Delta + V_{\text{av}}(r)$$

$$V_{\text{av}}(r) = (2\pi)^{-1} \int_0^{2\pi} V(r, \Theta) d\Theta$$

- $\Phi_2(r, \theta, N\theta)$, E_2 - 1st microstructure correction
 $\text{There is no } N^{-1} \text{ correction.}$

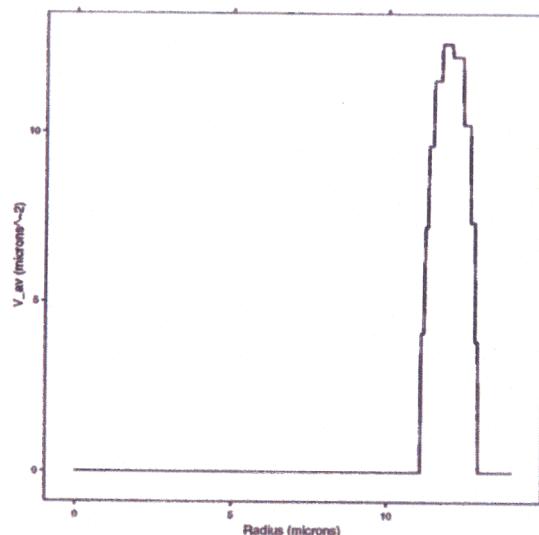
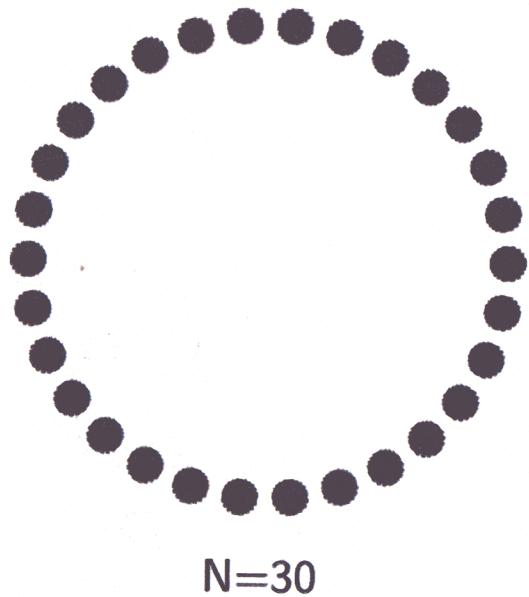
$$(H_{av} - E_{av}) \Phi_{av} = 0$$

$$(H_{av} - E_{av}) \Phi_2^{(h)} = \left\{ E_2 + \frac{r^2}{2\pi} \int_0^{2\pi} |f_p| \partial_p^{-1} [V(r, \theta, p) - V_{av}(r, \theta)] \right\} \times \Phi_0(r, \theta)$$

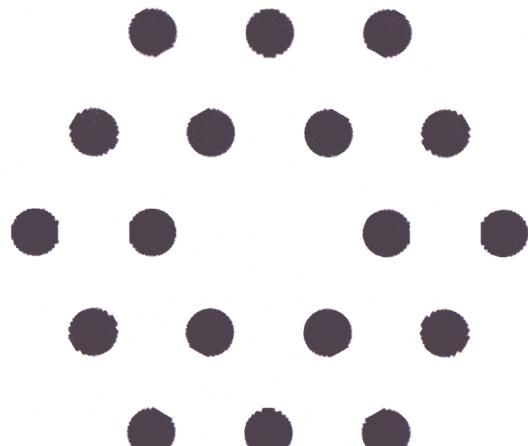
$\Phi_2^{(h)}$ outgoing $\Rightarrow E_2$

$$H_{av} = -\Delta + V_{av}(r)$$

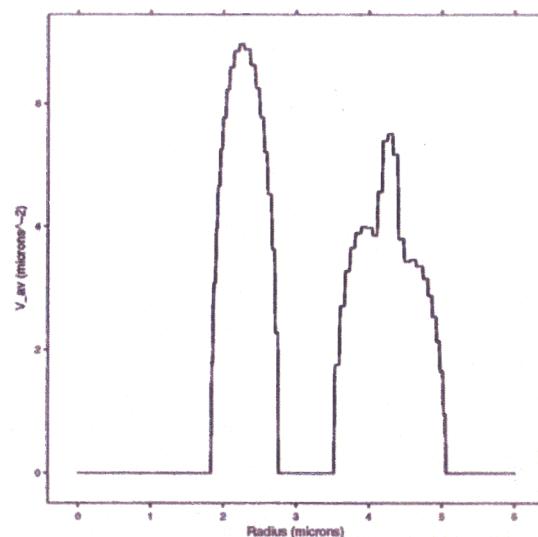
$V(r, N\theta)$ and $V_{av}(r)$



$V(r, N\theta)$ and $V_{\text{av}}(r)$



N=6



Numerical Computations and Comparisons

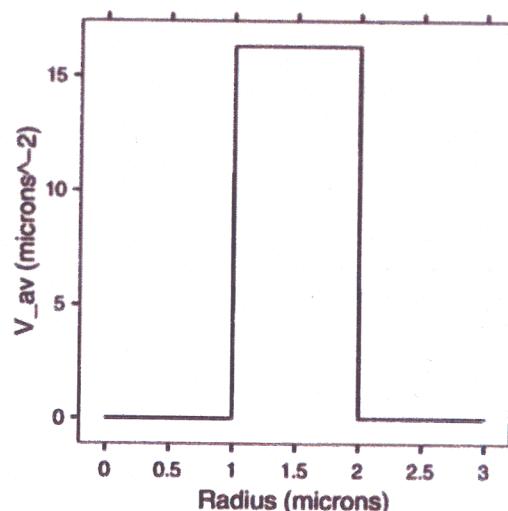
- Averaged theory (classical homogenization) $\Phi_{av}(r, \theta)$, E_{av}
- Higher (2nd) order homogenization theory

$$\Phi_{av}(r, \theta) + \frac{1}{N^2} \Phi_2(r, \theta, N\theta), \quad E_{av} + \frac{1}{N^2} E_2$$

- "Exact theory" (DNS - Fourier, Multipole methods)



$N=6$



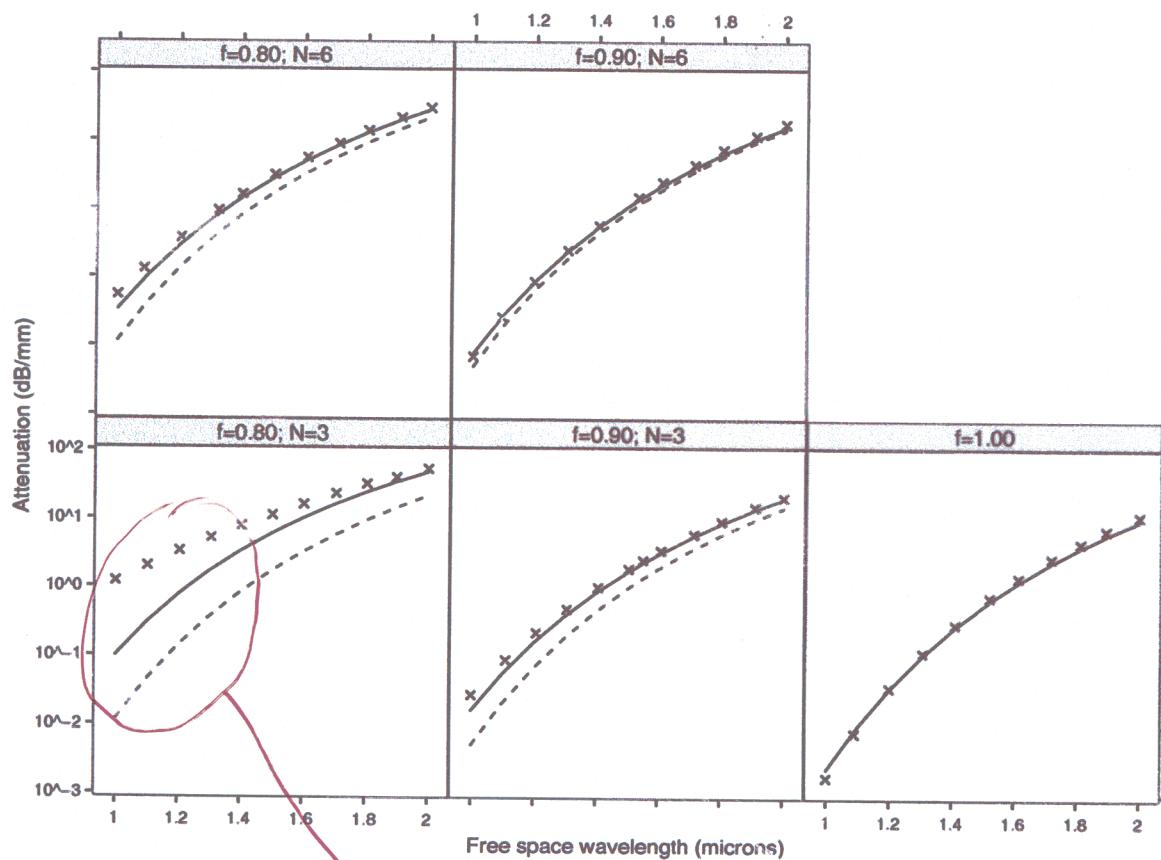
$V_{av}(r)$

Comparisons of computed attenuation rates $\sim \text{Imag } E_j \text{ Imag } \beta$

$\times \times \times = \text{DNS / "Exact Theory"}$

solid curve = 2nd order homogenization

dashed curve = Zeroth order homogenization (average theory)

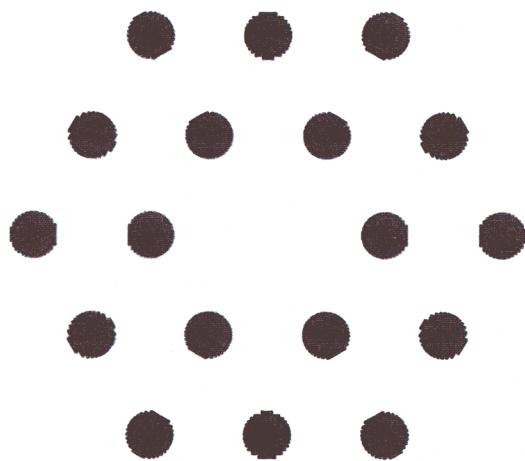


Convergence for:

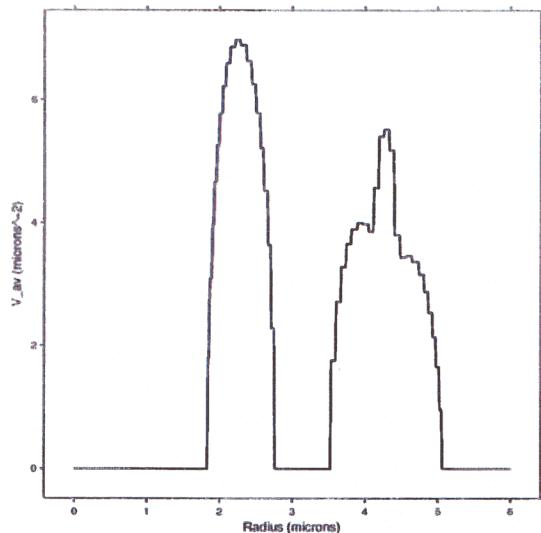
$$\frac{\lambda}{d} < 1$$

(fixed N , $\lambda \uparrow$) and (fixed λ , $N \uparrow$)

$N=3, 6$ not so large



$N=6$



$V_{av}(r)$

	Average theory	2nd Order Homogenization	DNS - "Exact"
Loss Rate	0.92 dB/cm	14 dB/cm	16 dB/cm

↑
scalar

↑
vector

DNS - McPhedran et. al.

Comparison of results for the first two leaky modes

mode	0 th Order Homogenization		2 nd Order Homogenization		Fourier	
	n_{eff}	atten.	n_{eff}	atten.	n_{eff}	atten.
LP01	1.3712+0.0000387i	1.36	1.3727+0.0000644i	2.27	1.365+0.000071i	2.48
LP11	1.2468+0.000436i	15.3	1.2515+0.000832i	29.3	1.255+0.00075i	27

Dependence on detailed microstructure:

- Extreme sensitivity of Imag E

Averaged potential gives large underestimation of attenuation rates.

Tunneling + Diffraction \neq Effective Tunneling, V_{av}

- Insensitivity of Real E

Averaged potential yields good approximation to Real E

- • $\lambda/d > 3/2$

Analytical Theory \implies

- Validity of homogenization expansion for N -fold symmetric structures, $V(r, N\theta)$ provided

$$\frac{1}{\lambda^2} \cdot \frac{1}{N} \cdot (\text{avg index contrast}) \cdot C_*(\text{geometry}) < \varepsilon_*$$

- Error bounds for numerical method:

$$E - E_{\text{approx}}, \varphi - \varphi_{\text{approx}} \sim N^{-2-\tau}$$

$\tau > 0$ dependent on "smoothness" properties of V_{micro}
(decay of \hat{V}_{micro})

- Theory (\implies general expansion) applies to high contrast microstructures

$V_{\text{micro}} \equiv V(x) - V_{\text{av}}(x)$ pointwise large

- Based on intrinsic measure of size of a microstructure perturbation: $\|V_{\text{micro}}\| \implies$

{ procedure to numerically compute properties of microstructure perturbations of V_{av} (homogenized effective medium) when there's no explicit parameter like N^{-1}

$$\text{SRP}(V) \quad (I + T_{\mathcal{R}}(\mu) T_V) \Psi = 0, \quad \Psi \in L^2$$

Perturbation Theorem:

Assume $\text{SRP}(V_0)$ has a solution Ψ_0, E_0 and consider $\text{SRP}(V_0 + V_{\text{micro}})$ where V_{micro} is a microstructure (oscillatory) perturbation which is high contrast (pointwise large).

$$\text{If } ||| V_{\text{micro}} ||| \equiv \| \langle D \rangle^{-1} V_{\text{micro}} \langle D \rangle^{-1} \|_{L^2 \rightarrow L^2}$$

is sufficiently small, then

$\text{SRP}(V_0 + V_{\text{micro}})$ has a solution: $\Psi(V_0 + V_{\text{micro}}), E(V_0 + V_{\text{micro}})$ with

$$\Psi(V_0 + V_{\text{micro}}) - \Psi_0 = \delta \Psi(V_{\text{micro}}),$$

$$E(V_0 + V_{\text{micro}}) - E_0 = \delta E(V_{\text{micro}})$$

analytic (convergent power series) in the norm: $||| V_{\text{micro}} |||$.

Special case: If $n_{\text{micro}} = n_{\text{micro}}(r, N\theta)$, then $||| V_{\text{micro}} ||| \sim \lambda^{-2} N^{-1}$

Theorem Validity of 2nd order
homogenization + error estimate

$$|\delta E^{(2)} - N^{-2} E_2^{\text{(homog)}}| \leq \frac{C}{N^{2+\gamma}},$$

$\gamma = 2$, if V_{micro} is C^2 w.r.t. r

$\gamma = 1$, if V_{micro} has jump
discontinuities in r

[Boundary layers, Santosa-Vogelius
Morkov - Vogelius]

Scattering Resonance Problem

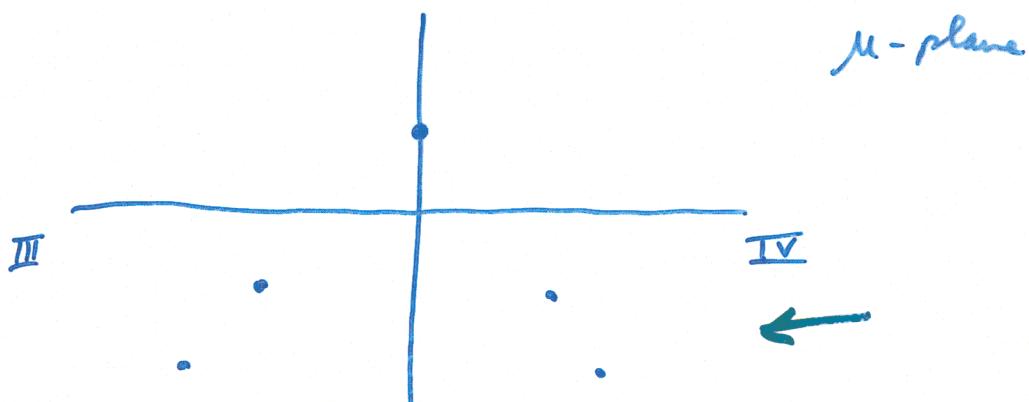
$$H_0 = -\Delta + V_0 \quad (V_0 = \text{Average Potential})$$

$$H = -\Delta + V \quad (V = V_0 + V_{\text{micro}})$$

Prop $R_{V_0}(\mu) = (-\Delta + V_0 - \mu^2)^{-1}$

can be analytically continued from $\text{Im } \mu > 0$

to lower half plane $\text{Im } \mu < 0$



Analytic continuation is meromorphic
with finite rank op. poles (scattering resonances):

$$(-\Delta + V_0 - \mu^2) u = 0$$

$\mu \in \text{II}$ u incoming

$\mu \in \text{IV}$ u outgoing

Seek a pert. thg valid for large contrast $\|V_{\text{micro}}\|_\infty$

①

Lippmann-Schwinger Eqn

$$(-\Delta - \mu^2) \varphi = -V\varphi, \quad \varphi \text{ outgoing}$$

$$R_0(\mu) = (-\Delta - \mu^2)_+^{-1}$$

$$R_0(\mu; x, y) = \frac{1}{4\pi} \left(\frac{\mu}{2\pi|x-y|} \right)^{\frac{n-2}{2}} H_{\frac{n-2}{2}}^{(1)}(\mu|x-y|)$$

L-S eqn:

$$\varphi = -R_0(\mu) V \varphi$$

Expect φ exponentially growing ($\text{Im}\mu < 0$)

Let X be exp. decaying $X \sim e^{-\alpha|x|}$

$$(X\varphi \in L^2)$$

(2) Reformulation in L^2

$$\underbrace{\chi(x)\varphi}_{\tilde{\phi}} = - \underbrace{\chi(x) R_0(\mu) \vee \chi^{-1}(x)}_{A_V(\mu)} \underbrace{\chi(x)\varphi}_{\tilde{\phi}}$$

$$(I + A_V(\mu)) \tilde{\phi} = 0, \quad \tilde{\phi} \in L^2$$

Only \mathcal{Y}_2 -way there: although formulation is in L^2

$$V_{av} \rightarrow V_{av} + V_{micro}$$

(V_{micro} , pointwise large + oscillatory)

$$\Rightarrow A_{V_{av}}(\mu) - A_{V_{av} + V_{micro}}(\mu) \text{ LARGE}$$

Key point: smallness required for pert. thy
comes from rapid oscillation
of $V_{micro}(x)$

③ Formulation as preconditioned Lippmann - Schwinger egn

$$\chi \varphi = -\chi R_0(\mu) V \chi^{-1} \chi \varphi$$

$$\text{Apply } \langle D \rangle = (I - \Delta)^{\frac{1}{2}}$$

$$\text{Define } \Psi \equiv \langle D \rangle \chi \varphi , \quad \chi \sim e^{-\alpha/|\chi|}$$

$$\bar{\Psi} = - \underbrace{\langle D \rangle \chi R_0(\mu) \chi \langle D \rangle}_{T_R(\mu)} \quad \underbrace{\langle D \rangle^{-1} \chi^{-1} V \chi^{-1} \langle D \rangle^{-1} \Psi}_{T_V}$$

Reformulation of SRP $(-\Delta + V) \varphi = \mu^2 \varphi , \varphi \text{ outgoing}$

$$\rightarrow \boxed{(I + T_R(\mu) T_V) \Psi = 0}$$

$$\Psi \in L^2$$

$$F[\Psi, \mu, v] = 0$$

$$\Rightarrow D_{\Psi, \mu} F \cdot D_v \left(\frac{\Psi}{\mu} \right) + D_v F = 0$$

Need:

(A) Invertibility of the operator

Jacobian $D_{\Psi, \mu} F$ near Ψ_0, μ_0, V_0

→ (B) Size of allowable perturbations

should not shrink rapidly for

Increasing contrast $\|V - V_0\|_\infty = \|V_{\text{micro}}\|_\infty$

$\Rightarrow \|\| \cdot \| \|$

Framework for analysis

$$(I + T_R(u) T_V) \Psi = 0, \quad \Psi \in L^2$$

$$F[\Psi_0, \mu_0, V_0] = 0$$

$$F[\Psi, \mu_0, V_0 + V_{\text{micro}}] = 0 ?$$

Implicit function theorem -

expansion about Ψ_0, μ_0, V_0

in an appropriate space.

A simple exercise in Fourier analysis

$$(V \sim e^{iN\theta}, \quad \langle D \rangle \sim |D|^{-1})$$

$f \mapsto e^{iN\theta} f$ has L^2 norm 1, BUT

$f \mapsto \langle D \rangle^{-1} e^{iN\theta} \langle D \rangle^{-1} f$ has L^2 norm $\sim \frac{1}{N}$

IDEA: Use the operator $\langle D \rangle$ as a "preconditioner"

$f(\theta)$, a 2π periodic function with Fourier series $f(\theta) = \sum_m \hat{f}(m) e^{im\theta}$.

$$\text{Let } \langle a \rangle \equiv (1 + |a|^2)^{\frac{1}{2}}$$

$$\begin{aligned} \langle D \rangle^{-1} f &\equiv (I - \partial_\theta^2)^{-\frac{1}{2}} f = \sum_m \frac{1}{\langle m \rangle} \hat{f}(m) e^{im\theta} \\ e^{iN\theta} \langle D \rangle^{-1} f &= \sum_m \frac{1}{\langle m \rangle} \hat{f}(m) e^{i(m+N)\theta} \\ \langle D \rangle^{-1} e^{iN\theta} \langle D \rangle^{-1} f &= \sum_m \frac{1}{\langle m+N \rangle \langle m \rangle} \hat{f}(m) e^{i(m+N)\theta} \end{aligned}$$

Compute L^2 norm:

$$\begin{aligned} \| \langle D \rangle^{-1} e^{iN\theta} \langle D \rangle^{-1} f \|_{L^2} &= \sqrt{\sum_m \frac{1}{\langle m+N \rangle^2 \langle m \rangle^2} |\hat{f}(m)|^2} \\ &\sim \frac{1}{N} \sqrt{\sum_m |\hat{f}(m)|^2} \\ &= \frac{1}{N} \|f\|_{L^2} \end{aligned}$$

$$(I + T_R(\mu) \underline{T_V}) \underline{\Psi} = 0$$

Estimates

Estimate (A):

$T_{V_{\text{micro}}}$ is small in norm for large amplitude and oscillatory V_{micro}
(high contrast microstructures)

For $V_{\text{micro}} \sim V_{\text{micro}}(r, N\theta)$

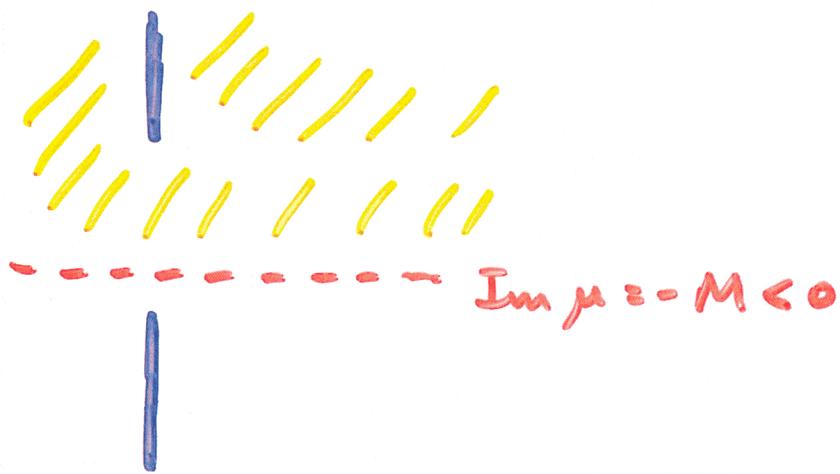
$$\|V_m\| \sim \| \langle D \rangle^{-1} V_{\text{micro}} \langle D \rangle^{-1} \|_{L^2 \rightarrow L^2} \leq C N^{-1}$$

$$(I + \underline{\underline{T_R(\mu) T_V}}) \Psi = 0$$

(B) $T_R(\mu) = \langle D \rangle \chi R_0(\mu) \chi \langle D \rangle$

Prop $k \geq 0$ $\partial_\mu^k T_R(\mu) : L^2 \rightarrow L^2$

is defined and analytic and
bounded for $\operatorname{Im} \mu > -M$



Pf: $T_R(\mu) = \chi \langle D \rangle R_0(\mu) \langle D \rangle \chi \leftarrow$
 $+ \sum \text{commutators}$

[can't analytically continue in F.T.]

Lemma

Let $\mathcal{L}_j(\mu, x, y) = \text{kernel of } R_0(\mu) \langle D \rangle^j, \text{Im } \mu > 0$

Then, $\mathcal{L}_j(\mu; -)$ has an analytic continuation to the LHP w/
branch cut from $-im$ to $-i\infty$

$$\begin{aligned} \underline{\mathcal{L}}_j(\mu, x, y) &= \sum \text{polynom}(\mu) (-iI - \Delta)^{-\alpha}(x, y) \\ &\quad + R_j(\mu, x, y) \end{aligned}$$

Error estimate on

2nd order homogenization correction

to resonance energy, $E^{(N)}$, due
to microstructure, $V_m(r, N\theta)$:

Thm

$$\left| \delta E^{(2)} - N^{-2} E_2^{(\text{homog})} \right| \leq \frac{C}{N^{2+\tau}}$$

where

$\tau = 2$, if V_m is C^2 w.r.t. r

$\tau = 1$, if V_m has jump discontinu-

$$R_j(\mu, x, y)$$

$$= (m^2 + \mu^2)^2 \int_{S^{n-1}} d\omega \int_0^\infty dp p^{n-1} e^{ip\omega \cdot (x-y)} \underbrace{\frac{1}{(p^2 - \mu^2)(m^2 + p^2)^2 - i\epsilon}}$$

Analytic in p except
for branch cut at $\pm im$
and poles at $p = \pm \mu$

Analytic continuation formula:

$$R_j(\mu, x, y) = R_j(-\mu, x, y)$$

$$+ (m^2 + \mu^2)^2 \int_{S^{n-1}} d\omega \oint dp p^{n-1} e^{ip\omega(x-y)} \frac{1}{(p^2 - \mu^2)(m^2 + p^2)^2}$$

$$= R_j(-\mu, x, y) + "(m^2 + \mu^2)^2 \text{Bessel}(\mu|x-y|)$$

(2)

$$\delta \bar{E}^{(2)} = \frac{1}{N^2} \bar{E}_2^{(\text{homog})}$$

requires estimation of $(l \gg 1)$

$$\left[(-\Delta_{\text{radial}} + \frac{l^2}{r^2} - \mu_0^2)^{-1} - \left(\frac{l^2}{r^2} \right)^{-1} \right]$$

R.P. Thy

$$V_{l,\text{micro}}(r)$$

homog.
thy

\rightarrow estimates depend on
regularity of V_{micro}
in the radial direction

proof

$$\delta E^{(2)} = \text{rigorous } O(\|V_{\text{micro}}\|^2)$$

Correction to resonance energy

= expression involving resolvents
at energies on 2nd sheet
(Im $\mu < 0$)

- ① Use analytic continuation to represent in terms of resolvents at energies on "physical sheet",
 $\text{Im } \mu > 0$
⇒ estimates involve operators with exp. decaying or oscillatory kernels

Summary

- Higher order homogenization / effective medium approach to scattering resonances
- Numerical method
 - efficient - "stiffness" due large range of scales removed ~~analytically~~
 - accurate - microstructure corrections are necessary for accurate computation of key spectral characteristics ($\text{Im}\beta$) due to very sensitive dependence on detailed microstructures
- Analytic theory - error bounds on approximation via preconditioned Lippman-Schwinger equation

$$V_{\text{micro}} = V_{\text{micro}}(r, N\theta) \implies$$

$$\left(\frac{2\pi}{\lambda}\right)^2 \frac{1}{N} \text{ (avg index contrast)} \quad C_*(\text{geometry}) < \varepsilon_*$$

- intrinsic parameter $(|V_{\text{micro}}|)$

general numerical method

$$- \frac{\delta E^{(2)}}{N^2} - \frac{1}{2} \delta E_2^{\text{homog}} = O\left(\frac{1}{N^{2+\varepsilon}}\right), \varepsilon > 0$$

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ε depends on regularity of V_{micro} .