

Resonance Problems in Photonics

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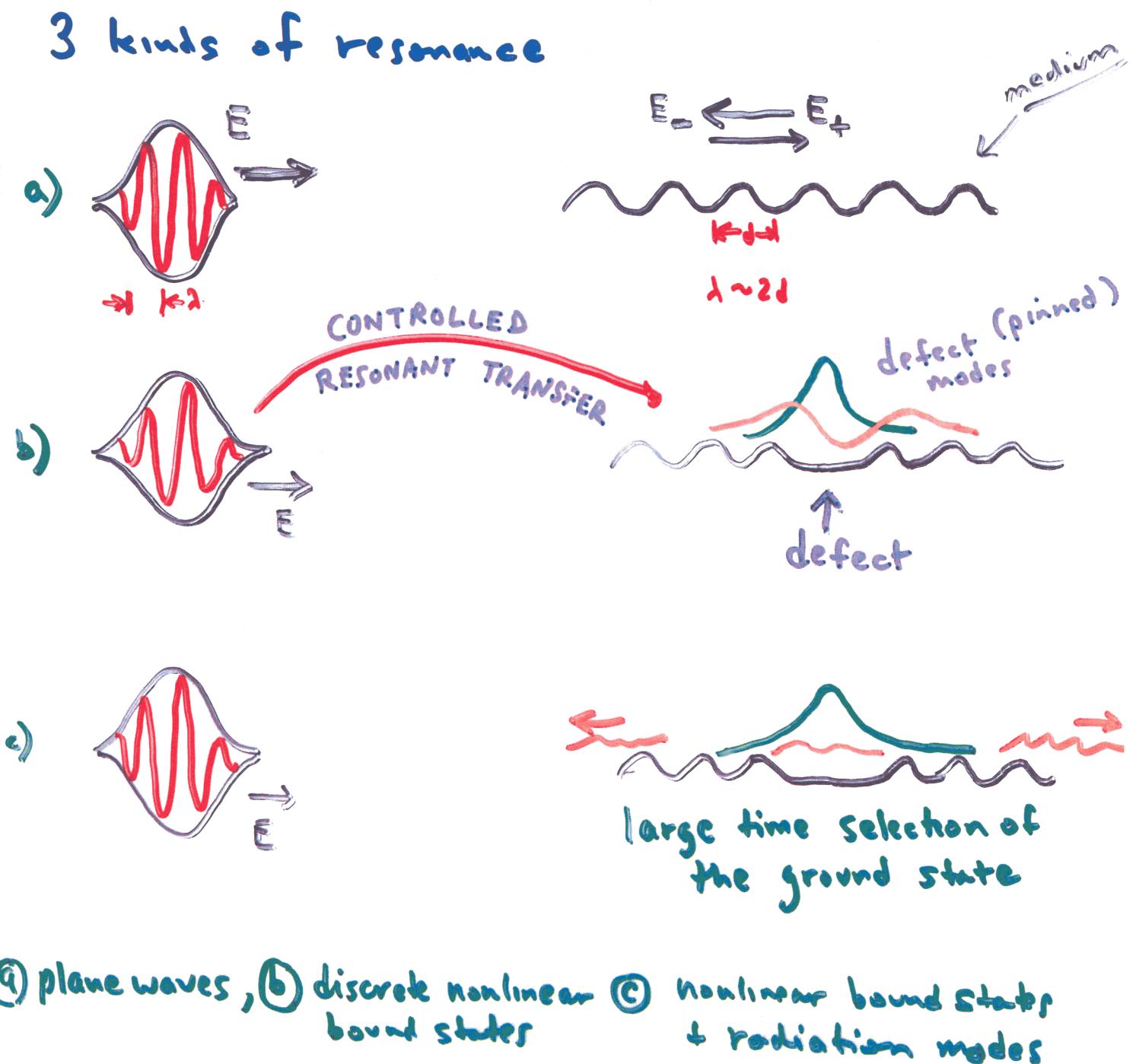
+ Bell Labs

Overview

Talk 1

Nonlinear propagation in inhomogeneous media

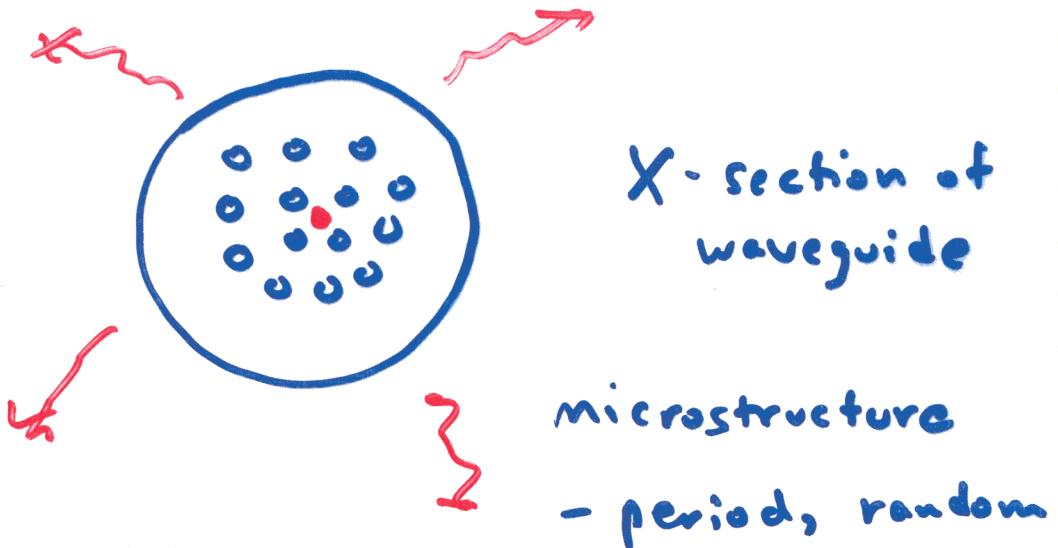
3 kinds of resonance



- (a) plane waves, (b) discrete nonlinear bound states
- (c) nonlinear bound states + radiation modes

Resonance Problems in Photonics

Talk 2



- Diffusion of energy in multimode systems random → w/ Kira
- Homogenization theory (higher order) and scattering resonances
(finiteness effects) → w/ Golowich

I. Nonlinear Propagation

in Inhomogeneous Media

- "Nonlinear periodic structures"

Co-workers:

R.E. Slusher (Optical Physics - Quant Info.
- Bell Labs)

R.H. Goodman (NJIT)

P.J. Holmes (Princeton)

A. Soffer (Rutgers)

E. Kirr (Chicago)

R. Jackson (Boston U.)

Outline

- 1) Resonant interaction of a high intensity light pulse with a periodic medium
- 2) Nonlinear dispersive system w/ Soliton-like states
- 3) Control Problem - design a defect in the periodic medium enabling capture of a propagating soliton (optical storage)
- 4) Modeling a trap , numerical simulations
- 5) Theorem on selection of the ground state
- asymptotic distribution of trapped energy .

- Bragg Resonance : $\lambda \sim 2d$ (strongly dispersive)

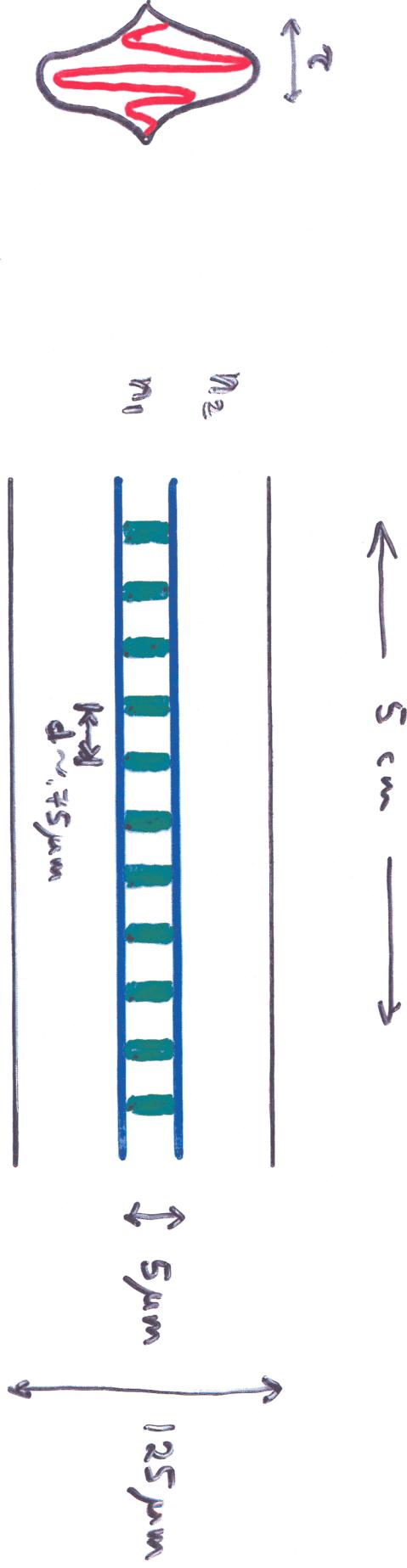
nonlinear Kerr effect

$$n^2 \approx n_1^2 + \Delta n \cos(2k_g z) + \chi^{(3)} E^2$$

\uparrow

$1 ps \leq 10^{-12} s$

light pulse
 $\tau \sim 30 \text{ ps}$
 $\lambda \sim 1.5 \mu\text{m}$



Electric Field, E ,
satisfies 1-D Maxwell eqn

with periodic and nonlinear
coefficients

$$\partial_t^2 \left(n^2(z, E) E(z, t) \right) = \partial_z^2 E(z, t)$$

$$n^2(z, E) = 1 + \epsilon \cos(2kz) + \chi^{(3)} E^2$$

Wave Packet data: $e^{ikz} A(\epsilon z)$

$$\text{Medium period} = \frac{2\pi}{2k} = \frac{\pi}{k}$$

$$\text{Carrier wavelength} = 2 \frac{\pi}{k} = 2 \times \text{Medium Period} \\ (\text{Bragg})$$

$E \sim \sqrt{\epsilon}$, balance of dispersion
+ nonlinearity

Bragg
Resonance

Strong coupling of forward traveling

and backward traveling waves

due to resonance between carrier wave
and periodic medium

$$\partial_t^2 \left[(1 + \epsilon \cos 2kz + \dots) E \right] = \partial_z^2 E$$

$$E_+ e^{i(kz-wt)} \rightarrow \epsilon \kappa (e^{2ikz} + e^{-2ikz}) + \dots$$

forward going

periodic structure

$$\rightarrow \epsilon \kappa E_+ e^{-i(kz+wt)} + \dots$$

backward going

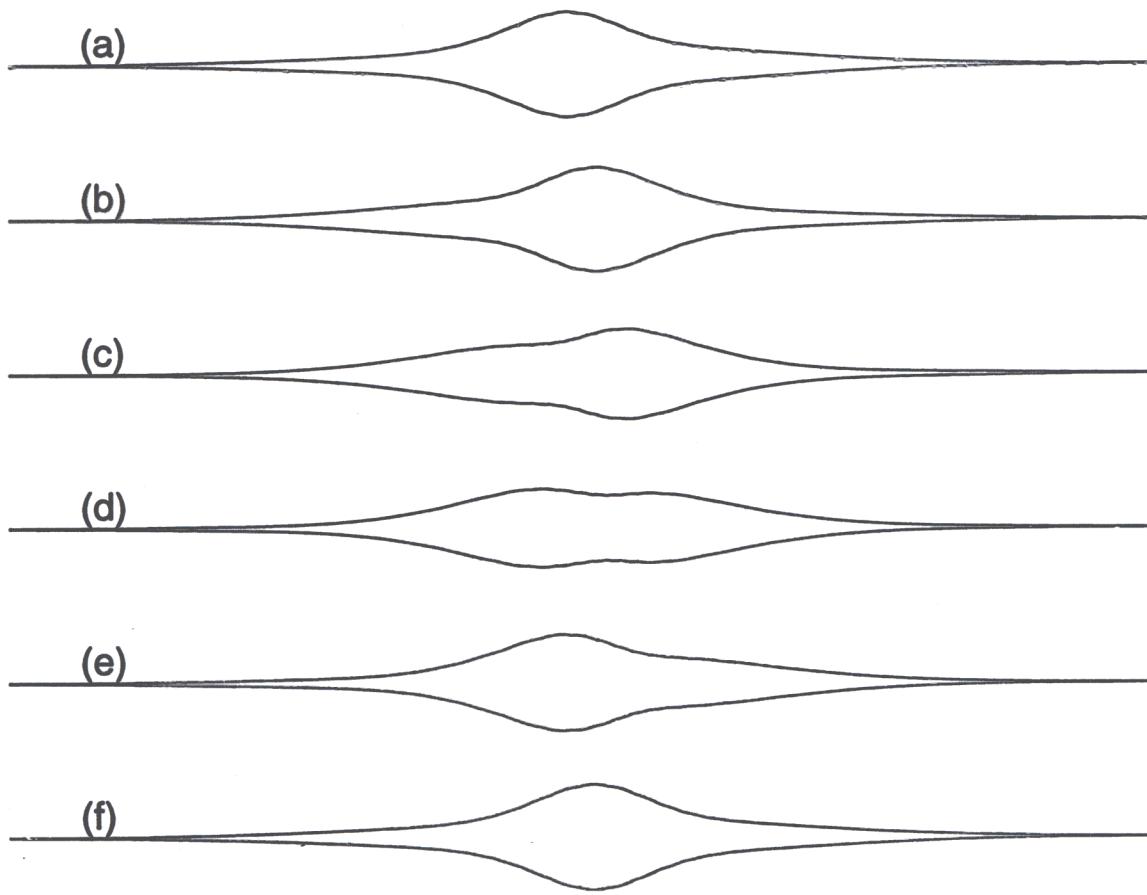
$$E_- e^{-i(kz+wt)} \rightarrow \epsilon \kappa (e^{2ikz} + e^{-2ikz}) + \dots$$

backward going

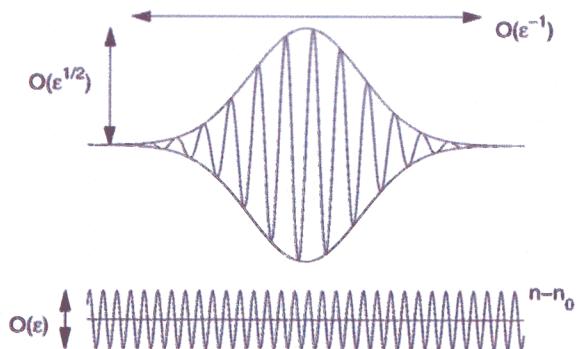
$$\rightarrow \epsilon \kappa E_- e^{i(kz-wt)} + \dots$$

forward going

Add nonlinearity + . . .



The Envelope Approximation



Multiple Scales Ansatz of the form

$$E = \sqrt{\epsilon} \left(E_+(\epsilon z, \epsilon t) e^{i(k_B z - \omega_B t)} + E_-(\epsilon z, \epsilon t) e^{-i(k_B z + \omega_B t)} \right)$$

\downarrow
 Nonlinear Maxwell (wave) eqn
 \downarrow

$$+ O(\epsilon^{3/2}) + c.c.$$

Nonlinear Coupled Mode Equations (NLCME)

$$i \frac{\partial E_+}{\partial T} + i c \frac{\partial E_+}{\partial Z} + \kappa E_- + \Gamma(|E_+|^2 + 2|E_-|^2) E_+ = 0$$

$$i \frac{\partial E_-}{\partial T} - i c \frac{\partial E_-}{\partial Z} + \kappa E_+ + \Gamma(|E_-|^2 + 2|E_+|^2) E_- = 0$$

AMLE ~ NLCME

Goodman-Holmes-W.
J. Nonlin. Sci.
2001

Theorem: There exists $\varepsilon_0 > 0$ such that for any $T_0 > 0$ and any $0 < \varepsilon \leq \varepsilon_0$, the solution with H^3 initial data is well approximated by a solution of NLCME in the sense that for all $t \in [0, T_0/\varepsilon]$ the following estimate holds:

$$\left\| \begin{pmatrix} E^\varepsilon \\ P^\varepsilon \end{pmatrix} - \begin{pmatrix} E_{NLCME}^\varepsilon \\ P_{NLCME}^\varepsilon \end{pmatrix} \right\|_{H^1(\mathbb{R})} \leq C(T_0; \omega_0, \nu, n) \varepsilon.$$

Nonlinear Coupled Mode Equations - NLCME

$$i(\partial_T + v_g \partial_Z) E_+ + \kappa E_- + \Gamma(|E_+|^2 + 2|E_-|^2) E_+ = 0$$

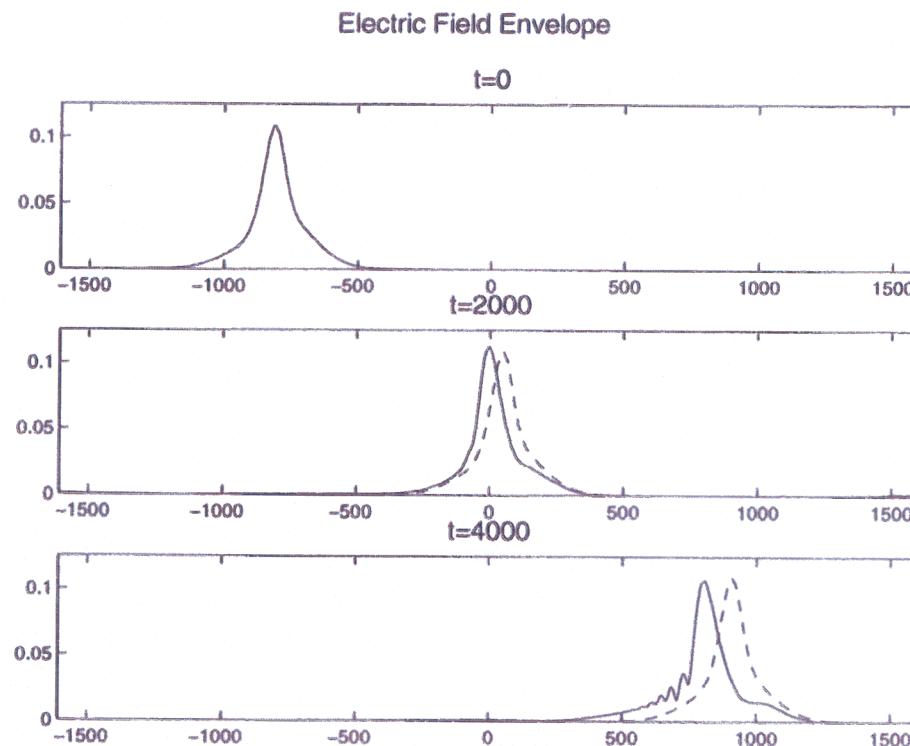
$$i(\partial_T - v_g \partial_Z) E_- + \kappa E_+ + \Gamma(|E_-|^2 + 2|E_+|^2) E_- = 0$$

$$\begin{pmatrix} E_{NLCME}^\varepsilon \\ P_{NLCME}^\varepsilon \end{pmatrix} \equiv$$

$$\sqrt{\varepsilon} \left(E_+(Z, T) e^{i(k_B z - \omega_B t)} + E_-(Z, T) e^{-i(k_B z + \omega_B t)} \right) \begin{pmatrix} 1 \\ \gamma_B \end{pmatrix}$$

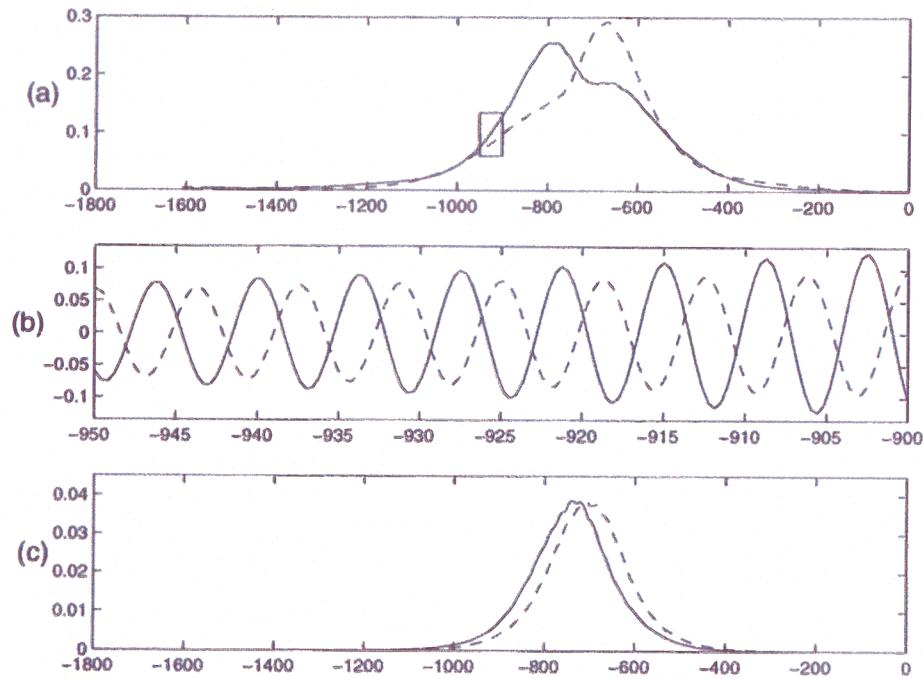
+ complex conjugate

Eventual steepening and break up



"Envelope
shock"

Breakdown of the theorem due to phase drift



Q.

Is there a theorem on validity for longer times
 $(o(\varepsilon^{-2})??)$ in weaker norms (*coarse-graining*)?

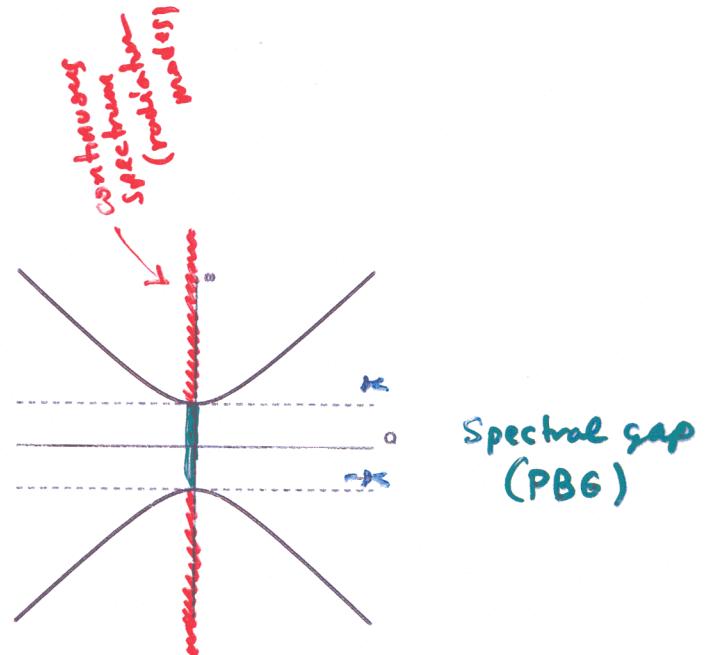
Linear Theory–Spectral Gap

(LCME)

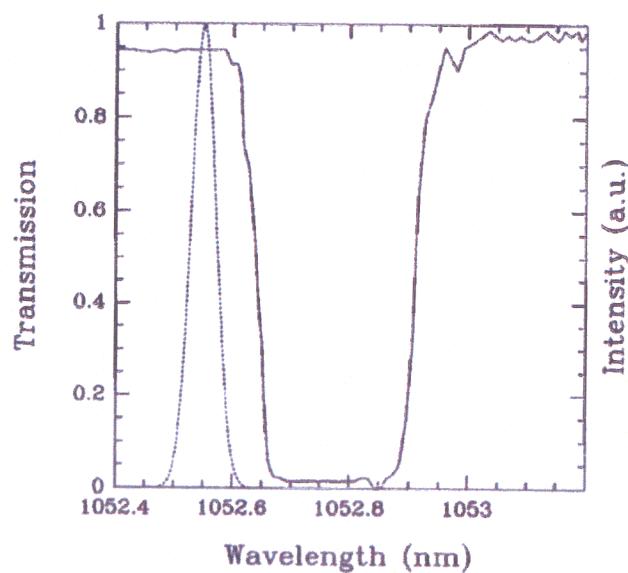
Plane waves of form:

$$E_{\pm} \sim e^{i(QZ - \Omega T)}$$

$$\Omega^2 = Q^2 + \kappa^2$$



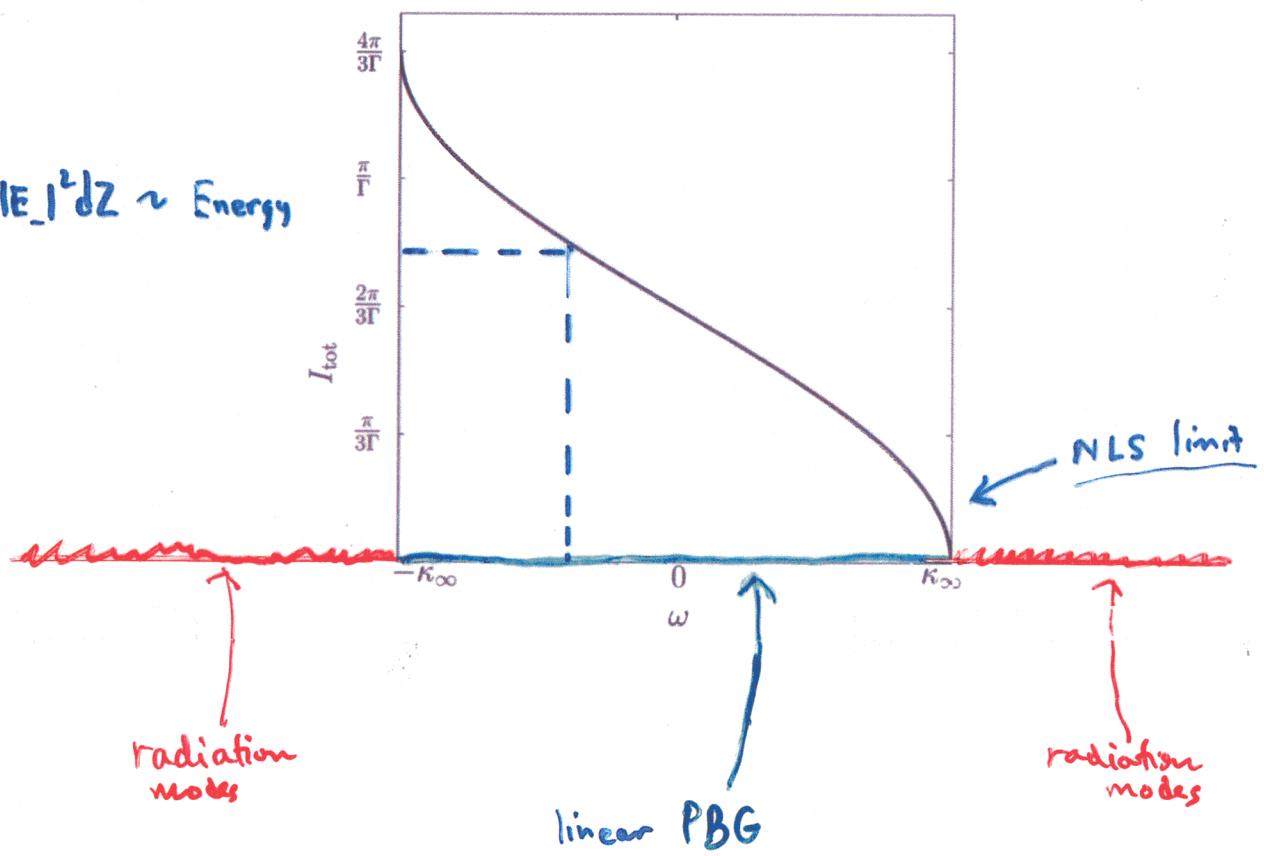
Experimental Data: (Eggleton et. al.)



Energy of gap soliton vs. frequency

(bifurcation diagram)

$$\int |E_+|^2 + |E_-|^2 dz \sim \text{Energy}$$



(Bifurcation from continuous spectrum)

$$\delta(H - \omega N) =$$

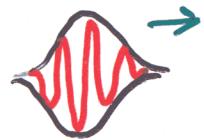
nonlinear stability

Gap solitons of NLCME

Christodoulides - Joseph
Aceres - Wabnitz } '89

← Solitons of
massive Thirring
model - 50's
Kamp-Newell
~'78

- $E_{\pm} \sim (\sin \delta) \operatorname{sech}((\sin \delta)(z - vt))$



- $e^{i(\kappa \cos \delta)} \frac{v z - t}{\sqrt{1-v^2}}$

any speed!

$0 \leq v < 1$ (c)

$\kappa \cos \delta \in [-\kappa, \kappa]$, spectral gap
of NLCME

- Experiments $v \approx \frac{1}{2} c$

Eggleton, Slusher et. al.

Broderick et.al.

'96, '97

- Linear stability / instability

Barashenkov - Pelinovsky - Zemlyanaya '98

A-W
C-J

- Gap solitons are nonlinear coherent structures which stably propagate in homogeneous periodic structures
- * • $0 \leq V_{gs} \leq c$ (Experiments: $V_{gs} \gtrsim \frac{1}{2}c$)
- For any $\omega \in \text{PBG}$ there is some gap soliton $E_\omega e^{-i\omega t}$
 $= \begin{pmatrix} E_+^\omega \\ E_-^\omega \end{pmatrix} e^{-i\omega t}$

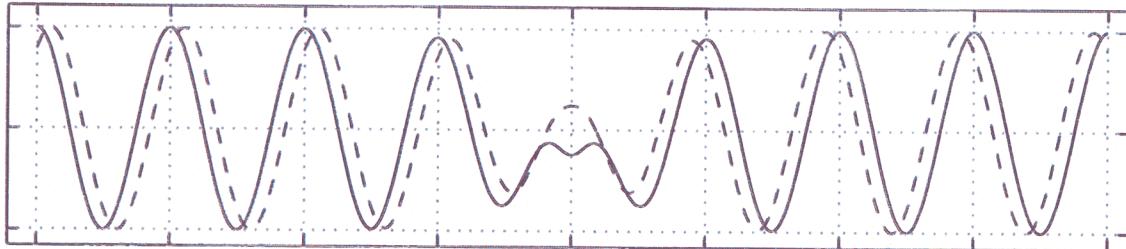
Q. Can a moving gap soliton be trapped?
(Slusher)

→ Strategy (use "designer defects")

Application of trapping: Optical storage, buffering

Trapping Gap Solitons

Introduce defects to Bragg grating



Modified NLCME

$$i\partial_T E_+ + i\partial_Z E_+ + \kappa(Z)E_- + V(Z)E_+ + \Gamma(|E_+|^2 + 2|E_-|^2)E_+ = 0$$

$$i\partial_T E_- - i\partial_Z E_- + \kappa(Z)E_+ + V(Z)E_- + \Gamma(|E_-|^2 + 2|E_+|^2)E_- = 0$$

We may construct $\kappa(Z)$ and $V(Z)$ and standing wave solutions of the form:

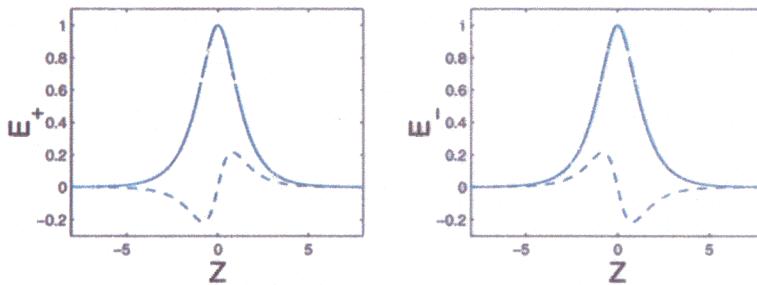
$$\begin{pmatrix} E_+ \\ E_- \end{pmatrix}(Z, T) = e^{-i\Omega T} \begin{pmatrix} E_+ \\ E_- \end{pmatrix}(Z)$$

in the linear case $\Gamma = 0$, for any $\Omega \in (-\kappa_\infty, \kappa_\infty)$.

Low intensity propagation of light
in a periodic structure with
localized defects :

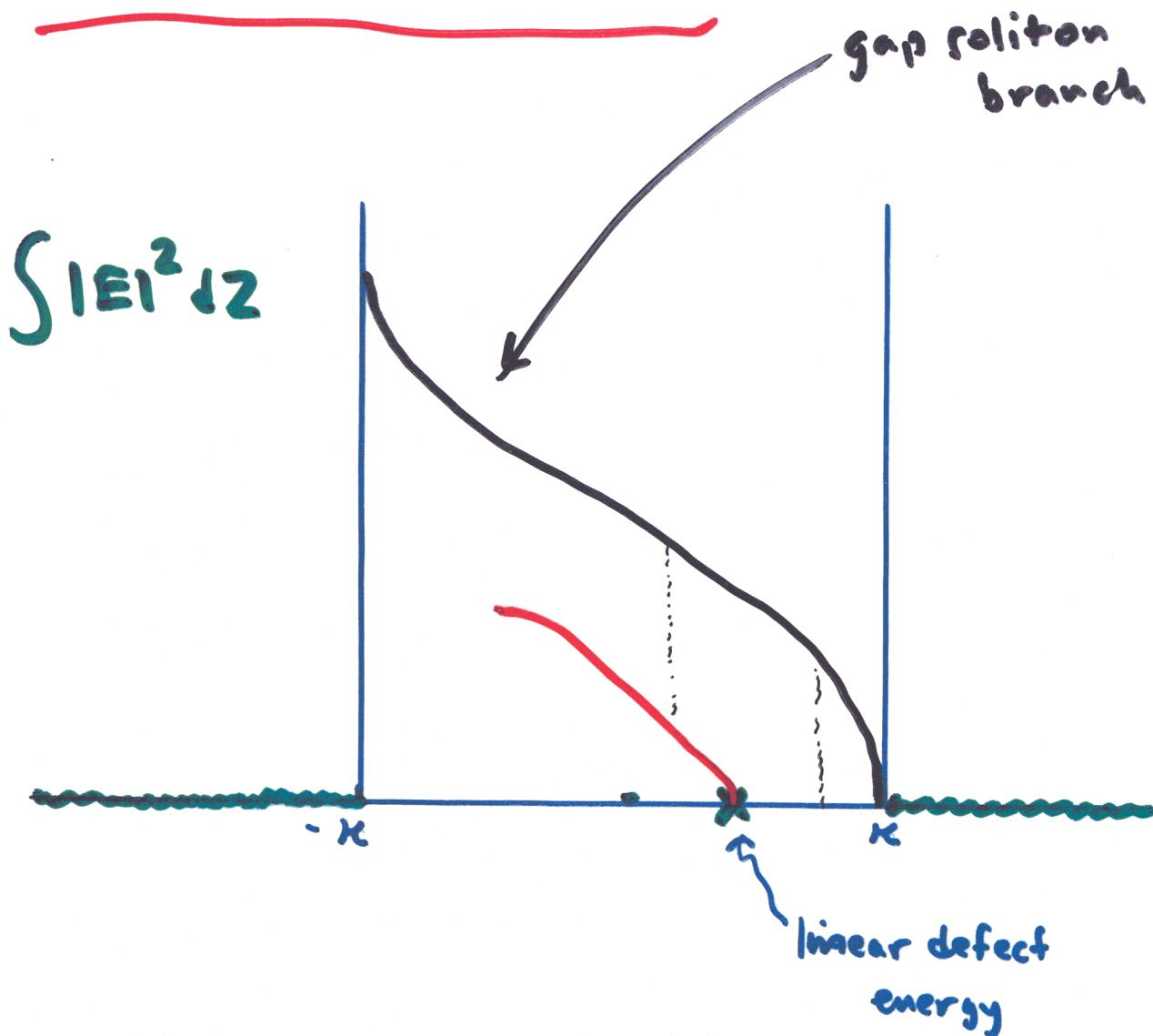
$$i(\partial_T + \sigma_3 \partial_Z) E + \chi(Z) \sigma_1 E + V(Z) E = 0$$

[$\chi(Z), V(Z)$ characterize defect]



"pinned" linear defect modes

Nonlinear Defect Modes



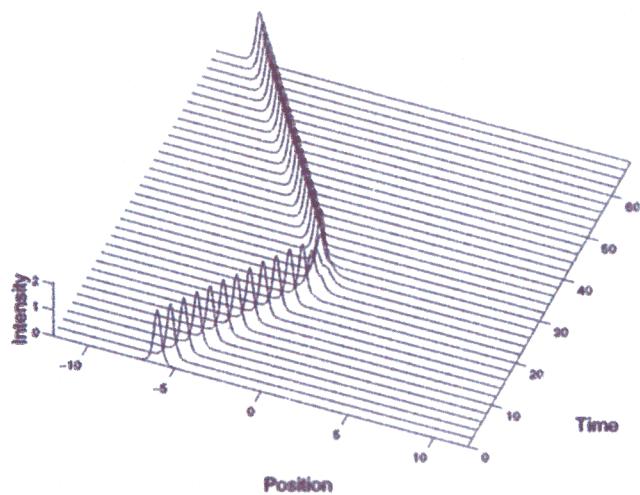
$$i(\partial_T + \sigma_3 \partial_Z) \begin{pmatrix} E_+ \\ E_- \end{pmatrix} + \sigma_1 \underline{\kappa(z)} \begin{pmatrix} E_+ \\ E_- \end{pmatrix} + \underline{V(z)} \begin{pmatrix} E_+ \\ E_- \end{pmatrix} + \underline{\Gamma N(E, E^*) E} = 0$$

$$\begin{pmatrix} \tilde{E}_+(z) \\ \tilde{E}_-(z) \end{pmatrix} e^{-i\omega T}$$

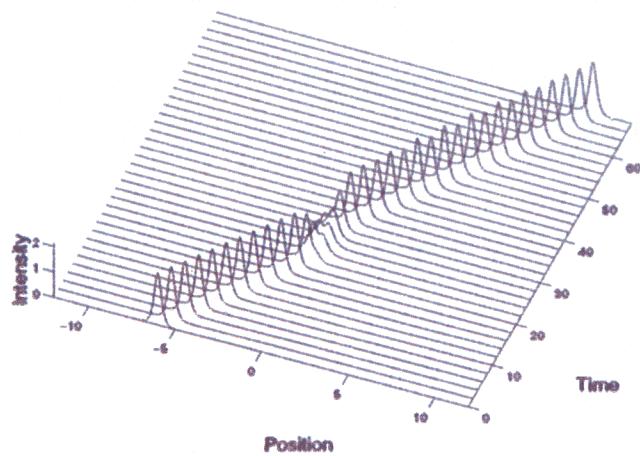
Theorem Linear defect modes deform into
Nonlinear defect modes -
Bifurcation from $E=0$ at linear defect energies

Case I: Reflection and Transmission

$v < v^*$

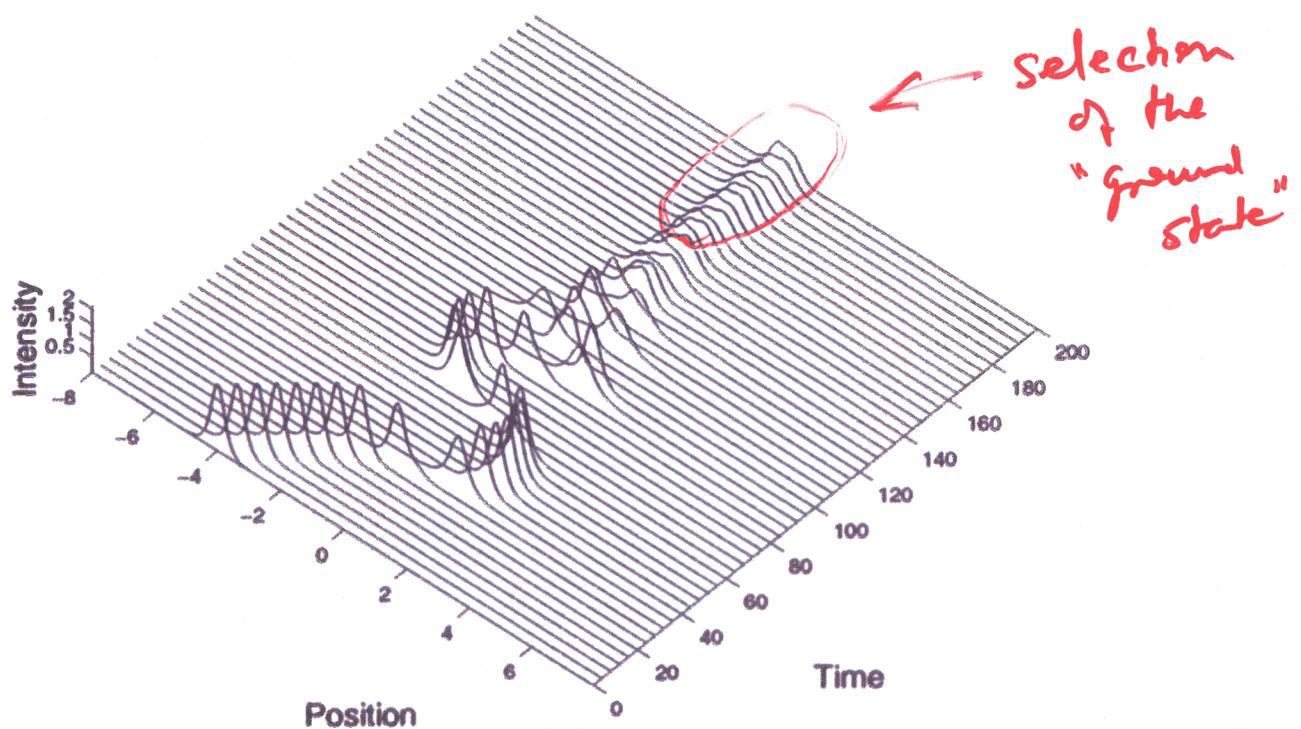


$v > v^*$



Case II: Capture

$$v < v^*$$



Soliton \leftrightarrow Defect interactions

are marked by

Complex energy exchange between
coherent pulse and the "modes"
of the defect

Gap soliton and defect mode families

Gap soliton ($v = 0$)

bifurcate from continuous spectrum band edge at $I_{\text{tot}} = 0$

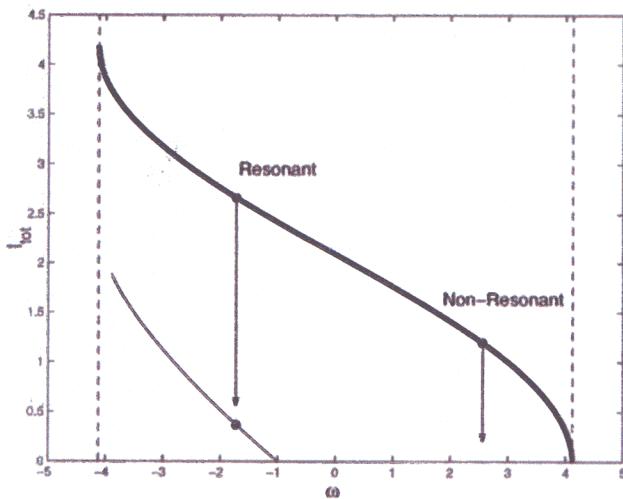
frequency: $\omega = \kappa_\infty \cos \delta$

L^2 norm: $I_{\text{tot}} = 4\delta/3$, $0 \leq \delta \leq \pi$.

Nonlinear defect modes:

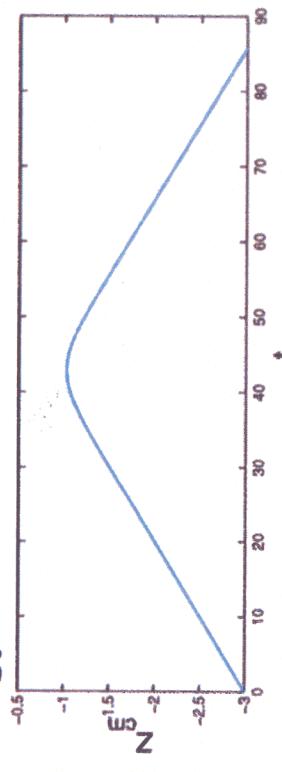
bifurcate from linear eigenvalue Ω at $I_{\text{tot}} = 0$

Resonant
+
Nonresonant
Cases

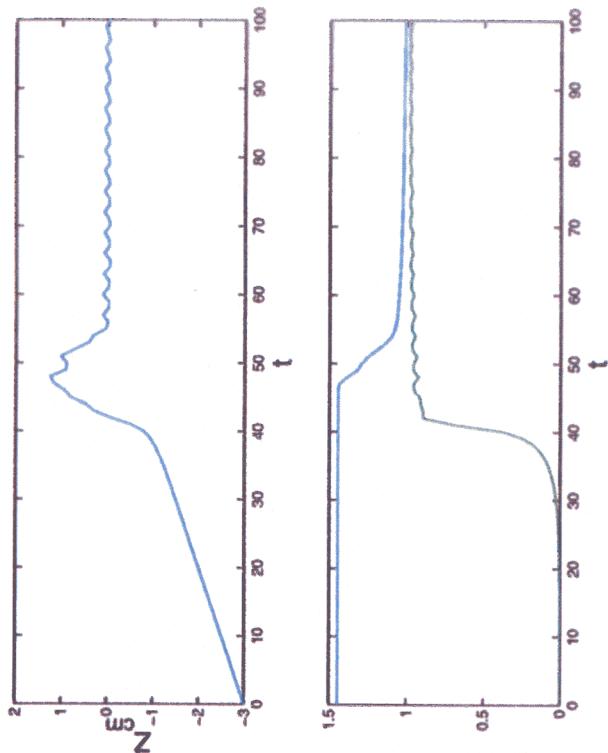


Contrast between resonant and nonresonant cases

Nonresonant case: No transfer of energy to defect mode.



Resonant case: Strong transfer of energy to defect mode.



Nonlinear Schrödinger Equation (NLS) with δ -function defect

A canonical model for wave-defect interaction

$$iu_t + \frac{1}{2}u_{xx} + |u|^2 u + \gamma\delta(x)u = 0.$$

Attractive features:

- NLS well studied and describes low-amplitude limit of NLCME equations.
- Solitons dependent on two parameters, η and V .

$$u = \eta \operatorname{sech}(\eta(x - Vt)) e^{i(Vx + \frac{1}{2}(\eta^2 - V^2)t)}$$

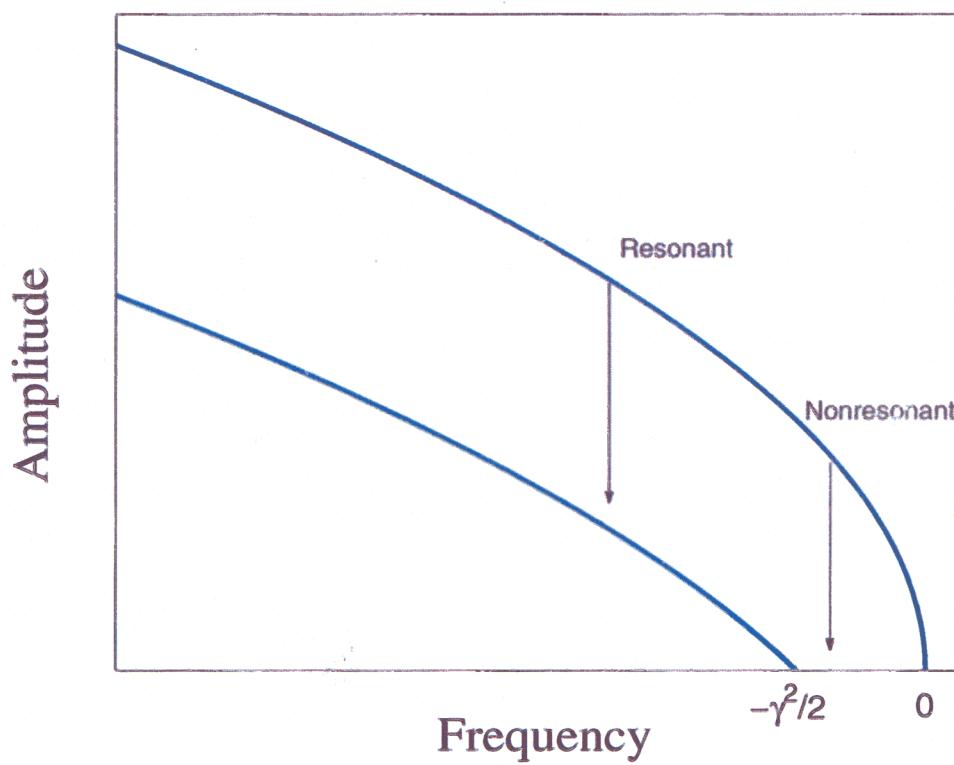
- Closed form nonlinear defect modes $a > \gamma$:

$$u(x, t) = ae^{ia^2 t/2} \operatorname{sech}(a|x| + \tanh^{-1} \frac{\gamma}{a})$$



Numerical Simulations of NLS

Reproduce NLCME behavior, i.e. capture when resonance exists between zero-velocity solitons and nonlinear defect mode.



Soliton Frequency $-\eta^2/2$

Collective Coordinate Analysis

Study a reduced system that captures the dynamics
Procedure

- Assume solution takes the form Soliton + DefectMode where each depends on a few time-dependent parameters:

$$u = \eta \operatorname{sech}(\eta x + Z) e^{i(Vx - \phi)} + a \operatorname{sech}(a|x| + \tanh^{-1} \frac{\gamma}{a}) e^{-i\psi - i\phi}$$

Need
 $d \geq 2$
d.o.f.

- Insert ansatz into the Lagrangian functional of the NLS system:

$$\mathcal{L} = \int_{-\infty}^{\infty} \frac{i}{2} (u^* u_t - u u_t^*) - \frac{1}{2} |u_x|^2 + \frac{1}{2} |u|^4 + \gamma \delta(x) |u|^2 dx.$$

- This yields:

$$\mathcal{L}_{\text{eff}} = 2\eta\dot{\phi} - 2Z\dot{V} + 2(a-\gamma)(\dot{\phi}+\dot{\psi}) + \frac{1}{3}\eta^3 - V^2\eta + \frac{1}{3}a^3 + \gamma\eta^2 \operatorname{sech}^2 Z + 2\gamma\eta\sqrt{a^2 - \gamma^2} \operatorname{sech} Z \cos\psi$$

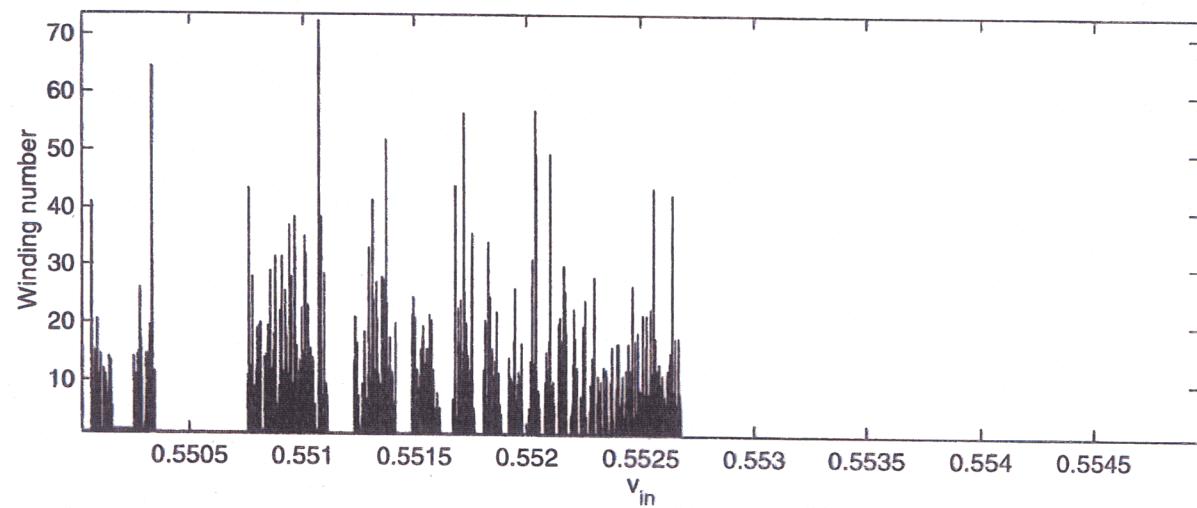
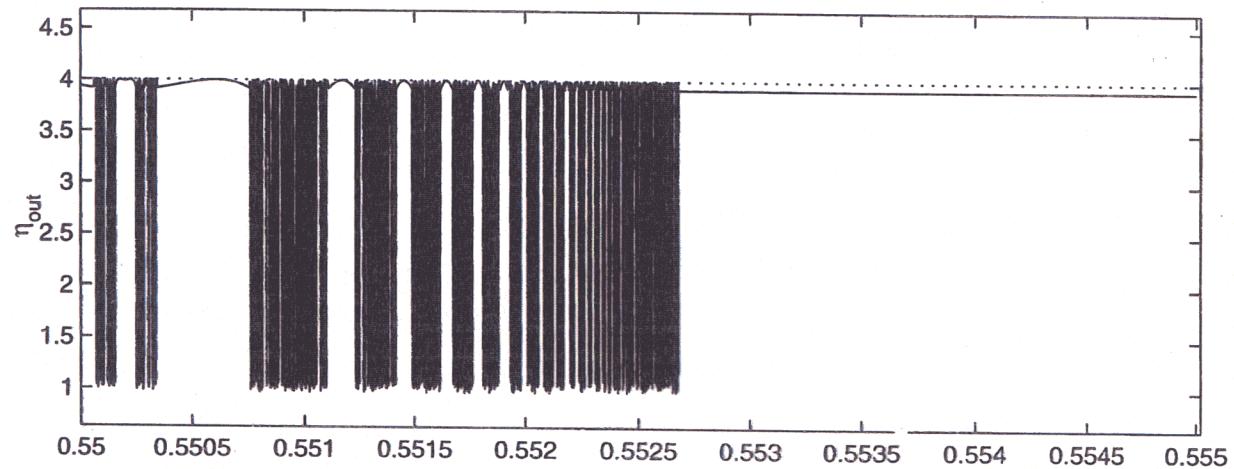
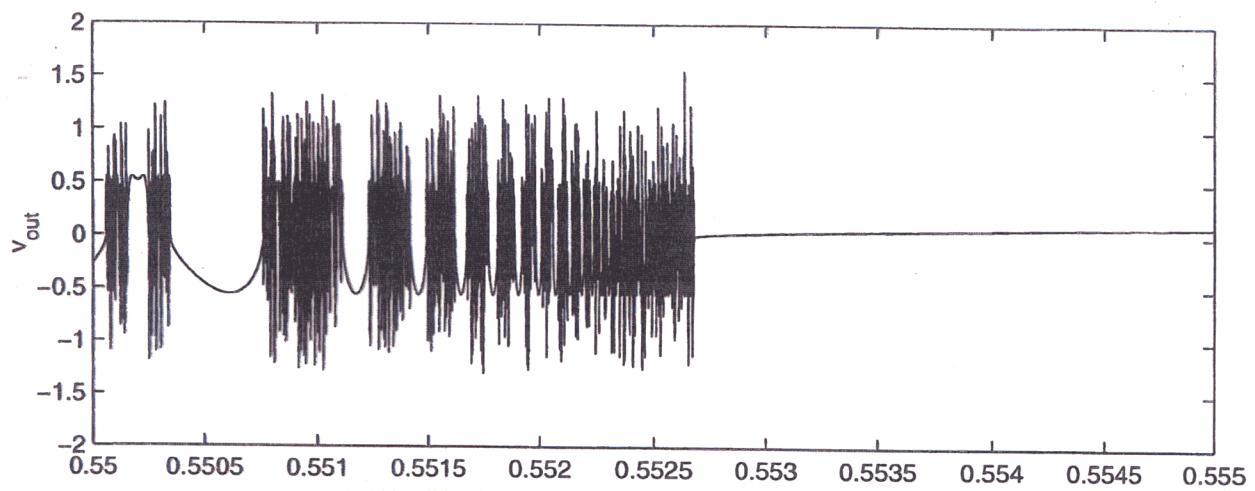
- Derive Euler-Lagrange equations for the evolution of the parameters
- Lagrangian independent of ϕ implies conserved total intensity $C = \eta + a$, allows us to reduce from 6 equations to 4.
- This can be rewritten as a **Hamiltonian system**

$$H = -\frac{1}{3}\eta^3 + V^2\eta - \frac{1}{3}a^3 - \gamma\eta^2 \operatorname{sech}^2 Z - 2\gamma\eta\sqrt{a^2 - \gamma^2} \operatorname{sech} Z \cos\psi$$

Resonant
Case

Capture / resonance

transmission



{ Captured
states

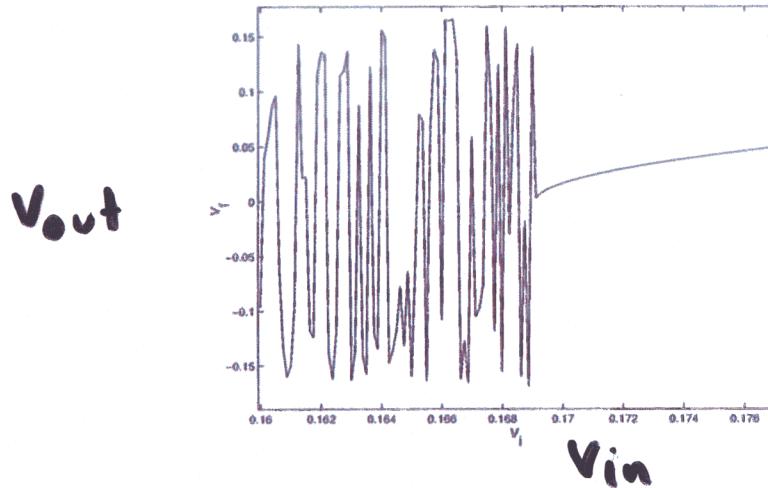


Figure 3.2: Capture, reflection and transmission of sine-Gordon kinks for the two-mode ODE model (2.11) of [6] with $\epsilon = 0.5$. V_f and V_i denote final and initial kink velocities. Compare with Figure 3.1.

{fig:ODEvf_vf.eps}

captured unless the velocity lies in certain reflection windows, with no evidence of transmission windows. There was rare evidence of transmission for subcritical velocities [12]. Fei et al. suggest a semi-heuristic formula for predicting the resonant velocities of the reflection windows, and their formula could easily be adapted to predict the transmission windows as well. Our simulations of equations (2.11) do not show any trapping for all time, and indeed, through study of (2.11) as a dynamical system, we show that the set of initial conditions that lead to trapping is nonempty but of measure zero.

4 Dynamical systems analysis

We modify Equation (2.11) by inserting a small coupling parameter $0 \leq \mu \leq 1$ in order to facilitate our analysis:

{sec:dynsys}

{eq:mumodel}

$$8\ddot{X} + U'(X) + \mu a F'(X) = 0; \quad (4.1a)$$

$$\ddot{a} + \Omega^2 a + \frac{\epsilon \mu}{2} F(X) = 0. \quad (4.1b)$$

We will perform perturbation theory for small μ and also consider the limiting case $\mu = 1$ of Equation (2.11). It will be convenient to rewrite Equations (4.1) in Hamiltonian form with momentum variables:

$$p_X = \frac{\partial L_{\text{eff}}}{\partial \dot{X}} = 8\dot{X}, \quad p_a = \frac{\partial L_{\text{eff}}}{\partial \dot{a}} = \frac{2}{\epsilon} \dot{a}, \quad (4.2) \quad \{eq:momenta\}$$

Radiative (damping) correction to
collective word. eqns

$$\ddot{\xi} + U'(x) + \alpha F'(\xi) = 0$$

$$\ddot{a} + \Omega^2 a + \frac{\epsilon}{\zeta} F(x) = -\epsilon^3 F(\xi) a^2 \dot{a}$$

=

Inclusion of
Radiation Damping
 \Rightarrow
positive measure
attracting set

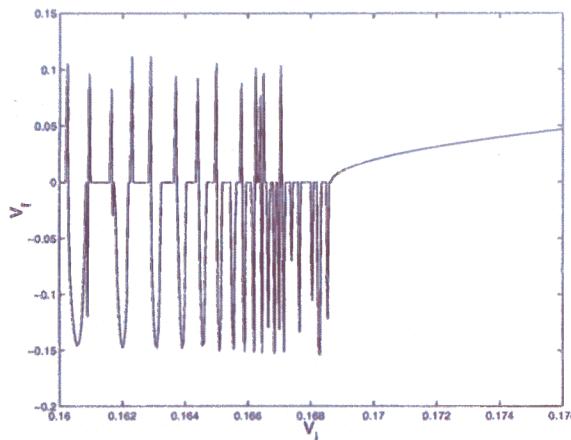


Figure 6.1: Output velocity V_f vs. input velocity V_i in the presence of radiation induced damping, with $\epsilon = 0.5$, $\Gamma = 16$. The behavior is simplified in comparison with Figure 3.2: below critical velocity, solutions may be captured, reflected, or transmitted, but the sensitivity to initial conditions is decreased. For smaller values of Γ , more reflection and transmission bands survive.

{fig:nldamp_vf}

where $\Gamma > 0$.

In Figure 6.1 we show the analog of Figure 3.2 with the nonlinear damping of (6.5b) active with $\epsilon = 0.5$ and $\Gamma = 16$. In the presence of damping, the output velocity is a much less sensitively dependent function of the input velocity than in the case without damping. In this case, below the critical velocity v_c , almost all trajectories are captured, except for a few intervals, almost all of which lead to reflection, as in Figure 3.1.

The incorporation of radiation damping effects present in the full dynamical system “smoothes out” the dynamics; many reflection resonance bands and most transmission bands appear to be eliminated. For smaller Γ values, more reflection and transmission bands survive. For all $\Gamma > 0$, trapping occurs on initial data sets of positive measure, since the flow of (6.5) is volume-contracting.

7 Conclusions

{sec:concl}

In this study we have examined a model of kink-defect interactions in the sine-Gordon equations. We use the model’s Hamiltonian structure to give a rather complete characterization of the dynamics. We demonstrate that a soliton propagating toward a defect may oscillate around the defect any integer number of times before being ejected, either in the original direction or in the opposite direction, and propagating off to infinity, leaving some of its energy in the stationary defect mode. This behavior is governed by phase-space transport in

* Limitations of finite dimensional model
in describing capture -

Finite dim. Hamiltonian system - volume preserving phase flow

\Rightarrow This { incoming solitons which are captured }
has measure zero!

Why is capture for the full (infinite dimensional)
PDE dynamics robust?

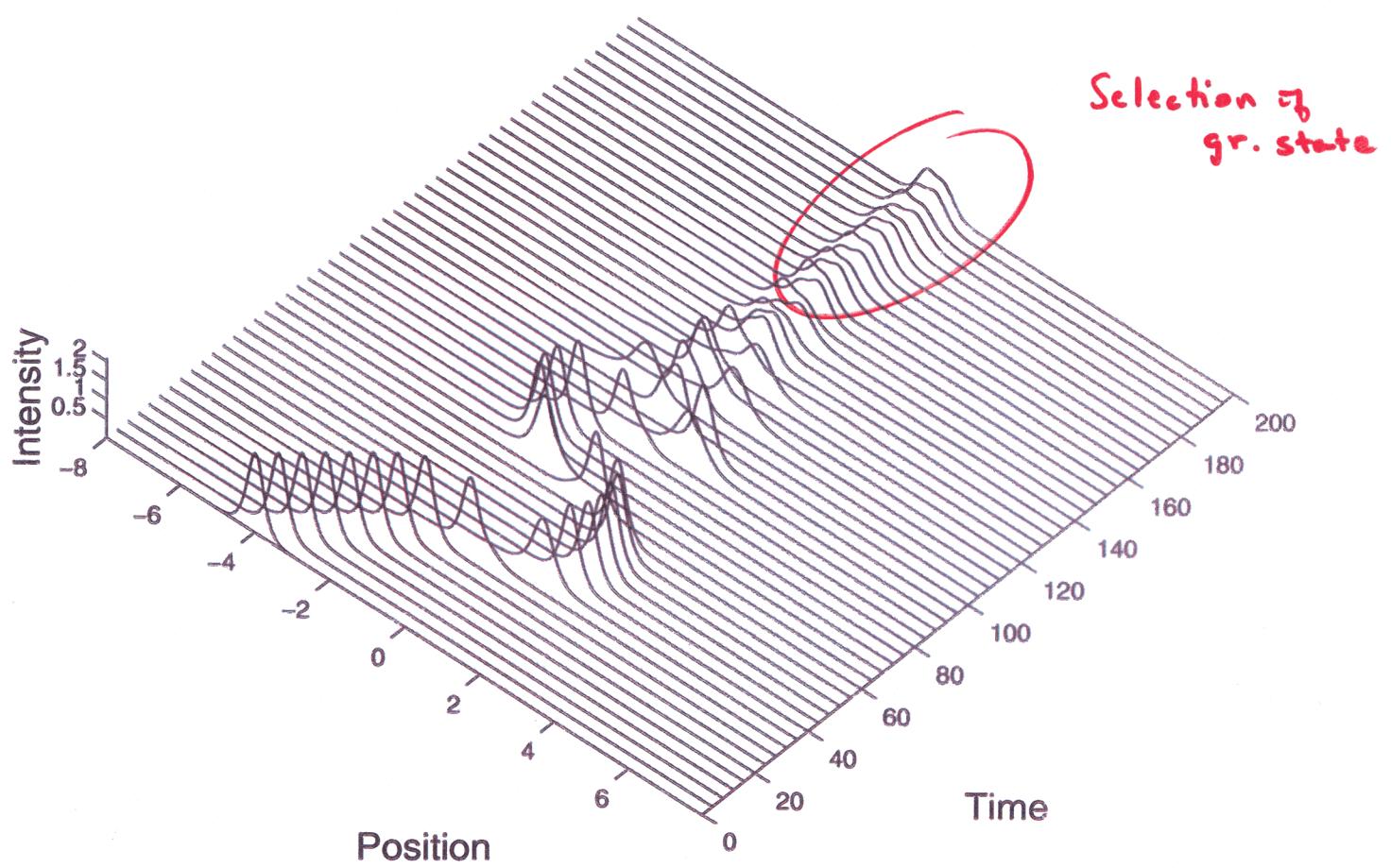
\rightarrow Finite dimensional dynamics

$$\dot{\vec{p}} = J \nabla H_{\text{particle}} [\vec{p}] \quad \leftarrow L_{\text{particle}}$$

Correct picture: $L = L_{\text{particle}} + L_{\text{field}} + L_{\text{coupling}}$

$$"i\partial_t \phi = -\partial_x^2 \phi"$$

$$\rightarrow \dot{\vec{p}} = J \nabla H_{\text{particle}} [\vec{p}] + \text{Radiation damping correction.} \quad |||$$



Goodman - Slusher - Weinstein '01
~~to appear in~~ JOSA B, 2002

$$i\partial_t \phi = H\phi + \lambda |\phi|^2 \phi$$

w/ A.Soffer

$$H = -\Delta + V(x)$$

$V(x) =$ potential well \rightarrow defect modes

\rightarrow nonlinear defect
modes of
 $H + \lambda |\phi|^2$

- Optical physics , Plasma (Rosen-W.)
- BEC - Gross-Pitaevskii egn

Related work:

Buslaev - Perelman

Buslaev - Sulem

Cuccagna

Fröhlich - Tsai - Yau

Tsai - Yau

Cuccagna - Kirr - Pelinovsky

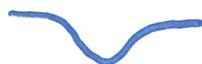
Kirr - W. ; almost periodic , random

Nonlinear Schrödinger Eqn

$$i\partial_t \phi = H\phi + \lambda |\phi|^2 \phi$$

$$H = -\Delta + V(x)$$

$V(x)$ = "potential well"



$$\lambda=0$$

Linear Schrödinger

H has 2 bound states ; $H\psi_{E_{j*}} = E_{j*} \psi_{E_{j*}}$

$$\|\psi_{E_{j*}}\|_{L^2} = 1 \quad j=0,1$$

Spectrum of H :



ϕ_0 arbitrary , localized data $\langle x \rangle^\sigma \phi_0 \in L^2$ $\sigma \geq 0 > 2$

$$\Rightarrow \phi(x,t) = e^{-iHt} \phi_0 =$$

$$= c_0 e^{-iE_{0*}t} \psi_{E_{0*}}(x) + c_1 e^{-iE_{1*}t} \psi_{E_{1*}}(x) + \underbrace{O(t^{-\frac{n}{2}})}_{L^\infty(\mathbb{R}^n)}$$

An arbitrary state , ϕ_0 ,

$\rightarrow \phi(x,t)$ which resolves into a

quasiperiodic(t) + radiative decay

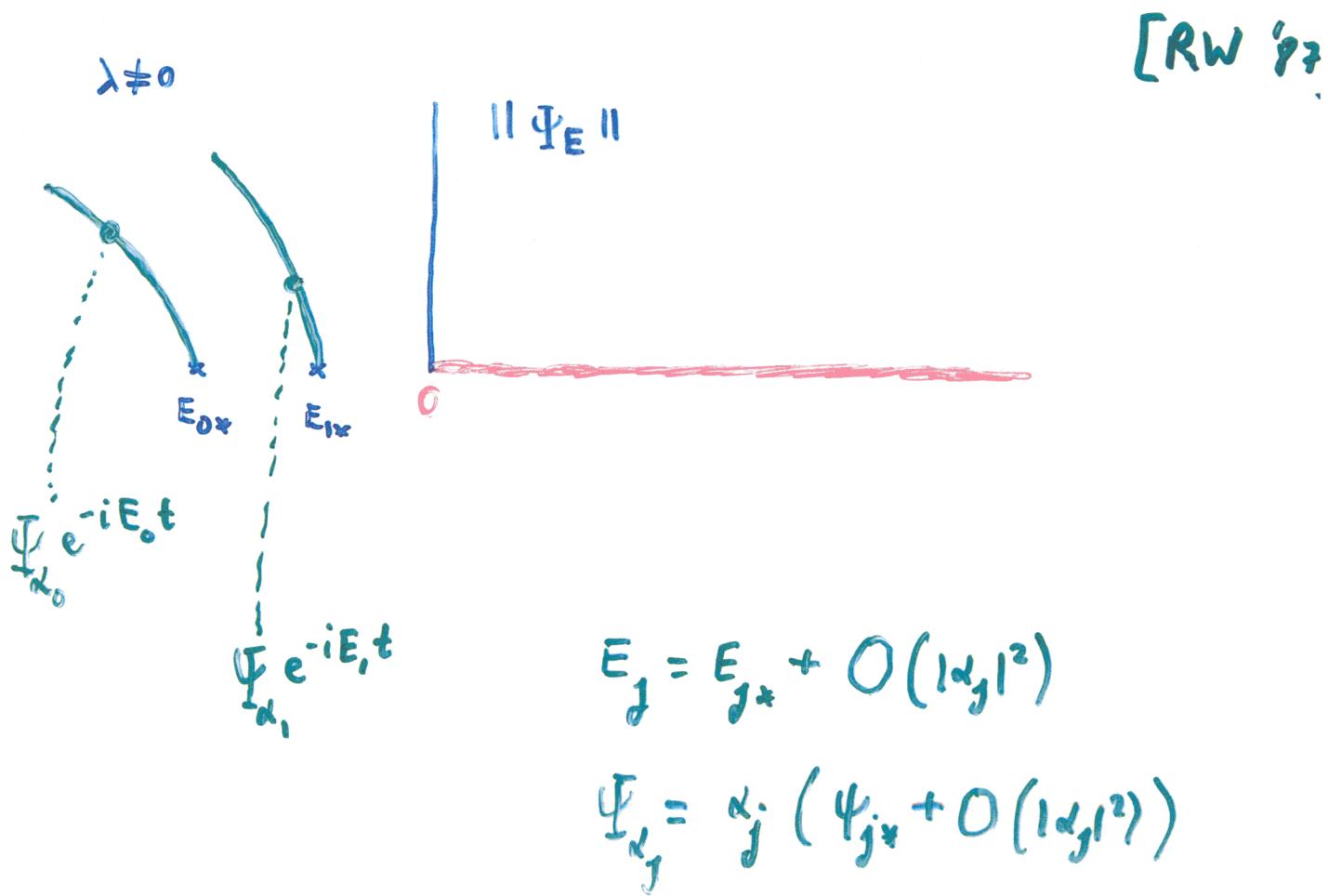
Nonlinear Schrödinger eqn

$$i\partial_t \phi = H\phi + \lambda |\phi|^2 \phi$$

Seek Ψe^{-iEt}

$$H\Psi + \lambda |\Psi|^2 \Psi = E\Psi$$

Theorem \exists nonlinear bound states bifurcating from linear ($\lambda=0$) states



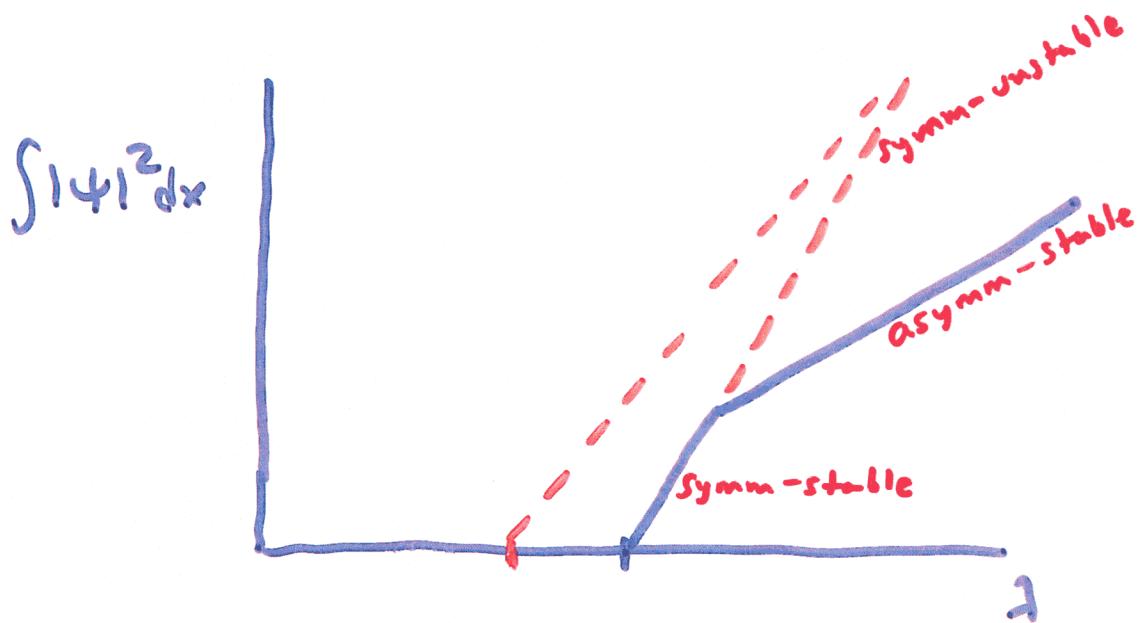
w/ RK Jackson

$$i\psi_t = -\psi_{xx} - 2|\psi|^2\psi + \epsilon V(x)\psi$$

$$V(x) = -\delta(x-L) - \delta(x+L)$$

double-well

$$\psi_b(x,t) = e^{i\omega t} \psi(x; 2)$$



Symmetry-Breaking Bifurcation

See also: AFGST, MKR, ...

↓
3-D Nonlinear Hartree

$\lambda=0$ linear Schrödinger \rightarrow 2 bound states

\rightarrow spatially localized, time periodic states

which both participate in large time

asymptotics $\phi(x, t) \sim \sum_j c_j e^{-i E_j t} \Psi_{j*}(x)$
 $+ O(t^{-n/2})$

$\lambda \neq 0$ NLS \rightarrow 2 nonlinear bound states

$$\Psi_{\alpha_j} e^{-i E_j t} \quad j=0,1$$

(No superposition principle)

QUESTIONS

* How does the solution of NLS resolve as $t \uparrow \infty$?

* How do the nonlinear bound states

Ψ_{d_0} and Ψ_{k_1} participate in the dynamics?

Theorem $i\partial_t \phi = (-\Delta + V(x))\phi + 2|\phi|^2\phi, x \in \mathbb{R}^3$

$$\phi(x, 0) = \phi_0(x)$$

Assumptions

1) $H = -\Delta + V(x)$ has 2 bound states

$$H \psi_{jx} = E_{jx} \psi_{jx} ; e^{-iE_{jx}t} \psi_{jx}(x)$$

2) $\phi_0(x)$ small (weakly nonlinear case)

3) $\Gamma_{\omega_x} \sim \left| \Im[\phi_{0x} \psi_{1x}^2] (2E_{1x} - E_{0x}) \right|^2 > 0 \quad (\text{FGR})$

Conclusion Generic selection of the ground state

$$\phi(t) \underset{t \rightarrow \infty}{\sim} e^{-i\omega_j t} \psi_{\alpha_j(\infty)}(x) + O(t^{-1/2}),$$

where either $j=0$ (nonlinear gr. state)

or $j=1$ (nongeneric)



Dynamical systems picture / Finite dimensional "reduction"

$$\phi(x, t) = \Psi_{\alpha_0(t)} e^{-i\theta_0(t)} + \Psi_{\alpha_1(t)} e^{-i\theta_1(t)} + \gamma_{\text{rad.}}(t, x)$$

$\alpha_0(t), \alpha_1(t)$ coordinates on nonlinear bound state in fields of equilibria

$$P_0(t) \simeq |\alpha_0(t)|^2, \quad P_1(t) \simeq |\alpha_1(t)|^2$$

$$\frac{dP_0}{dt} = \dots, \quad \frac{dP_1}{dt} = \dots$$

+ interacting particles

$$i \frac{\partial}{\partial t} \gamma = \mathcal{L} \gamma + \text{Resonant forcing by bound state "oscillators"}$$

radiation field

→ "Solve" for radiation as functional of particle dynamics (taking dominant resonance into account)
oscillatory integrals, l.a.p.

$$\frac{dP_0}{dt} \simeq \underbrace{\Gamma P_1^2 P_0}_{\text{in}} + \text{higher order coupling to radiation}$$

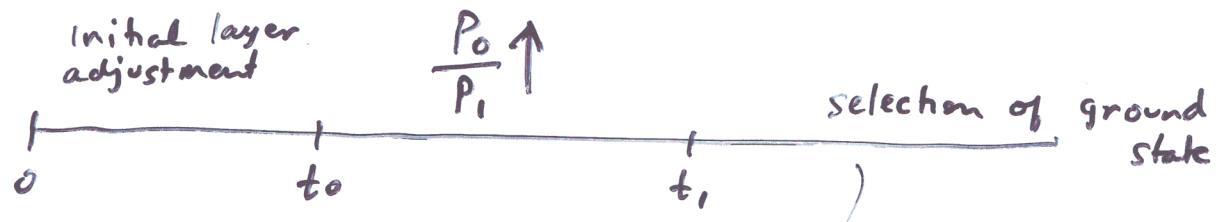
Radiation damping

$$\frac{dP_1}{dt} \simeq -2\Gamma P_1^2 P_0 + O(P_1^3 \sqrt{P_0}) + \text{higher order coupling to radiation}$$

3 time scales

$$\frac{dP_0}{dt} \approx \Gamma P_1^2 P_0 + \text{h.o. radiation}$$

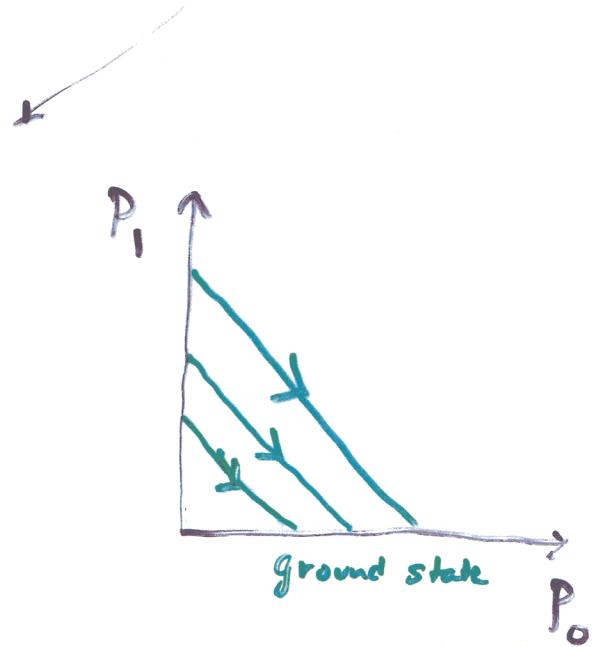
$$\frac{dP_1}{dt} \approx -2\Gamma P_1^2 P_0 + O(P_1^3 \sqrt{P_0}) + \text{h.o. radiation}$$



$t > t_1$

$$\frac{dP_0}{dt} \approx \Gamma P_1^2 P_0$$

$$\frac{dP_1}{dt} \approx -2\Gamma P_1^2 P_0$$



$$P_0(t) \sim |\bar{\Psi}_{\alpha_0(t)}|^2$$

$$P_1(t) \sim |\bar{\Psi}_{\alpha_1(t)}|^2$$

NLKG eqn

$$(\partial_t^2 + B^2) u = 2u^3$$

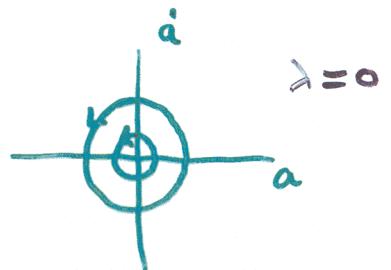
$$B^2 = -\Delta + 1 + V(x); \quad \Gamma(B^2)$$

$$\dot{u} \underset{\lambda=0}{\underset{\Omega^2}{\underset{1}{\text{---}}}} \underset{\lambda \neq 0}{\underset{\Omega^2}{\underset{1}{\text{---}}}}$$

$$\lambda=0 \Rightarrow u_0(x, t) = a(t) \varphi(x)$$

$$\partial^2 \varphi = \Omega^2 \varphi,$$

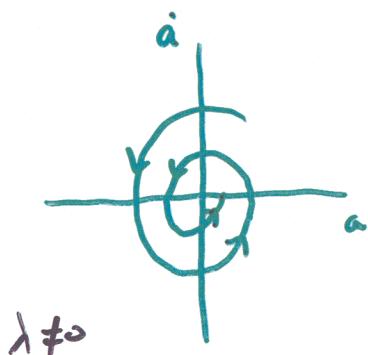
$$\ddot{a} + \Omega^2 a = 0$$



$\lambda \neq 0$
(small data)

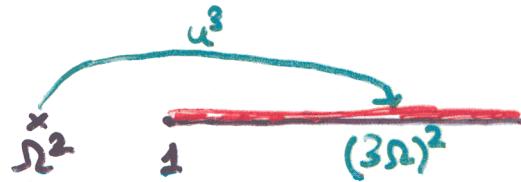
$$u_\lambda(x, t) = a(t) \varphi(x) + \eta(x, t)$$

$$\langle \varphi, \eta(t) \rangle = 0$$



$$\ddot{a} + (\Omega^2 + O(|a|^2)) a = -P a^4 \dot{a} + \dots$$

$$P \sim \left| \frac{\Im[\varphi^3](3\Omega)}{B} \right|^2$$



Nonlinear resonance w/
continuum modes

radiation
damping

$$a(t) \sim t^{-\frac{1}{4}}$$

$$t \gg 1$$