

CAMP Seminar

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Resonances in Hamiltonian PDE

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Invariant tori for
Hamiltonian PDE

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(2) Hamiltonian PDE

$$(1) \quad \partial_t \mathcal{N} = J \delta_u H(\omega)$$

$$\mathcal{N}(x, 0) = \mathcal{N}_0(x)$$

- phase space \mathcal{X} - Hilbert space
- symplectic form $\omega(X, Y) = \langle X, JY \rangle_{\mathcal{H}}$
 $J^T = -J$
- flow of the dynamical system

$$\mathcal{N}(x, t) = \varphi_t(\mathcal{N}_0(x)) ,$$

tracing a curve in \mathcal{X} through \mathcal{N}_0 .

Contents:

- Hamiltonian PDE - examples
- Invariant tori - a variational problem
- Results
- Estimates of the linearized problem

1) Principal examples

(1) nonlinear wave equations

$$(1) \quad \partial_t^2 u - \Delta u + g(x, u) = 0$$

$x \in \mathbb{T}^d = \mathbb{R}^d / \Gamma$ periodic boundary cond
or else

$x \in \Omega \subseteq \mathbb{R}^d, x \in \partial\Omega \Rightarrow u(x, t) = 0$ Dirichlet

The Hamiltonian functional

$$H(u, p) = \int_{\mathbb{T}^d} \frac{1}{2} p^2 + \frac{1}{2} |\nabla u|^2 + G(x, u) dx$$

then

$$\partial_t u = p = \delta_p H$$

$$\partial_t p = \Delta u - \partial_u G(x, u) = -\delta_u H$$

$$g(x, \cdot) = \partial_u G(x, \cdot)$$

This system has the form

$$\partial_t \begin{pmatrix} u \\ p \end{pmatrix} = J \delta H$$

with

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

Darboux coordinates

- Assume that

$$G(x, m) = \frac{1}{2} g_1(x) m^2 + \frac{1}{3} g_2(x) m^3 + \dots$$

Then $H = H^{(2)} + H^{(3)} + \dots$ Taylor expansion about $\omega = (\frac{m}{p}) = 0$.

- The quadratic Hamiltonian is

$$\begin{aligned} H^{(2)} &= \int_{\mathbb{T}^d} \frac{1}{2} p^2 + \frac{1}{2} |\nabla u|^2 + \frac{1}{2} g_1(x) u^2 dx \\ &= \sum_n \frac{1}{2} |p_n|^2 + \frac{\omega_n^2}{2} |u_n|^2, \end{aligned}$$

expansion in terms of eigenfunctions

$$\left(\begin{array}{c} u(x) \\ p(x) \end{array} \right) = \sum_n \left(\begin{array}{c} u_n \\ p_n \end{array} \right) \psi_n(x)$$

satisfying

$$L(g_n) \psi_n = (-\Delta + g_n(x)) \psi_n = \omega_n^2 \psi_n;$$

$(\psi_n(x), \omega_n^2)$ eigenfunction, eigenvalue pair

- This is a harmonic oscillator, with frequencies ω_n

- Solutions of the linearized equations

$$(3) \quad \dot{\varphi}_t \begin{pmatrix} u \\ p \end{pmatrix} = J \delta H^{(2)} \begin{pmatrix} u \\ p \end{pmatrix}$$

are of the form

$$\begin{pmatrix} u(x,t) \\ p(x,t) \end{pmatrix} = \sum_n t_n \begin{pmatrix} \cos(\omega_n t + \vartheta_n) & \frac{1}{\omega_n} \sin(\omega_n t + \vartheta_n) \\ -\omega_n \sin(\omega_n t + \vartheta_n) & \cos(\omega_n t + \vartheta_n) \end{pmatrix} \begin{pmatrix} u^0 \\ p^0 \end{pmatrix}$$

$$= \Phi_t \begin{pmatrix} u^0 \\ p^0 \end{pmatrix} \quad \text{the linear flow.}$$

- Facts:

- The Hamiltonian is preserved along the flow

$$H^{(2)}(\Phi_t(u)) = H^{(2)}(u)$$

- Actions are preserved along the flow

$$I_n = \frac{1}{2} (\omega_n |u_n|^2 + \frac{1}{\omega_n} |p_n|^2)$$

such that

$$I_n(\Phi_t(u)) = I_n(u).$$

- Angles evolve linearly in time; $\vartheta_n \rightarrow \omega_n t + \vartheta_n$
- All solutions are

+ periodic

all actions $\omega_n = j_n \omega_0$ j_n

+ quasi-periodic, or

$\omega_n = (j_k \tilde{\omega}_0) j_k \epsilon \mathbb{Z}$

+ almost-periodic, if no finite # suffice.

- Basic questions :

+ (1) Whether some solutions of the nonlinear problem have the same properties:

- + periodic
- + quasi-periodic
- + almost-periodic

(KAM theory)

+ (2) Whether all solutions with $\omega_0 \in B_R(\omega) \subseteq \mathcal{X}$ remain in $B_{2R}(\omega)$

(well-posedness)

• Whether action variables are preserved for long time intervals

$$|I_k(\omega(t)) - I_k(\omega_0)| < \varepsilon^\alpha$$

for $|t| < T(\varepsilon) \sim \exp(\gamma/\varepsilon^\beta)$

(Nekhoroshev stability)

+ (3) Upper and lower bounds on growth of the action variables, or on higher Sobolev norms

(Arnold diffusion)

(II) nonlinear Schrödinger equation

$$(4) i \partial_t u - \frac{1}{2} \Delta u + Q(x, u, \bar{u}) = 0$$

$x \in \mathbb{R}^d / \Gamma = \mathbb{T}^d$ periodic boundary conditions
 or else Dirichlet boundary conditions
 $x \in \Omega \subseteq \mathbb{R}^d, \quad u(x) = 0 \text{ for } x \in \partial\Omega.$

Initial data

$$u(x, 0) = u_0(x) \in \mathcal{H}$$

The flow of the dynamical system

$$u(x, t) = \Phi_t(u_0(x)).$$

- The Hamiltonian is

$$H(u) = \int_{\mathbb{T}^d} \frac{1}{2} |\nabla u|^2 + G(x, u, \bar{u}) dx$$

where

$G(x, z, w)$ is real valued when $w = \bar{z}$,

and $\partial_{\bar{z}} G = Q$.

Then (4) can be expressed as

$$(5) \partial_t u = i \mathcal{J} H(u)$$

where $\mathcal{J} = i \mathbf{I}$ complex symplectic coordinate

(8)

(III) Korteweg deVries equation

$$(5) \quad \partial_t q_t = \frac{1}{6} \partial_x^3 q_t - \partial_x (\partial_x G(x, q_t))$$

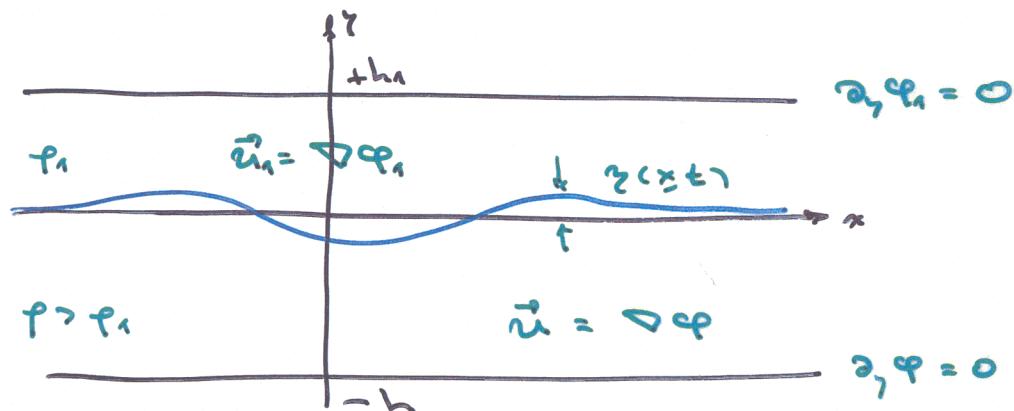
$$x \in \mathbb{R}' / \Gamma = \mathbb{T}'$$

The Hamiltonian

$$H = \int_{\mathbb{T}'} \frac{1}{2} (\partial_x q_t)^2 + G(x, q_t) dx,$$

symplectic form given by $J = -\partial_x$.

(IV) Large amplitude long waves in an interface



Equations for the interface

$$(6) \quad \partial_t \begin{pmatrix} \gamma \\ u \end{pmatrix} = \begin{pmatrix} 0 & -\partial_x \\ -\partial_x & 0 \end{pmatrix} \begin{pmatrix} \delta_q H \\ \delta_u H \end{pmatrix}$$

symplectic form

$$\omega(x, Y) = \int (\partial_x^i X_q) Y_m + (\partial_x^i X_u) Y_q dx$$

same as for the Boussinesq equation.

Hamiltonian

$$H = \sum_{\mathbf{q}} \frac{1}{2} m R_0(\mathbf{q}) u + g \frac{(e - e_0)}{2} \mathbf{q}^2$$

$$+ R_1(\mathbf{q}) (\partial_x u)^2 + \frac{1}{4} (\partial_x u^2) R_2(\mathbf{q}) \partial_x \mathbf{q} + R_3(\mathbf{q}) u^2 (\partial_x$$

with rational coefficients

$$R_0(\mathbf{q}) = \frac{(h + \gamma)(h_i - \gamma)}{\tau_i(h + \gamma) + \tau(h_i - \gamma)}$$
nonlinear propagation velocity

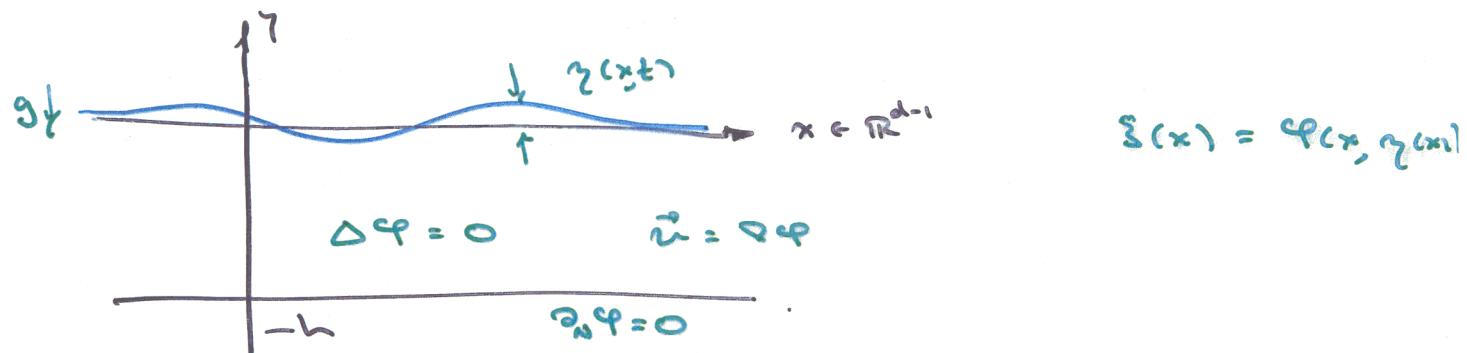
$$R_1(\mathbf{q}) = -\frac{1}{3} \frac{(h + \gamma)^2 (h_i - \gamma)^2 [\tau_i(h_i - \gamma) + \tau(h + \gamma)]}{[\tau_i(h + \gamma) + \tau(h_i - \gamma)]^2}$$

$$R_2(\mathbf{q}) = -\frac{1}{3} \frac{\tau \tau_i (h + h_i) (h + \gamma) (h_i - \gamma) [(h_i - \gamma)^2 - (h + \gamma)^2]}{[\tau_i(h + \gamma) + \tau(h_i - \gamma)]^3}$$

$$R_3(\mathbf{q}) = -\frac{1}{3} \frac{\tau \tau_i (h + h_i)^3 [\tau_i(h + \gamma)^3 + \tau(h_i - \gamma)^3]}{[\tau_i(h + \gamma) + \tau(h_i - \gamma)]^4}$$

nonlinear dispersion coefficients.

(IV) surface water waves



Euler's equations are equivalent to

$$(7) \quad \partial_t \begin{pmatrix} \gamma \\ \xi \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \delta H$$

where the Hamiltonian is given by

$$H(\gamma, \xi) = \int \left(\frac{1}{2} \int G(\gamma) \xi + \frac{g}{2} \gamma^2 \right) dx .$$

The Dirichlet-Neumann operator $G(\gamma)$ satisfies

$$\xi_{\text{ext}} \mapsto \varphi(x, \gamma) \quad \text{harmonic extension}$$

$$\mapsto N \cdot \nabla \varphi \, dS_{\gamma} \quad \text{normal derivative}$$

$$:= \underbrace{\int G(\gamma) \xi \, dx}_{\text{---}}$$

(41)
2) An invariant torus

Torus $S(s) : \mathbb{T}^m \rightarrow \mathcal{X}$ phase space

$$S(s + t\omega) = \Phi_t(S(s)) \text{ flow invariant}$$

The frequency $\omega \in \mathbb{R}^m$.

This implies that

$$(8) \quad \partial_t S = J \delta_{\omega} H(S) \text{ and } \partial_t S = \omega \cdot \partial_S S.$$

Problem: solve this equation (8) for $(S(s), \omega)$

where ω is a bifurcation parameter

This is generally a small divisor problem.

Rewrite (8) as

$$(9) \quad J^{-1} \omega \cdot \partial_S S - \delta_{\omega} H(S) = 0$$

(12)

+ A variational problem

Consider the space of mappings

$$S \in X := \{ S(\xi) : \mathbb{T}^m \rightarrow \mathcal{X} \}$$

Define action functionals

$$+ I_j(S) = \frac{1}{2} \int_{\mathbb{T}^m} \langle S, J^{-1} \partial_{\xi_j} S \rangle d\xi$$

$$\delta_S I_j = J^{-1} \partial_{\xi_j} S$$

and the average Hamiltonian

$$+ \bar{H}(S) = \int_{\mathbb{T}^m} H(S(\xi)) d\xi$$

$$\delta_S \bar{H} = \partial_m H$$

Consider the subvariety of X defined by

$$\begin{aligned} \Pi_a &= \{ S \in X : I_1(S) = a_1, \dots, I_m(S) = a_m \} \\ &\subseteq X \end{aligned}$$

Variational problem: critical points of $\bar{H}(S)$ on the variety Π_a correspond to solutions of equation (a), with Lagrange multipliers Ω .

NB: invariance under the action of \mathbb{T}^m .

(13)

Two questions :

(i) Do critical points exist?

The functional $S_2 \cdot S_3 \cdot S$ is degenerate.

(ii) How to understand multiplicity?

Topology of $\mathbb{P}_n / \mathbb{P}^m$ and
its equivariant cohomology.

Answers - in some cases ;

(i) study the linearized problem -

Frohlich - Spencer estimates.

Nash - Nirenberg method.

(ii) Morse - Bott theory of critical points / orbits.

(3) The linearized problem

- The linearized problem around $\sigma = 0$ is given by the quadratic Hamiltonian

$$\begin{aligned} H^{(2)}(\sigma) &= \sum_n \frac{\omega_n}{2} \left(\frac{1}{\omega_n} p_n^2 + \omega_n q_n^2 \right) \\ &= \sum_n \omega_n I_n, \end{aligned}$$

where $v = \begin{pmatrix} q \\ p \end{pmatrix}$.

- Representing mappings $S: T^m \rightarrow \mathcal{X}$ of a torus into phase space,

$$\begin{aligned} S = S(x, \xi) &= \sum_n S_n(\xi) t_n(x) \\ &= \sum_{j,n} S_{jn} t_n(x) e^{ij \cdot \xi} \quad j \in \mathbb{Z}^m \end{aligned}$$

- The linearized equation can be written

$$\begin{aligned} (\delta_{\sigma}^2 H^{(2)} - \Omega \cdot \delta_{\sigma}^2 I^{(2)}) S \\ = \sum_{j,n} \begin{pmatrix} \omega_n & i\Omega \cdot j \\ -i\Omega \cdot j & \omega_n \end{pmatrix} \begin{pmatrix} q_{jn} \\ p_{jn} \end{pmatrix} t_n(x) e^{ij \cdot \xi}. \end{aligned}$$

- Eigenvalues of this 2×2 block diagonal problem are

$$\mu(j, n) = \omega_n \pm \Omega \cdot j$$

Null space :

- Choose $\Omega^0 = (\Omega_{1,}^0, \dots, \Omega_{m,}^0) \in \mathbb{R}^m$ a frequency vector which is a solution of the many resonance

$$(10) \quad \omega_{k_e} - \Omega^0 \cdot j_e = 0 \quad e=1, \dots, m.$$

This identifies an eigenspace $X_1 \subseteq X$ within the space of mappings X_1 spanned by $\{\phi_{i(m)} e^{ij\cdot}\}$.

Proposition : $X_1 \subseteq X$ is even dimensional $\geq 2m$:
(it could be infinite dimensional, but it is symplectic).

- The non-resonant case is when $\dim(X_1) = 2m$. Otherwise $\dim(X_1) > 2m$, the case is resonant.

The other eigenvalues are

$$\{ \mu_{jk} = \omega_k \pm \Omega^0 \cdot j \neq 0 \}$$

which typically forms a dense set in \mathbb{R} , these are the small divisors.

Lyapunov - Schmidt decomposition:

In the space X of mappings $S: \mathbb{T}^m \rightarrow \mathbb{R}$

decompose the equation $X = X_1 \oplus X_2$

$$QX = X_1 \quad \text{null space}$$

$$PX = (I - Q)X = X_2$$

The equations to solve are therefore

$$(g') \quad Q \left[J^{-1} \nabla \cdot \partial_S S - \delta_{\alpha} H(S_1) \right] = 0 \quad \text{bifurcation eqn}$$

$$P \left[J^{-1} \nabla \cdot \partial_S S - \delta_{\alpha} H(S_1) \right] = 0 \quad \text{small divisors}$$

Decompose mappings into $S = S_1 + S_2$.

If one can solve for $S_2 = S_2(S_1, \mu)$ there is a reduced variational problem (Weinstein - Moser).

$$I_j'(S_1) = I_j(S_1 + S_2)$$

$$\bar{H}'(S_1) = \bar{H}(S_1 + S_2)$$

$$\Pi_a' = \{S_1 \in X_1 : I_j'(S_1) = a, j=1 \dots m\}$$

critical pts of \bar{H}' on Π_a' are solutions of $(g')_Q$.

- The linearized operator about an approximate torus embedding s_0 :

$$(11) \quad (\delta_{\nu}^2 H(s_0) - \omega \cdot \delta_{\nu}^2 I) V = \left[\text{diag}_{2m} \begin{pmatrix} \omega_n & i\omega_j \\ -i\omega_j & \omega_n \end{pmatrix} + W(j_h, j_k) \right] V$$

where the index $(j, k) \in \mathbb{Z}^m \oplus \mathbb{Z}^d$.

Definition: A lattice site $(j, k) \in \mathbb{Z}^m \oplus \mathbb{Z}^d$

is do - singular for ω when

$$d_0 > |\omega_n \pm \omega_j|$$

and regular otherwise.

Proposition: For $A \subseteq \mathbb{Z}^m \oplus \mathbb{Z}^d$ having only regular sites, and for $|W|_{op} < \frac{d_0}{2}$ then

$$\|(\delta_{\nu}^2 H(s_0) - \omega \cdot \delta_{\nu}^2 I)^{-1}\|_A < \frac{4}{d_0}$$

- Fröhlich - Spencer estimates are used to add in the regular sites

(17)

+ Fröhlich - Spencer estimates:

Depend upon two properties of the operator

$$(\delta_n^2 H(s_0) - \Omega \cdot \delta_n^2 I) = D(\omega) + W.$$

(ii) nonresonance : if $(j, k) = \gamma \in \mathbb{Z}^m \oplus \mathbb{Z}^d$
 then $(j', k') = \gamma'$

$$\begin{aligned} d_n &< |\omega_n \pm \Omega \cdot j| < d_0 \\ (12) \quad \dots & \quad |\omega_n \pm \Omega \cdot j| \dots & \text{singular sites} \\ \text{whenever} \quad R_n &< |\gamma|, |\gamma'|. \end{aligned}$$

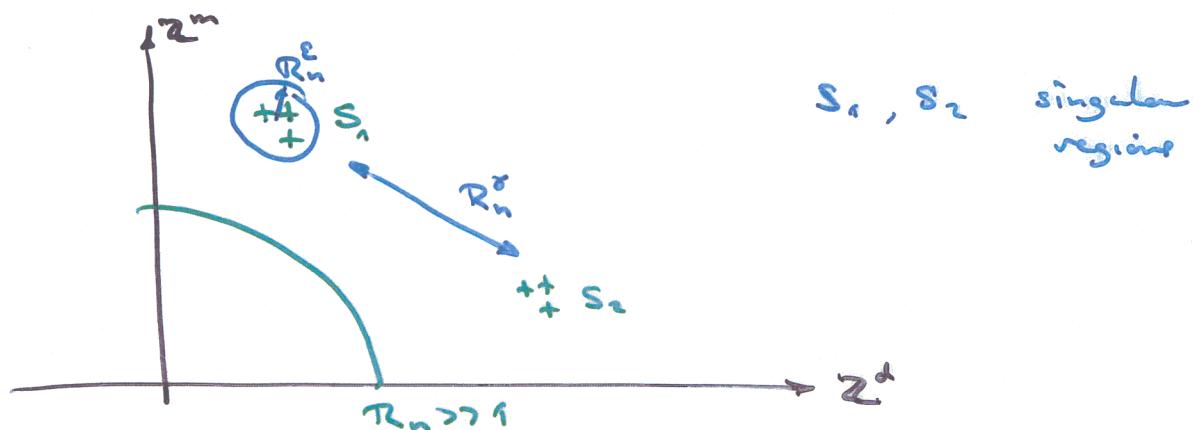
(iii) separation : if γ, γ' are singular sites

with $R_n < |\gamma|, |\gamma'|$, then either

$$(1) \quad \text{dist}(\gamma, \gamma') < R_n^{\epsilon} \quad 0 < \epsilon \ll 1$$

or else

$$(2) \quad \text{dist}(\gamma, \gamma') \gg R_n^{\epsilon} \quad 0 \ll \epsilon$$



(4) Results: existence of KAM tori

(i) nonlinear wave equations

- S. Kuksin (1988) E. Wayne (1989); Dirichlet b.c.
- W.C & E. Wayne (1993) periodic b.c. $m \neq 1$
- J. Bourgain (1994), $d > 1$.
- D. Bambusi (2000)

nb separation imposed by diophantine conditions.

(ii) nonlinear Schrödinger equation

- S. Kuksin (1988)
- W.C. & E. Wayne (1994) $m = 1$
- S. Kuksin & J. Pöschel (1996)
- J. Bourgain (1996) $d > 1$

nb separation imposed by the dispersion relation ω_n .

(iii) perturbations of KdV

- S. Kuksin (1988)
- T. Kappeler & J. Pöschel (2003)

(iv) Birkhoff-Lewis orbits for the nonlinear Schrödinger eqn

- W.C., J. Dolbeault & E. Séré (2003)

(v) Standing water waves

- Plotnikov & Toland ($h < \infty$) (2001)
- Iooss, Plotnikov & Toland ($h = \infty$) (2003 CRAS)