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Euclidean Symmetry and the Dynamics of Spiral Waves

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Outline

- **Waves**
- **Spirals**
- **Euclidean symmetry**
- **Relative equilibria**
- **Bifurcation to meandering**
- **Broken symmetry**
- **Conclusions and ongoing work**

Waves

- **Wave = Pattern in space which evolves over time**
- **Pervade every aspect of our lives:**

    light, microwaves, radio waves ...

    atmospheric pressure waves influence weather

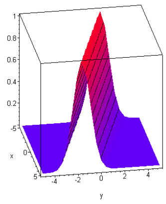
    physiological waves keep you alive!

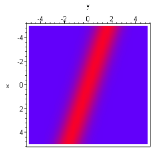
        e.g. electric wave causes heart beat

Some examples of 2-d waves:

- **Plane waves**

$$t = 0 \; \xrightarrow[]{\textit{snapshot}}$$

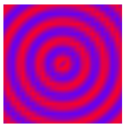




e.g. $u(x, y, t) = e^{-\frac{1}{10}(x+3y-t)^2}$

- **Target pattern waves**

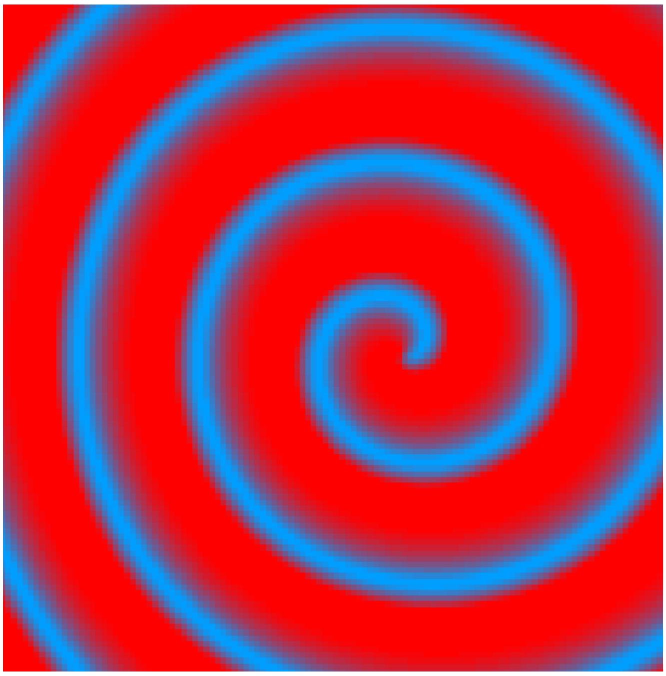
$$t = 0 \; \xrightarrow[]{\textit{snapshot}}$$



e.g. $u(x, y, t) = \sin\left(\sqrt{x^2 + y^2} - t\right)$

Spirals

Snapshot of a spiral wave (seen from above):



Spiral waves are observed in:

- **certain types of chemical reactions (Belousov-Zhabotinsky)**
- **slime-mold aggregates**
- **cardiac tissue (may lead to fatal arrythmias)**
- **other excitable media (including biological tissue)**

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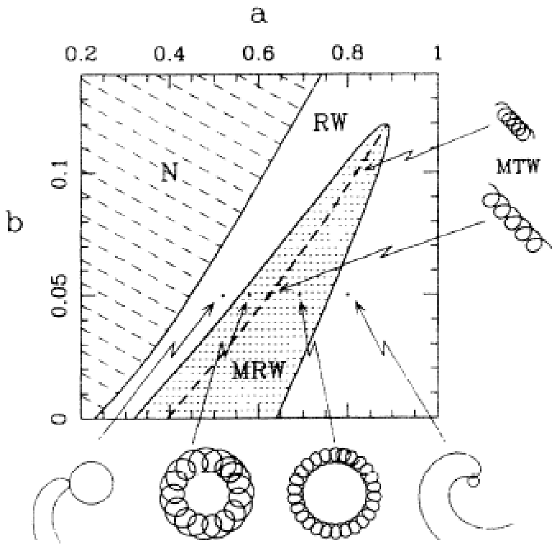
"...spirals on the heart are fatal, spirals on the cerebral cortex may lead to epileptic seizures, and spirals on the retina may cause hallucinations " — Mathematical Physiology
J. Keener & J. Sneyd, P. 305

How do (isolated) spirals move? ***(A. Winfree)***

- **rigid uniform rotation**
- **quasi−periodic meandering**
- **linear drifting**
- **hypermeander (chaotic?)**

$\vdots$

Transition to meandering



D. Barkley (1994) PRL

Numerical simulation of a model for cardiac electrophysiology

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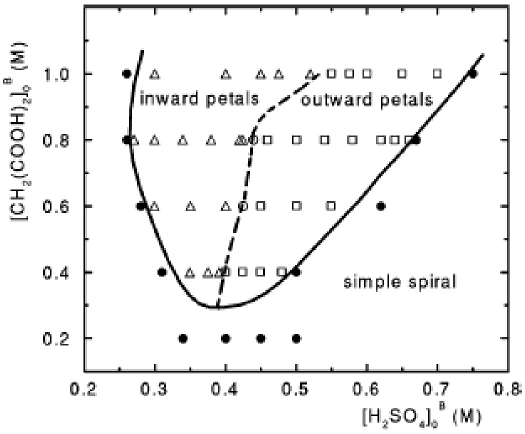

FIG. 4. Dynamics of spirals as a function of $[H_2SO_4]_0^B$ and $[CH_2(COOH)_2]_0^B$ (with other conditions as in Fig. 1). The solid line marks the transition from simple spirals (●) to meandering spirals with inward (△) and outward (□) petals. Traveling spirals (○) exist along the dashed line that separates the two types of meandering spirals.

Li, Ouyang, Petrov & Swinney (1996) PRL

Actual chemical reaction

Transition to meandering

Phenomenon appears to be model independent!

Euclidean symmetry

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$\longrightarrow$ *nonlinear! (can not solve them exactly)*

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e.g. Belousov−Zhabotinsky chemical reaction

two species, concentrations $\begin{array}{l} u = u(x,y,t) \\ v = v(x,y,t) \end{array}$

$$\frac{\partial u}{\partial t} = D_u \nabla^2 u \; + \; \frac{1}{\varepsilon}\left(u - u^2 - fv\frac{u-q}{v-q}\right)$$

$$\frac{\partial v}{\partial t} = D_v \nabla^2 v \; + \; (u - v)$$

N.B. $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

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diffusion *reaction*

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e.g. FitzHugh-Nagumo (electric waves in biological tissue)

electric potential $\Phi = \Phi(x, y, t)$

recovery function $v = v(x, y, t)$

$$\frac{\partial \Phi}{\partial t} = \nabla^2 \Phi + \frac{1}{\varepsilon}\left(\Phi - \frac{\Phi^3}{3} - v\right)$$

$$\frac{\partial v}{\partial t} = \varepsilon(\Phi + \beta - \gamma v)$$

N.B. $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

General class of models

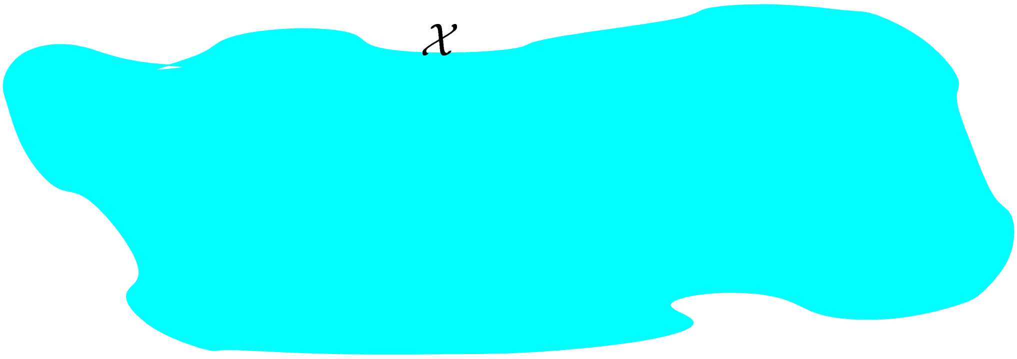
$$\vec{u} = (u_1(x,y,t), \ldots, u_n(x,y,t))$$

$$\boxed{\frac{\partial \vec{u}}{\partial t} = D \cdot \nabla^2 \vec{u} + \mathcal{F}(\vec{u})} \qquad \textbf{(RD)}$$

$D = n \times n$ matrix of diffusion constants

$\mathcal{F} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ "smooth enough"

(RD) defines a dynamical system on a suitable space of functions : $\mathcal{X} = \{\, \vec{v} : \mathbb{R}^2 \longrightarrow \mathbb{R}^n \mid \vec{v} \text{ satisfies } \ldots \}$



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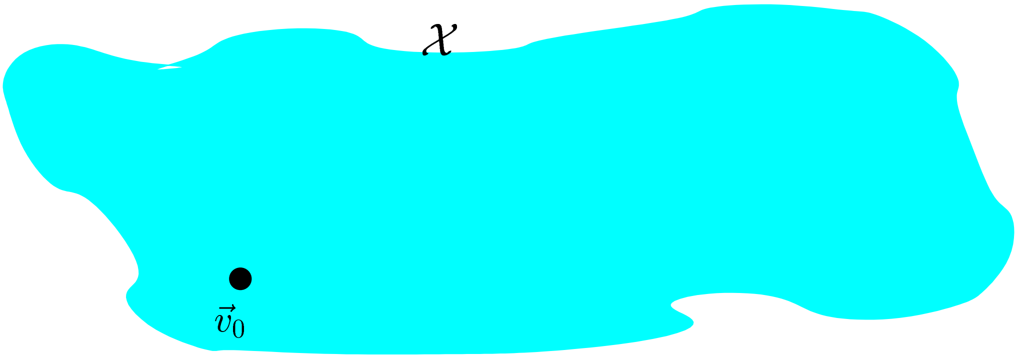

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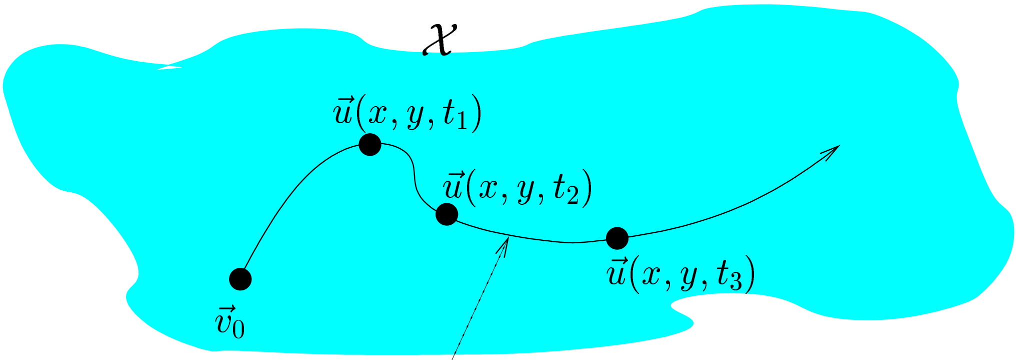

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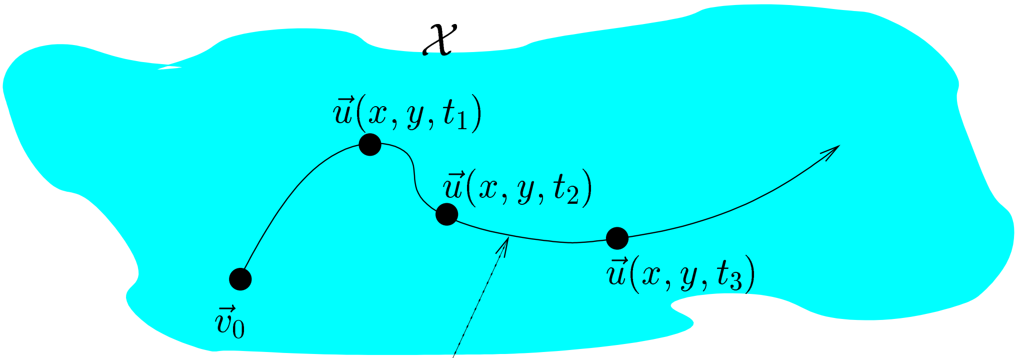

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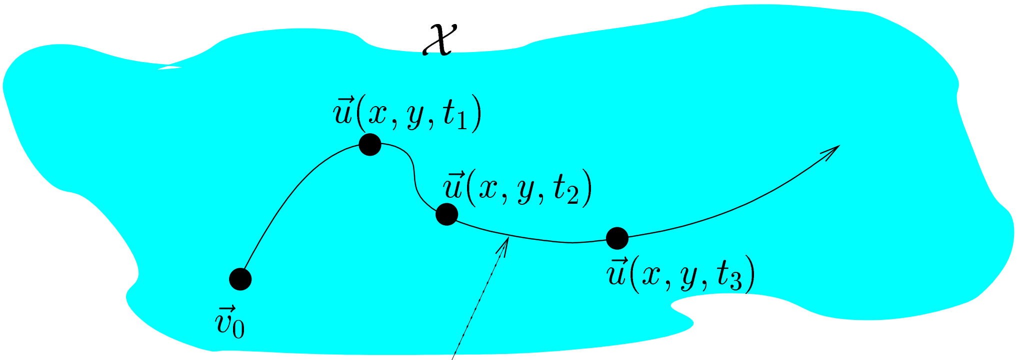

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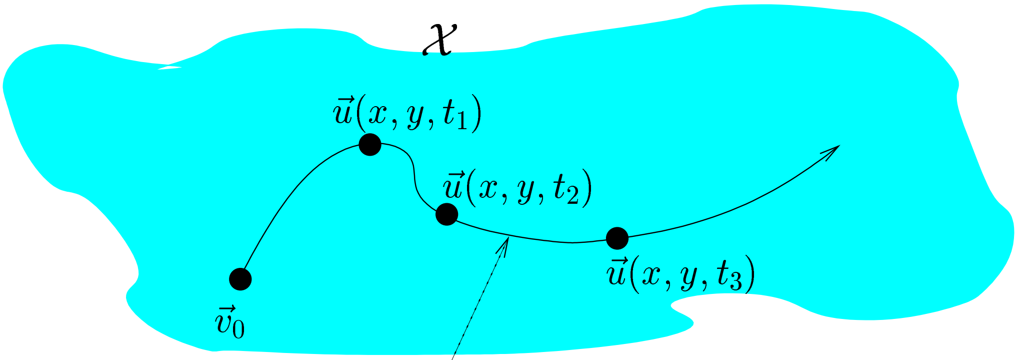

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$$\varphi(t_1 + t_2, \cdot) = \varphi(t_1, \varphi(t_2, \cdot)) \ \textit{(semi-flow property)}$$

This dynamical system has nice symmetry properties :

$$\mathbb{SE}(2) \equiv \text{ special Euclidean group action on } \mathbb{R}^2$$

$$(\gamma_{\theta,p_1,p_2}) \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\,\theta & -\sin\,\theta \\ \sin\,\theta & \cos\,\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}}_{rotation} + \underbrace{\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}}_{translation}$$

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Complex notation: $\quad z = x + iy,\ p = p_1 + ip_2$

$$\gamma_{\theta,p}\, z = e^{i\theta} z + p \qquad (\theta, p) \in \mathbb{S}^1 \times \mathbb{C}$$

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Group product: $(\theta_b, p_b) \cdot (\theta_a, p_a) = (\theta_a + \theta_b, p_b + e^{i\theta_b} p_a)$

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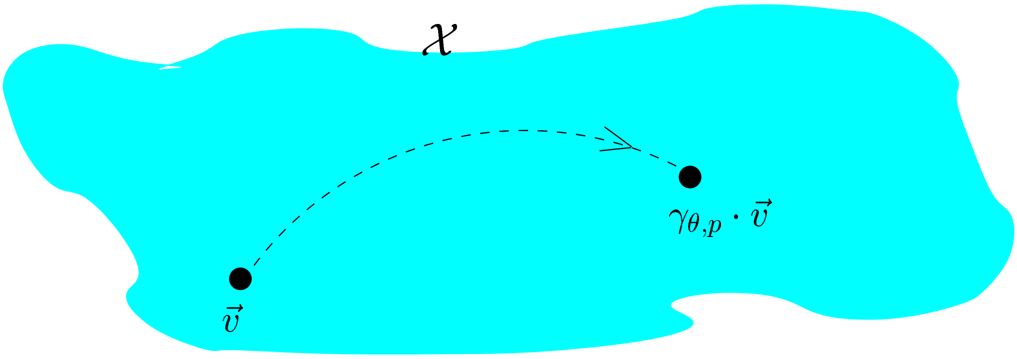
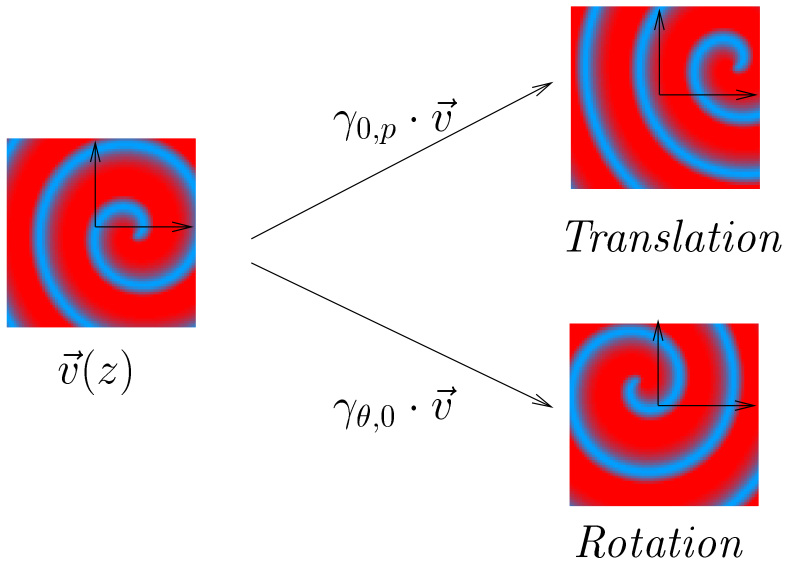
Induced $\mathbb{SE}(2)$ – action on $\mathcal{X}$



$$\boxed{(\gamma_{\theta,p} \cdot \vec{v})(z) \equiv \vec{v}(\gamma_{\theta,p}^{-1}\, z)}$$

Example of group action :



This dynamical system has nice symmetry properties :

$$\boxed{\varphi(t, \gamma \cdot \vec{v}) = \gamma \cdot \varphi(t, \vec{v}),\ \forall\, \gamma \in \mathbb{SE}(2),\ \forall\, \vec{v} \in \mathcal{X},\ \forall\, t > 0}$$

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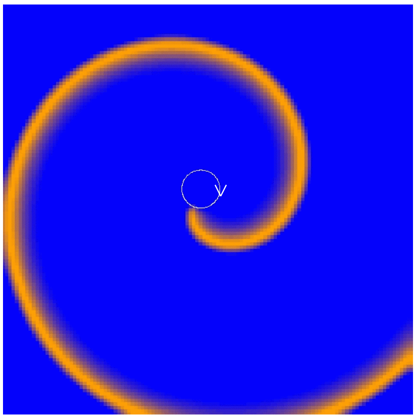
Consequence:

$$\boxed{\begin{array}{c} \text{If } \vec{u}(t, x, y) \text{ is a solution of (RD),} \\[2em] \text{then so is } \vec{u}_\gamma(t, x, y) \equiv \gamma \cdot \vec{u}(t, x, y),\ \forall\, \gamma \in \mathbb{SE}(2) \end{array}}$$

Relative equilibria

Rotating wave :



$$\vec{u}(t,z) = \gamma_{\omega\, t,0} \cdot \vec{u}^* = \vec{u}^*(e^{-i\omega\, t} z)$$

$$\text{for some } \vec{u}^* \in \mathcal{X}$$

Time evolution =

uniform rigid spatial rotation

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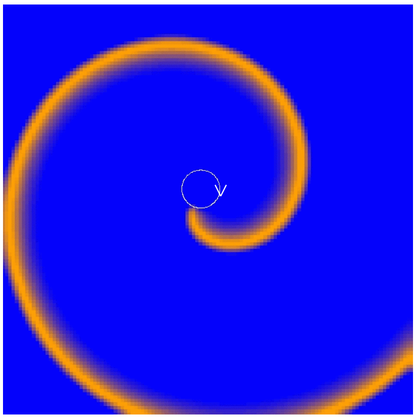

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$$\omega \left(y \frac{\partial \vec{u}^*}{\partial x}(x,y) - x \frac{\partial \vec{u}^*}{\partial y}(x,y) \right) = D \cdot \nabla^2 \vec{u}^*(x,y) + \mathcal{F}(\vec{u}^*(x,y))$$

Group orbit: $\mathcal{Y} \equiv \{\, \gamma \cdot \vec{u}^{*} \mid \gamma \in \mathbb{SE}(2) \,\} \subset \mathcal{X}$

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Dynamics of the semi-flow φ on $\mathcal{Y}$ are described by ODEs

$$\boxed{\begin{aligned} \dot{\theta} &= \omega \\ \dot{p} &= 0 \end{aligned}}$$

Describes motion of spiral "tip"

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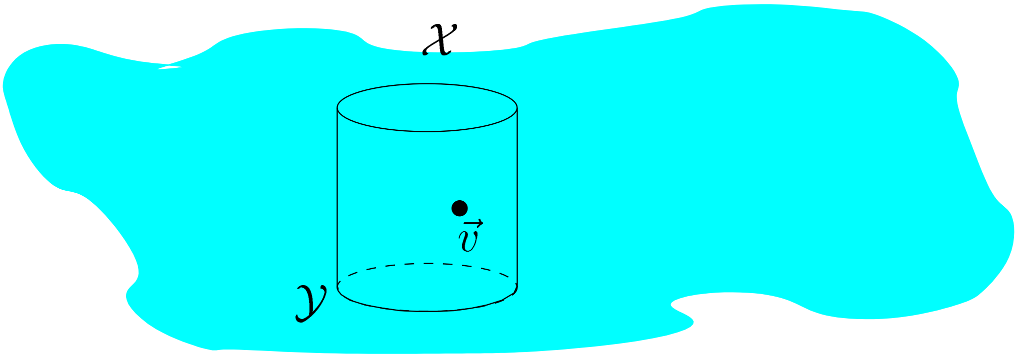
$$\theta(t) = \omega\, t + \omega_0, \;\; p(t) = p_0$$

Rotation about p_0 with frequency ω

Bifurcation to meandering

Linearized stability :



If you start (initial condition) on $\mathcal{Y}$ . . .

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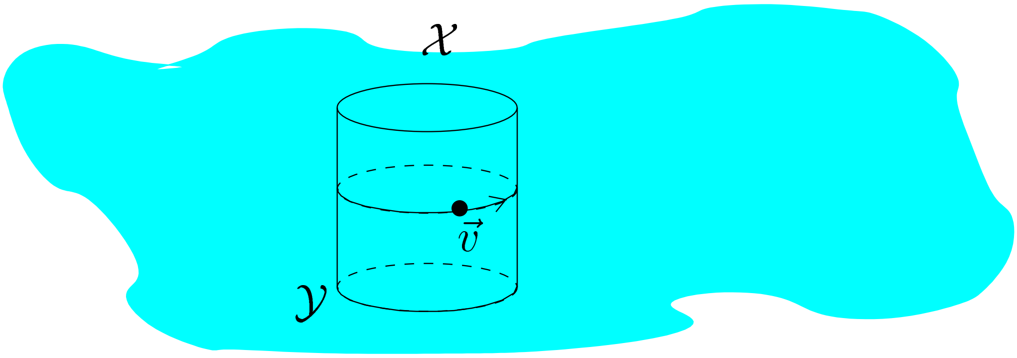
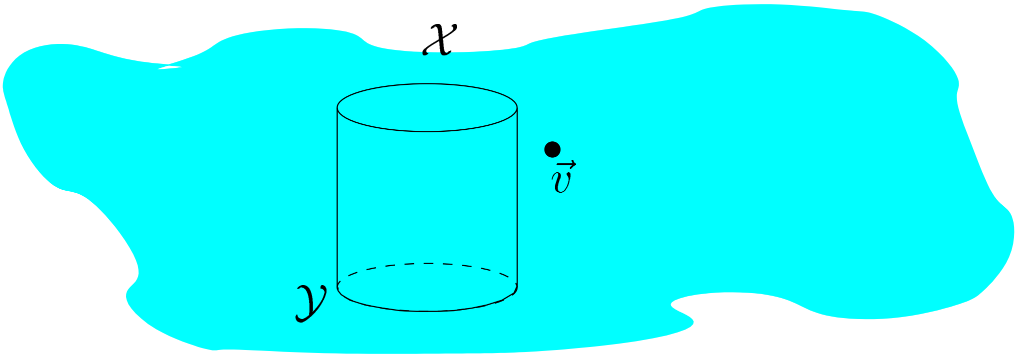


If you start (initial condition) on $\mathcal{Y}\ldots$

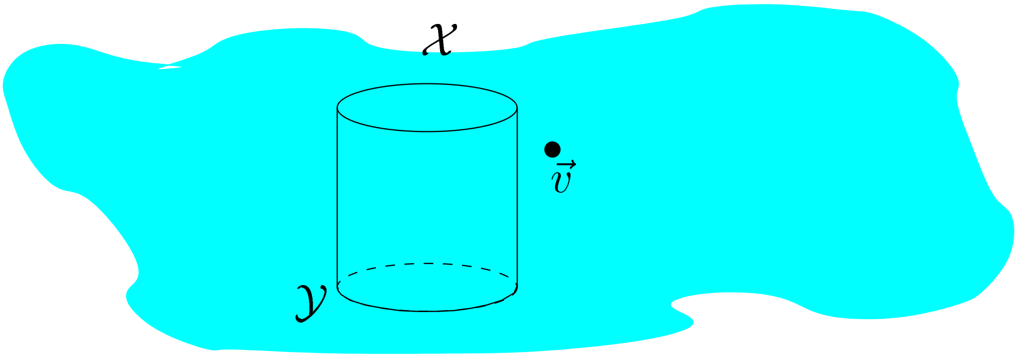
you stay on $\mathcal{Y}$ for all $t > 0$ ($\varphi(t, \mathcal{Y}) \subset \mathcal{Y}$)

Linearized stability :



What if the initial condition is off (but close to) $\mathcal{Y}$?

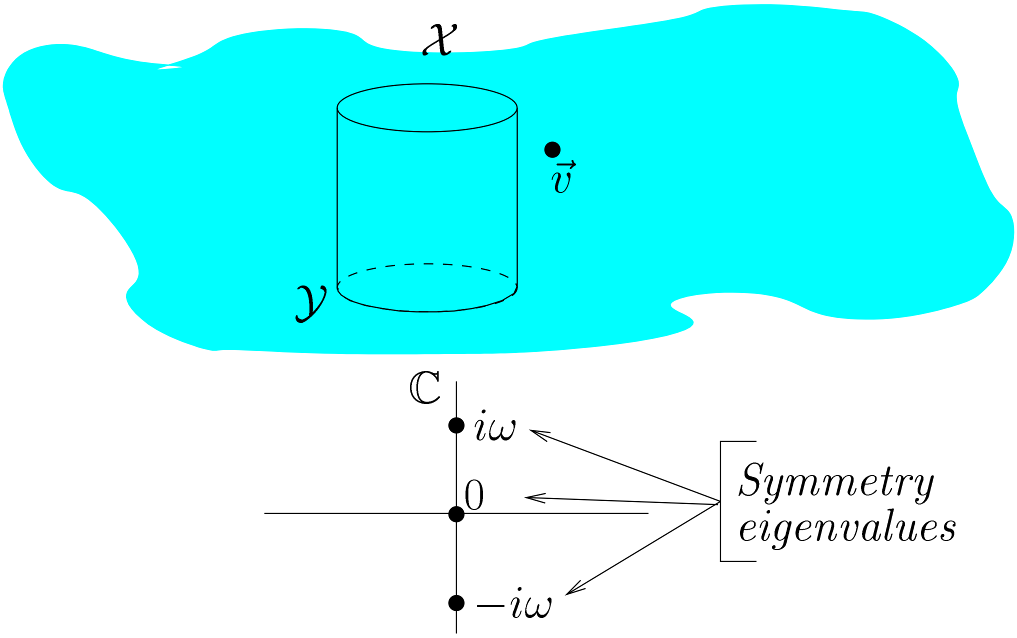
Linearized stability :



To answer this question, need to consider the eigenvalue problem $\mathcal{L}\vec{u} = \lambda\vec{u}$

$$\mathcal{L} = D \cdot \nabla^2 + d\mathcal{F}(\vec{u}^*) - \omega\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)$$

Linearized stability :



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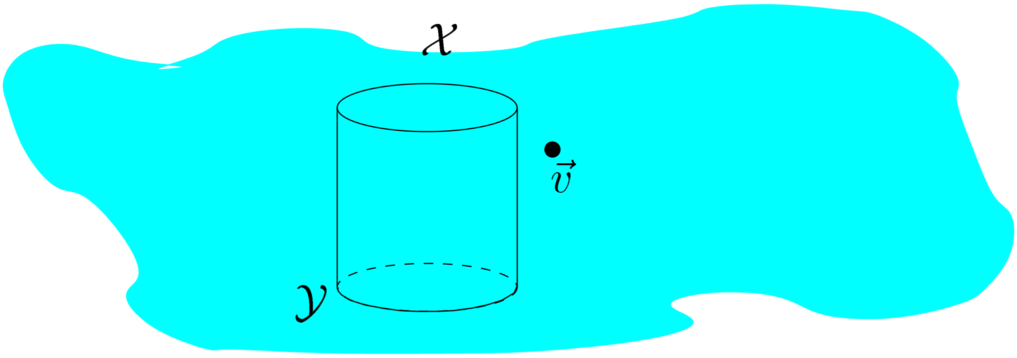

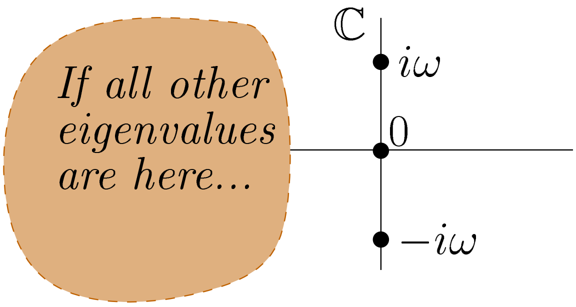

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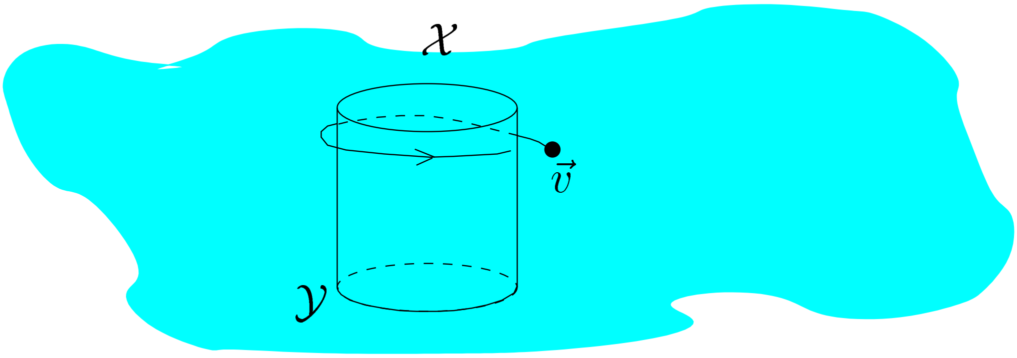
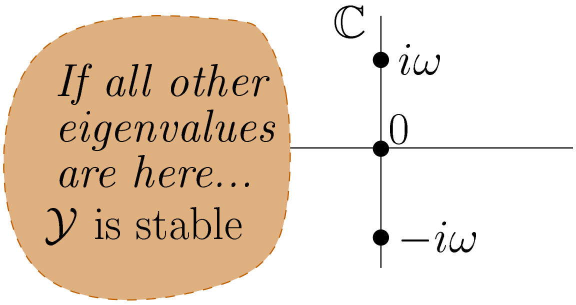
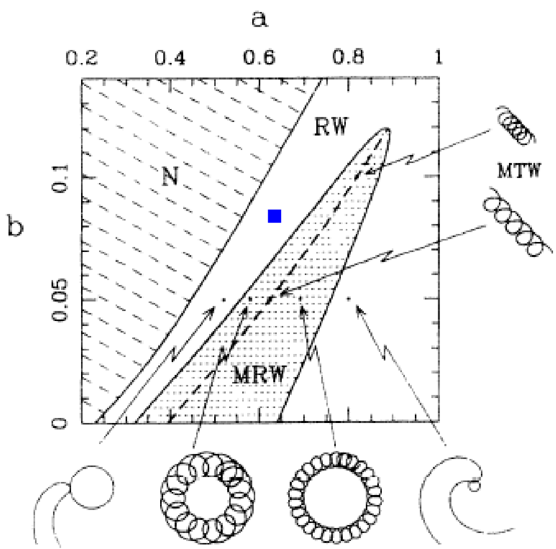
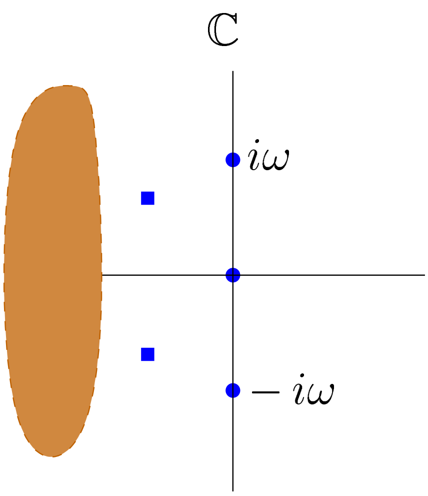




Transition to meandering : linearized stability





D. Barkley (1994) PRL

Numerical simulation of a model for cardiac electrophysiology

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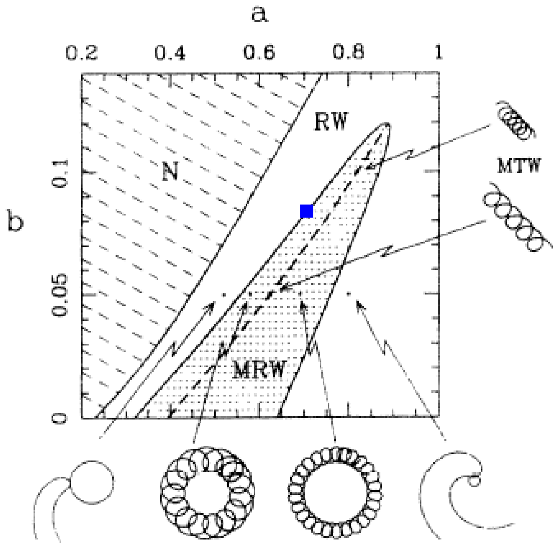

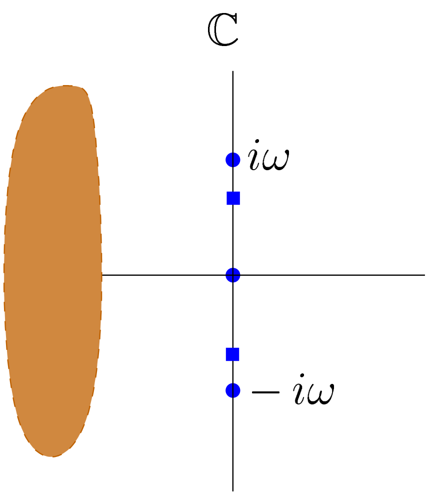

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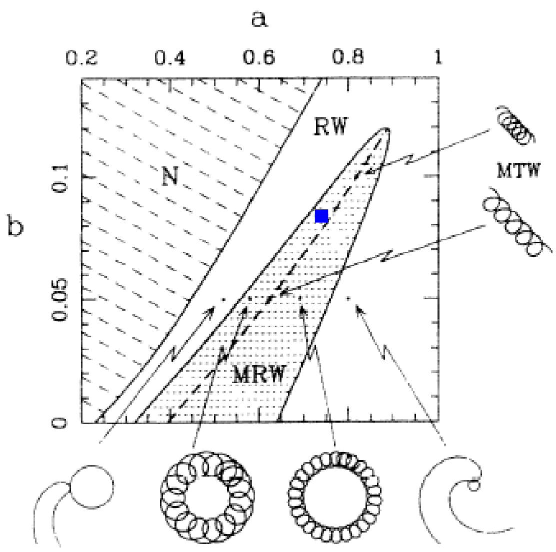

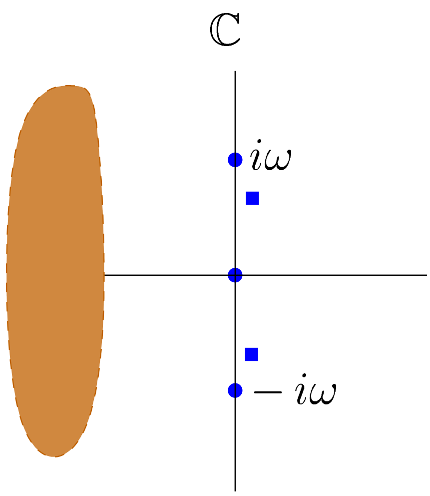

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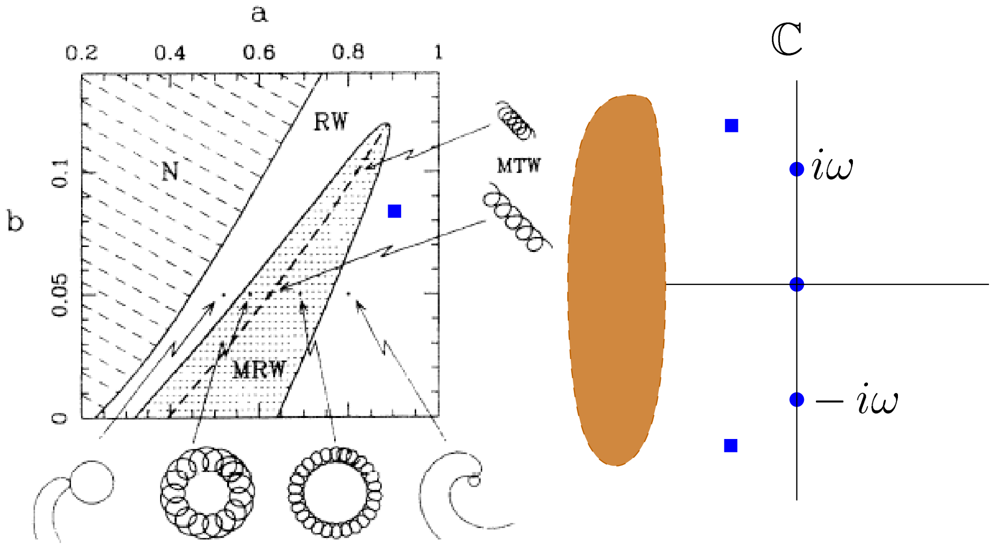

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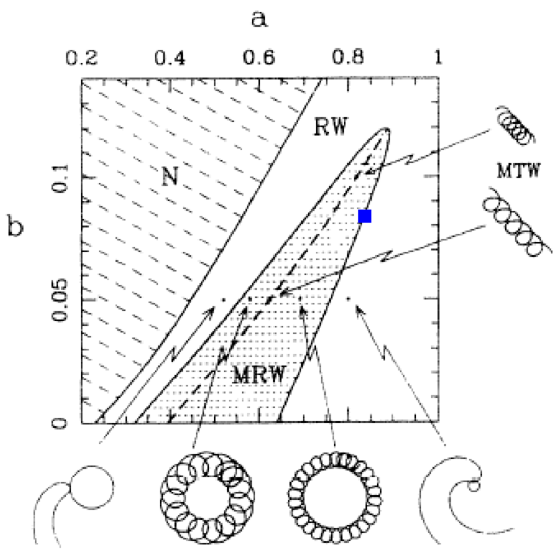

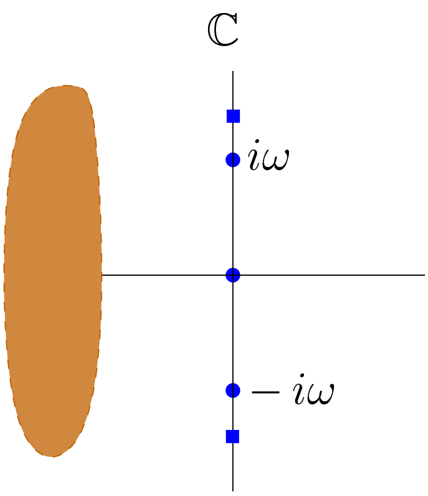

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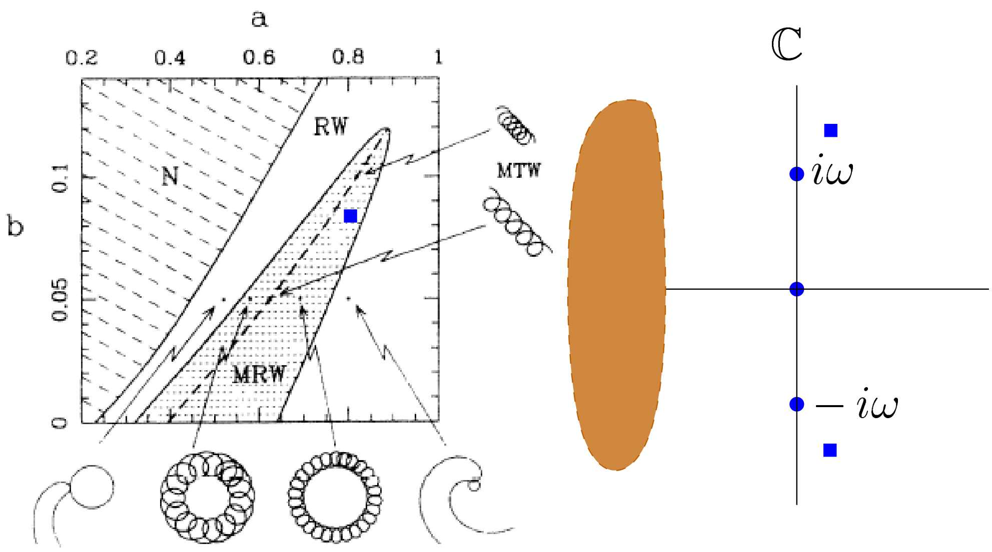

D. Barkley (1994) PRL

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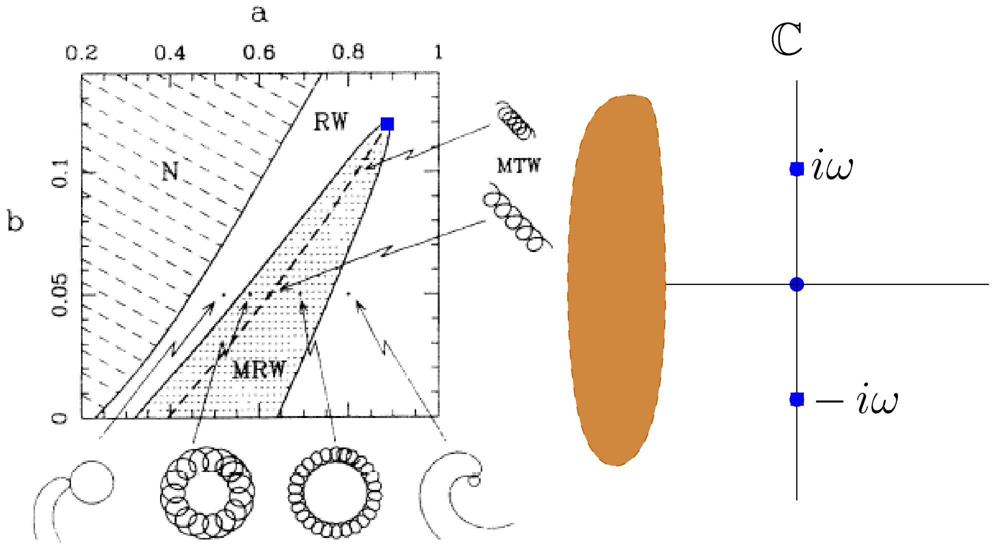

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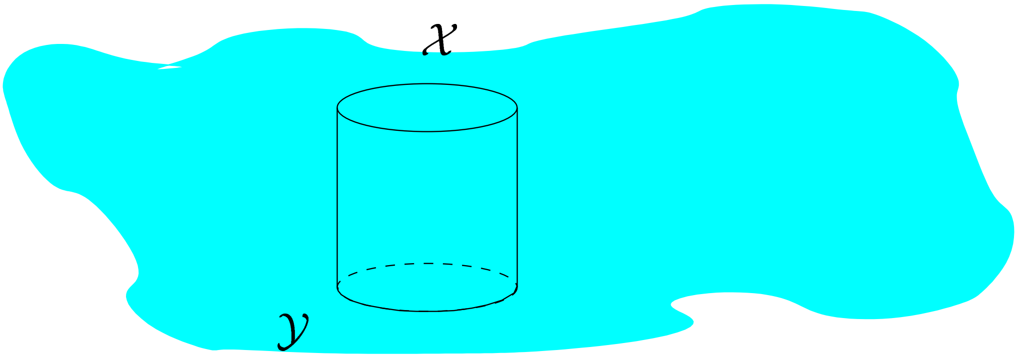
Hopf bifurcation from $\mathcal{Y}$ (Sandstede, Scheel, Wulff, JDE (1997))



If $\mathcal{L}$ has eigenvalues $0, \pm i\omega, \pm i\Omega$ and all other eigenvalues bounded away from imaginary axis in left-plane...

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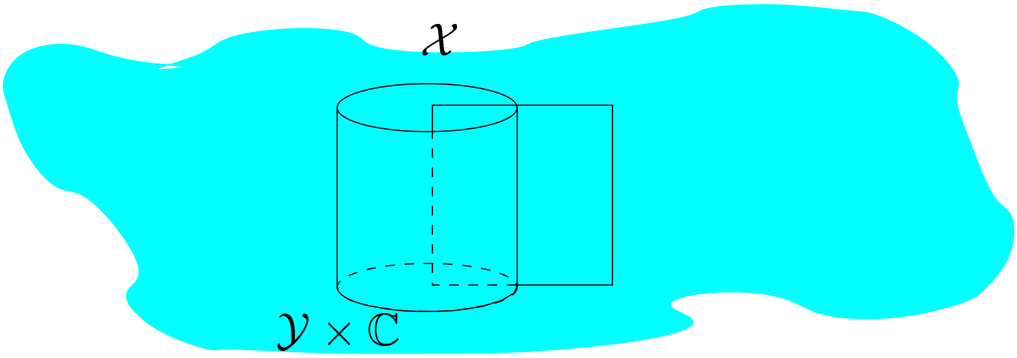

If $\mathcal{L}$ has eigenvalues $0, \pm i\omega, \pm i\Omega$ and all other eigenvalues bounded away from imaginary axis in left-plane...

$\exists$ invariant 5-d "center-bundle" $\mathcal{Y} \times \mathbb{C}$ which is stable,

and dynamics of φ on $\mathcal{Y} \times \mathbb{C}$ reduce to...

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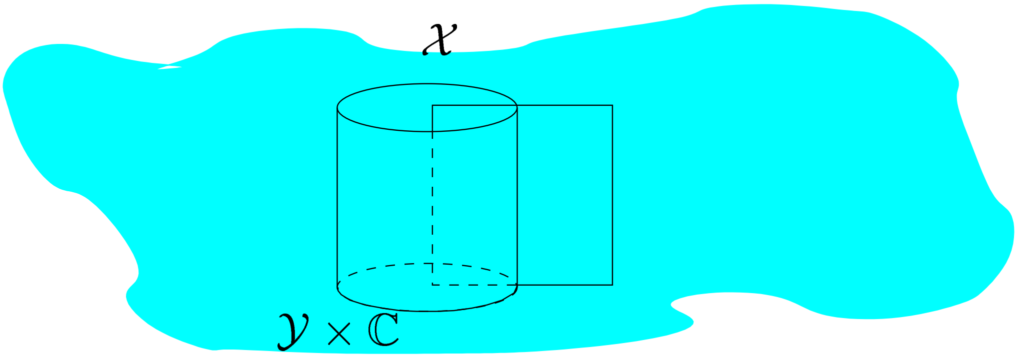

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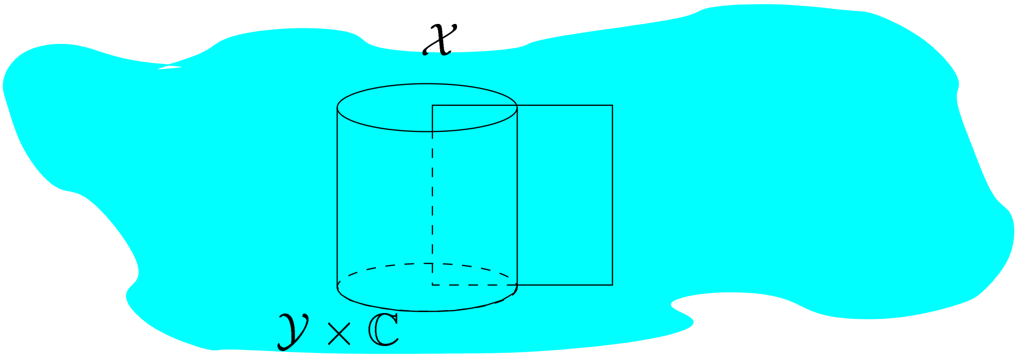

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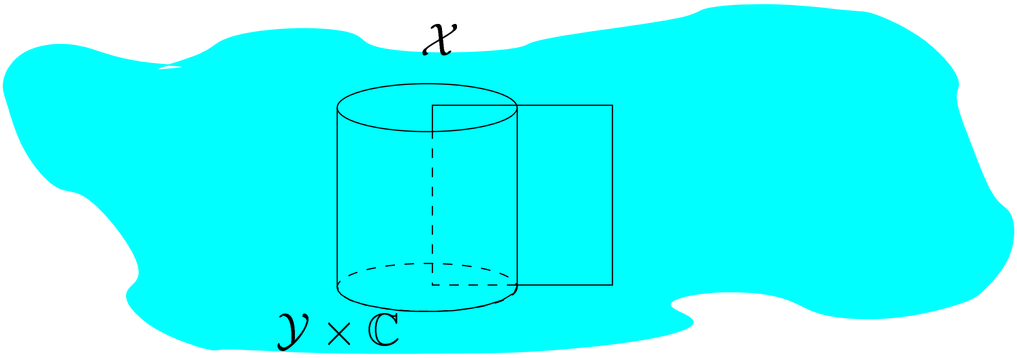

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Analysis of center−bundle ODEs

Golubitsky, LeBlanc & Melbourne, JNS (1997, 2000)

$$\boxed{\begin{aligned}\dot{\theta} &= \omega + F^{\theta}(q,\bar{q},\beta)\\ \dot{p} &= e^{i\theta}F^{p}(q,\bar{q},\beta)\\ \dot{q} &= F^{q}(q,\bar{q},\beta)\end{aligned}}$$

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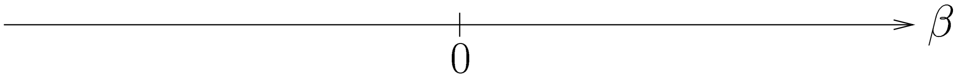

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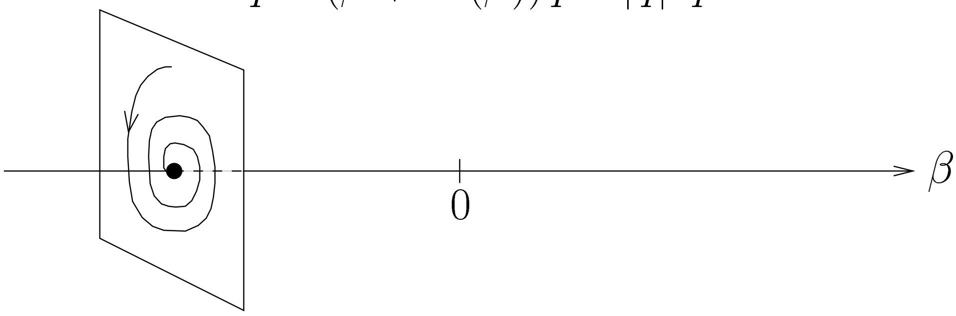

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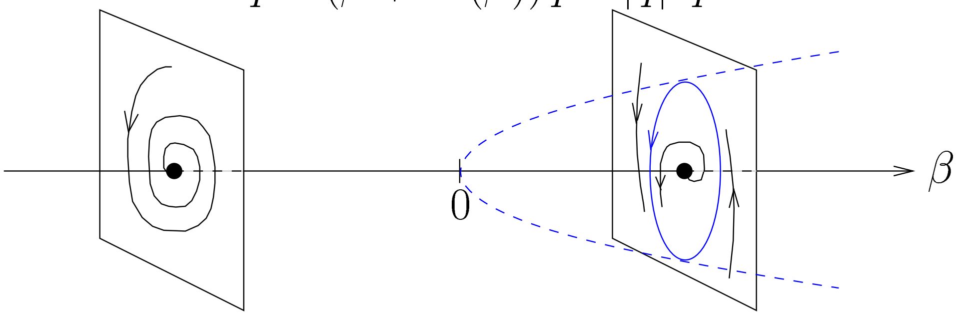

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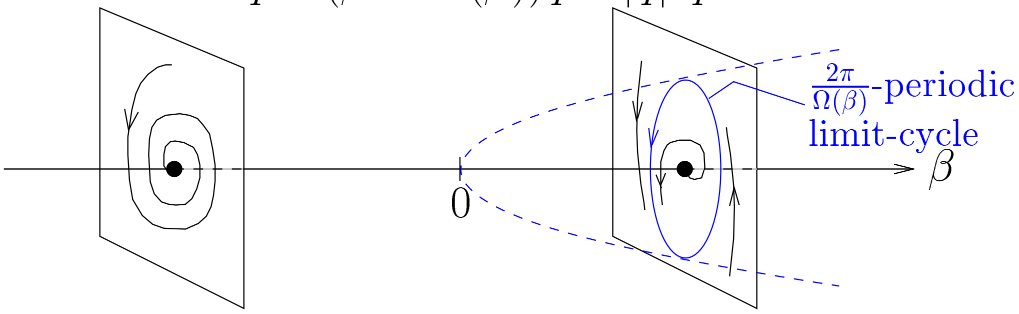
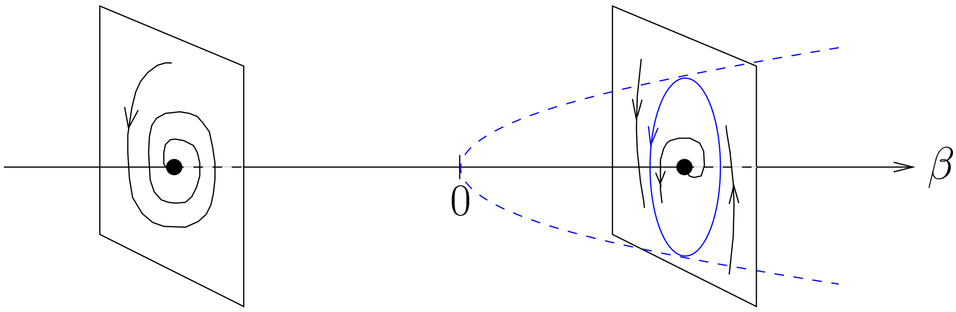


$$\begin{aligned} \dot{\theta} &= \omega + F^{\theta}(q(t), \bar{q}(t), \beta) \\ \dot{p} &= e^{i\theta(t)} F^{p}(q(t), \bar{q}(t), \beta) \end{aligned}$$



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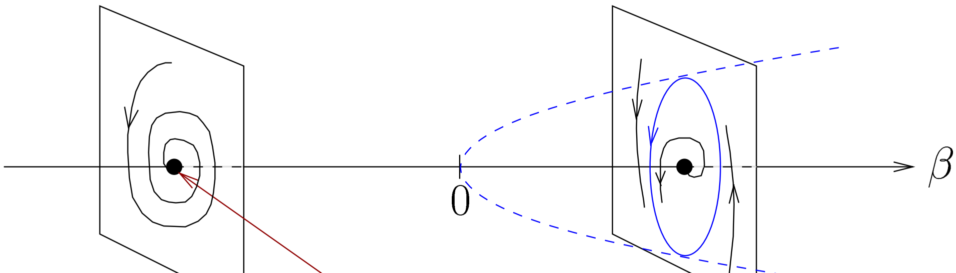

$q(t) = 0$

$\Rightarrow \dot{\theta} = \omega,\ \dot{p} = 0$

$\Rightarrow \theta(t) = \omega\, t + \theta_0,\ p(t) = p_0$

$\Rightarrow$ *rotating wave*

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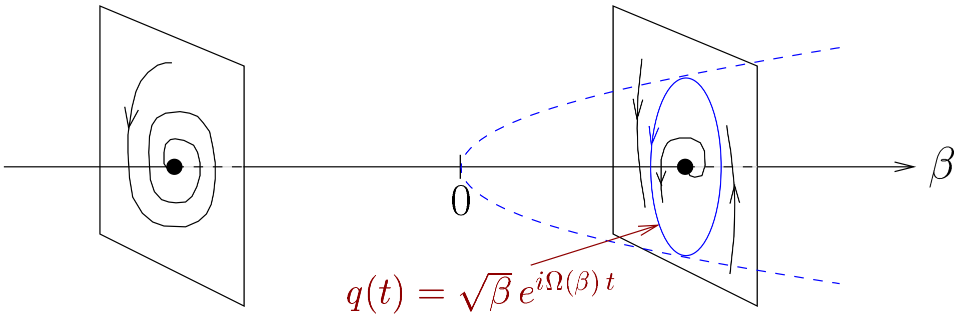

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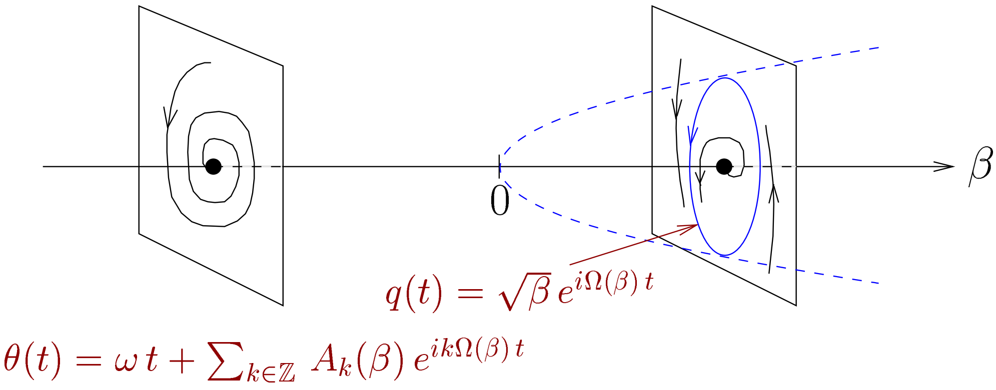

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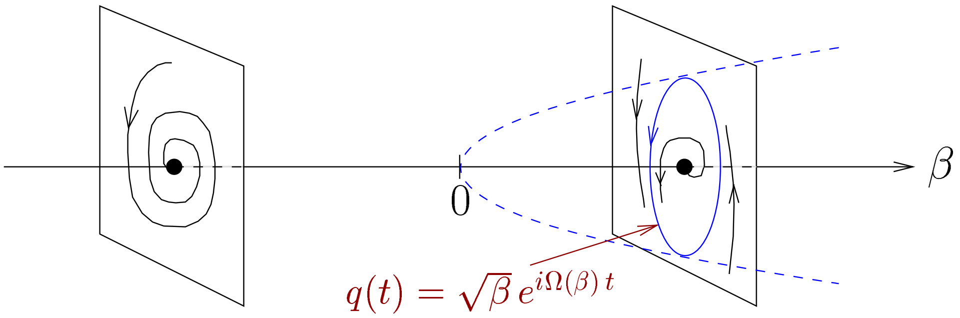

$\theta(t) = \omega\, t + \sum_{k\in\mathbb{Z}} A_k(\beta)\, e^{ik\Omega(\beta)\, t}$ $\quad \dot p(t) = e^{i\omega\, t} \sum_{k\in\mathbb{Z}} B_k(\beta)\, e^{ik\Omega(\beta)\, t}$

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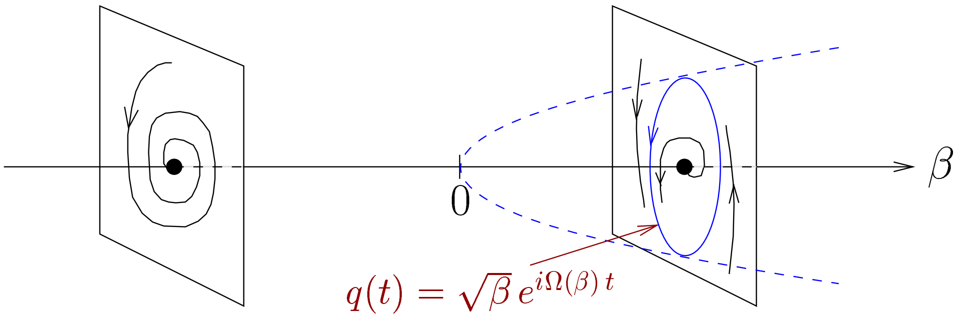

$$q(t) = \sqrt{\beta}\, e^{i\Omega(\beta)\, t}$$

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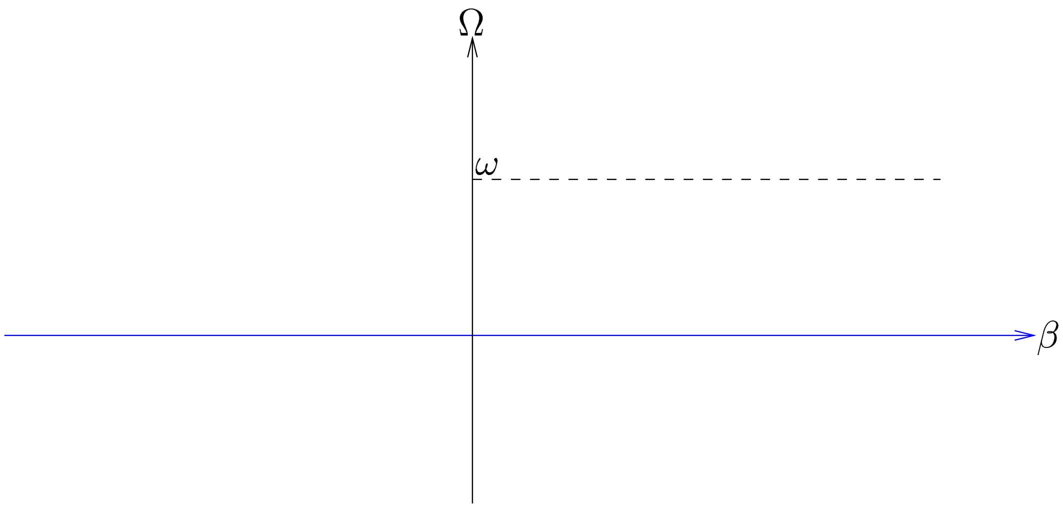
$$p(t) = \begin{cases} \displaystyle\sum_{k\in\mathbb{Z}} \frac{B_k(\beta)}{i(\omega + k\Omega(\beta))}\, e^{i(\omega+k\Omega(\beta))t} & \text{if } \ \omega/\Omega(\beta) \notin \mathbb{Z} \\ \displaystyle B_{-1}(\beta)t + \sum_{k\in\mathbb{Z}, k\neq -1} \frac{B_k(\beta)}{i(k+1)} e^{i(k+1)\omega\, t} & \text{if } \ \Omega(\beta) = \omega \end{cases}$$



Ω
ω
β

$$\theta(t) = \omega\, t + \theta_0$$

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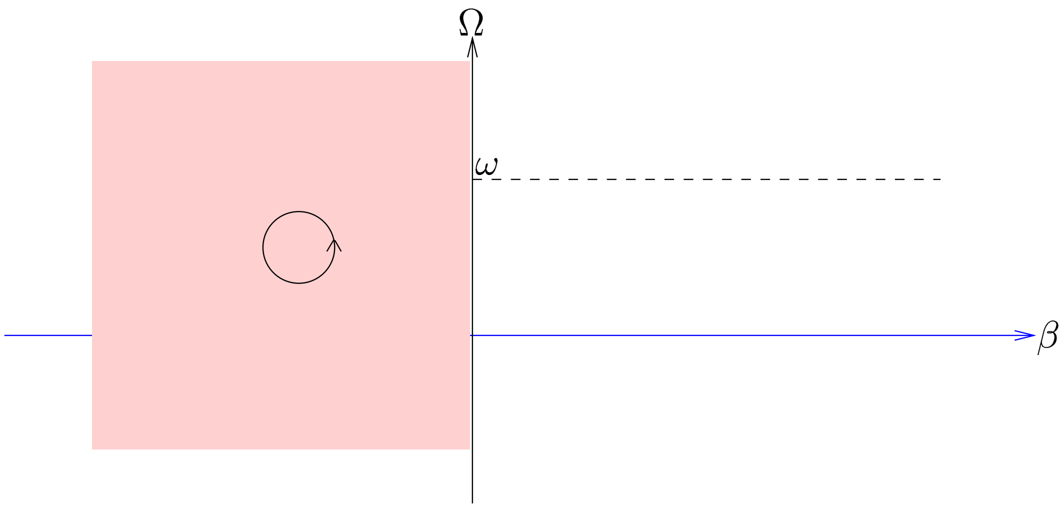

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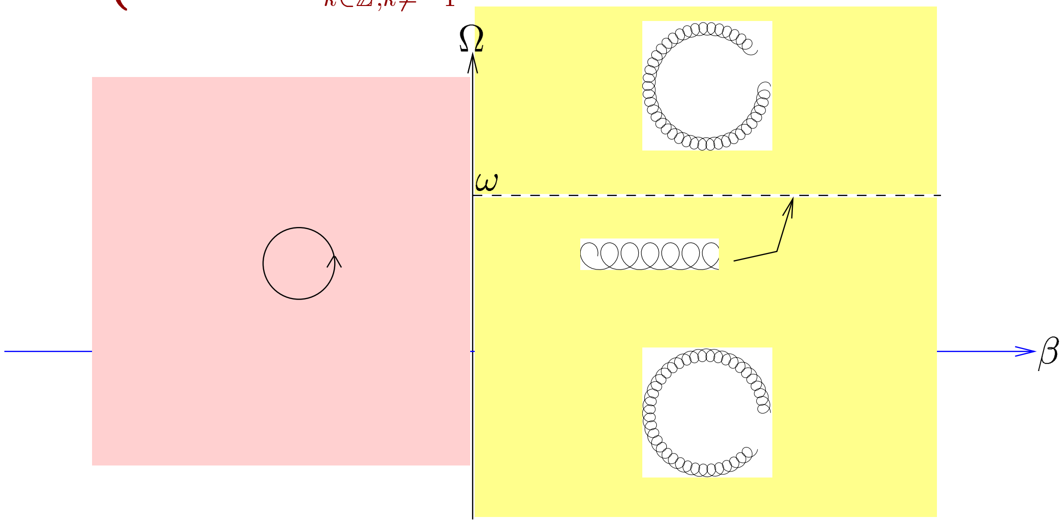

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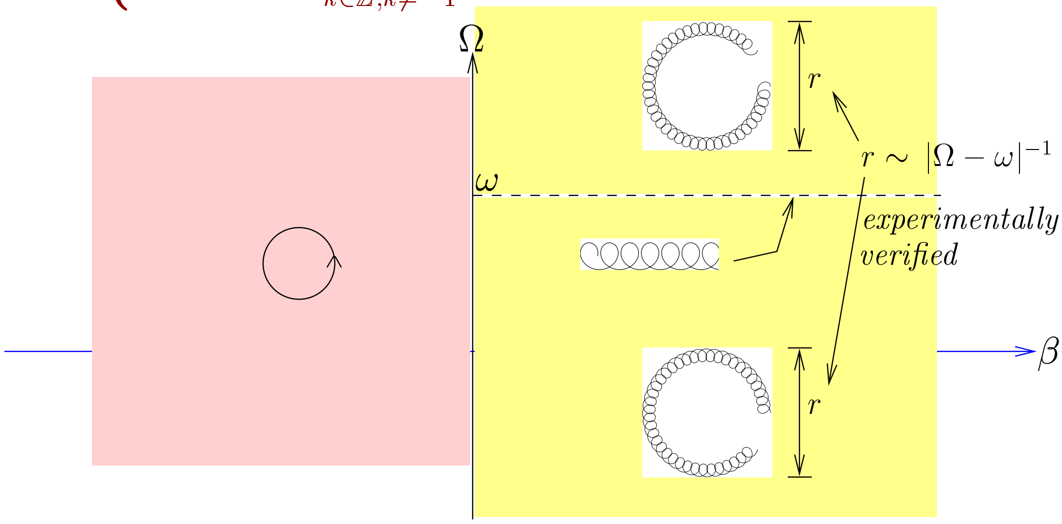
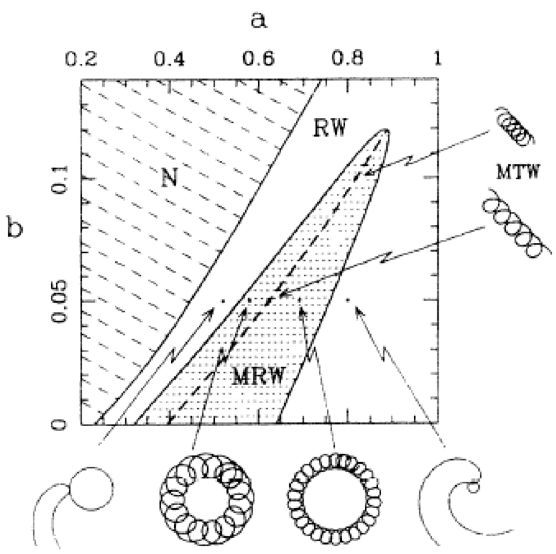


Transition to meandering



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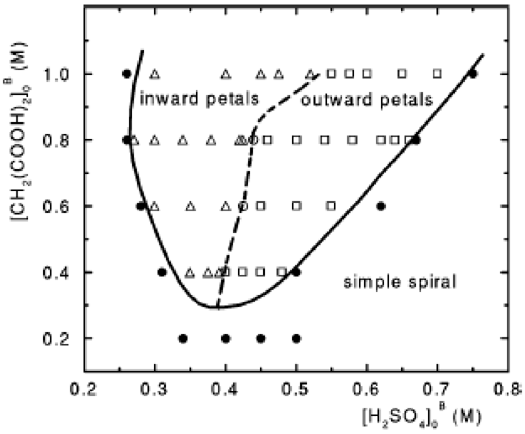

FIG. 4. Dynamics of spirals as a function of $[H_2SO_4]_0^B$ and $[CH_2(COOH)_2]_0^B$ (with other conditions as in Fig. 1). The solid line marks the transition from simple spirals (●) to meandering spirals with inward (△) and outward (□) petals. Traveling spirals (○) exist along the dashed line that separates the two types of meandering spirals.

Li, Ouyang, Petrov & Swinney (1996) PRL

Actual chemical reaction

Broken symmetry

In reality, Euclidean symmetry is NEVER exact :

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- **Boundaries and inhomogeneities break translational symmetry**
- **Anisotropy breaks rotational symmetry (especially relevant in cardiac tissue)**
- **Experiments confirm effects of symmetry breaking :**

Boundary drifting (Zykov & Muller)

Spiral anchoring (Munuzuri et al., Jalife et al.)

Phase locking / drifting of meandering waves in anisotropic tissue (Roth)

We can explain these experimentally observed phenomena using finite−dimensional center−bundle ODEs (forced symmetry−breaking) similarly to what was presented here

LeBlanc & Wulff, JNS (2000)
LeBlanc (2002)

and make some predictions which were verified experimentally

LeBlanc & Roth (2003)

Conclusions and ongoing work

- **Model−independent approach**

  characterize the fundamental dynamical properties
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- **Current and future work includes**

  combined forced symmetry−breaking
  spiral waves in spherical and quasi−spherical domains
  scroll waves in 3−d media

Ottawa−Carleton Institute

- **Applied Mathematics**
- **Logic and Foundations of Computing, Discrete Maths**
- **Algebra**
- **Analysis**
- **Stats and Probability**
- **Topology and Geometry**
- **Number Theory**

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Euclidean Symmetry and the Dynamics of Spiral Waves

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