Liquidity Risk and Arbitrage Pricing Theory

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X. Discrete approximations and liquidity risk

We have analyzed empirical results to determine the importance of liquidity to option pricing. Our findings are that liquidity risk is related to the moneyness of an option.

Out-of-money options have the lowest dollar denominated liquidity costs, but the greatest sensitivity to illiquidity in terms of percentage impact on price.

Feasible trading strategies are those which change only on a discrete time set. The *discrete trading strategies* are defined as the simple s.f.t.s. X_t where

$$\in \left\{ x_{t_0} 1_{\{t_0\}} + \sum_{j=0}^{N} x_{t_{j-1}} 1_{(t_{j-1}, t_j]} \;\middle|\; \begin{array}{l} \text{1. } x_t \text{ is } \mathcal{F}_t \text{ measurable} \\ \text{2. } t_j \text{ are } \mathcal{F}_t \text{ stopping times} \\ \text{3. } t_0 \equiv 0;\ t_j - t_{j-1} > \delta \\ \quad \text{for a fixed } \delta > 0. \end{array} \right.$$

Note that the minimum time between two successive trades is a fixed $\delta > 0$.

any discrete trading strategy, the liquidity cost is equal to

$$\sum_{j=-1}^{N} \left[x_{t_{j+1}} - x_{t_j}\right] \left[S(t_{j+1}, x_{t_{j+1}} - x_{t_j}) - S(t_{j+1}, 0)\right]$$

where $x_{t_{-1}} \equiv 0$. For a discrete trading strategy with $X_T = 0$, the hedging error is given by

$$\begin{aligned} C_T - Y_T = C_T \quad & (Y_0 + x_0 S(0, x_0) \\ & + \sum_{j=0}^{N} x_{t_j}[S(t_{j+1}, 0) - S(t_j, 0)]) + L_T \end{aligned}$$

Thus, there are two components to this hedging error. The first quantity

$$\left[Y_0 + x_0 S(0, x_0) + \sum_{j=0}^{N} x_{t_j}[S(t_{j+1}, 0) \quad S(t_j, 0)]\right]$$

is the error due to a discrete approximation of the Black Scholes hedging strategy and consequently denoted the *approximation error*. The second term results from the *liquidity cost* and is denoted L_T.

Estimation Results

Using our model we are lead to the regression equation:

$$\ln\left(\frac{S(t_2, x_{t_2})}{S(t_1, x_{t_1})}\right) = \alpha\left[x_{t_2} - x_{t_1}\right] + \mu(t_2 - t_1) + \sigma\epsilon_{t_2,t_1}$$

The error ϵ_{t_2,t_1} equals $\epsilon\sqrt{t_2 - t_1}$ with ϵ being distributed $\mathcal{N}(0, 1)$. Observe that the left side of the equation is the percentage return between two consecutive trades and this expression reduces to a standard geometric Brownian motion when α is identically zero.

For small α, a Taylor series expansion of the previous equation indicates the terms being summed

$$\left[x_{t_{j+1}} - x_{t_j}\right]\left[S(t_{j+1}, x_{t_{j+}} - x_{t_j}) - S(t_{j+}, 0)\right]$$

may be expressed as

$$\begin{aligned}(x_{t_{j+1}} - x_{t_j})S(t_{j+1}, 0)(\exp\{\alpha(x_{t_{j+1}} - x_{t_j})\} - 1)\\ \approx \alpha\, S(t_{j+1}, 0)\,(x_{t_{j+1}} - x_{t_j})^2\end{aligned}$$

We find preliminary evidence in Figure 2 that over long time periods, the α parameter and the stock price move inversely to each other. This empirical evidence suggests that market makers strive to obtain a constant dollar denominated fee per lot transacted.

Three strategies for discrete trading:

Binomial approximation

Black Scholes approximation with fixed time steps

Black Scholes approximation with random time steps

A common theme that emerges from Tables 2, 3, and 4 is the relevance of an option's moneyness to liquidity risk.

In-the-money options are subject to the lowest percentage impact from illiquidity, despite having the largest dollar denominated liquidity costs.

This large dollar denominated liquidity cost is partially attributed to the high initial cost of forming the replicating portfolio.

For **out-of-the-money options** with low initial prices, the impact of illiquidity is very significant despite a small dollar denominated liquidity cost.

At-the-money options lie between these extremes.

We found that liquidity costs are approximately the same across the three strategies, particularly for in-the-money options.

Therefore, the percentage impacts on option prices are quite stable across the five firms.

By implication, using trade volume or another proxy for liquidity is not appropriate when trying to ascertain the impact of illiquidity on option prices.

Of greater interest is the magnitude of the liquidity cost versus the approximation error. Despite the attention given to the approximation error in the previous literature, the economic importance of liquidity appears to be of greater significance.

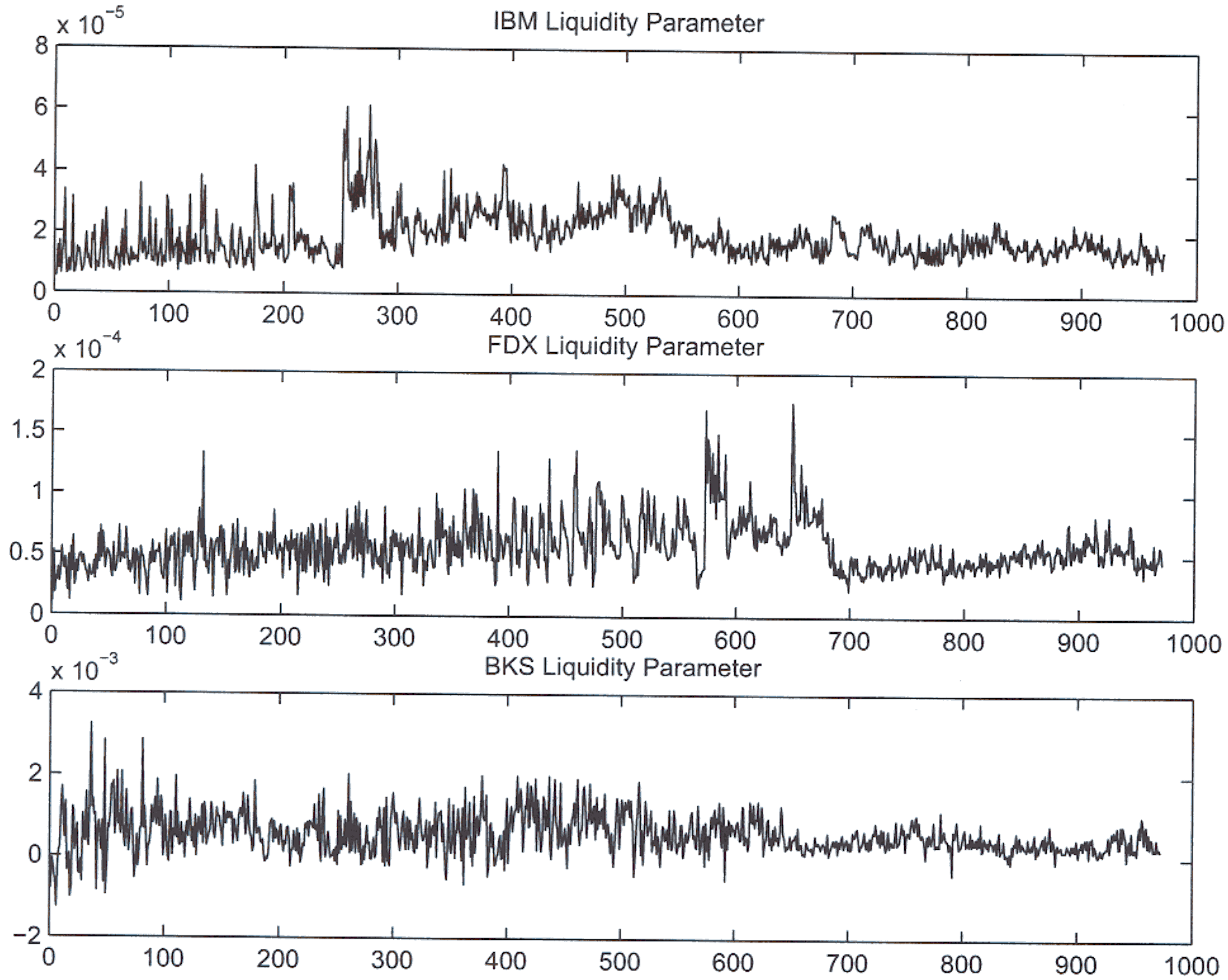

Figure 1: Plot of estimated α parameters each day of sample period from January 3, 1995 to December 31, 1998 based on equation (11) for IBM, Federal Express (FDX), and Barnes & Noble (BKS). Only days when the α estimates of all three companies are significant at the 5% level are plotted with details in Table 1. These three companies represent high, medium, and low liquidity firms with respect to NYSE stocks that have traded CBOE options.

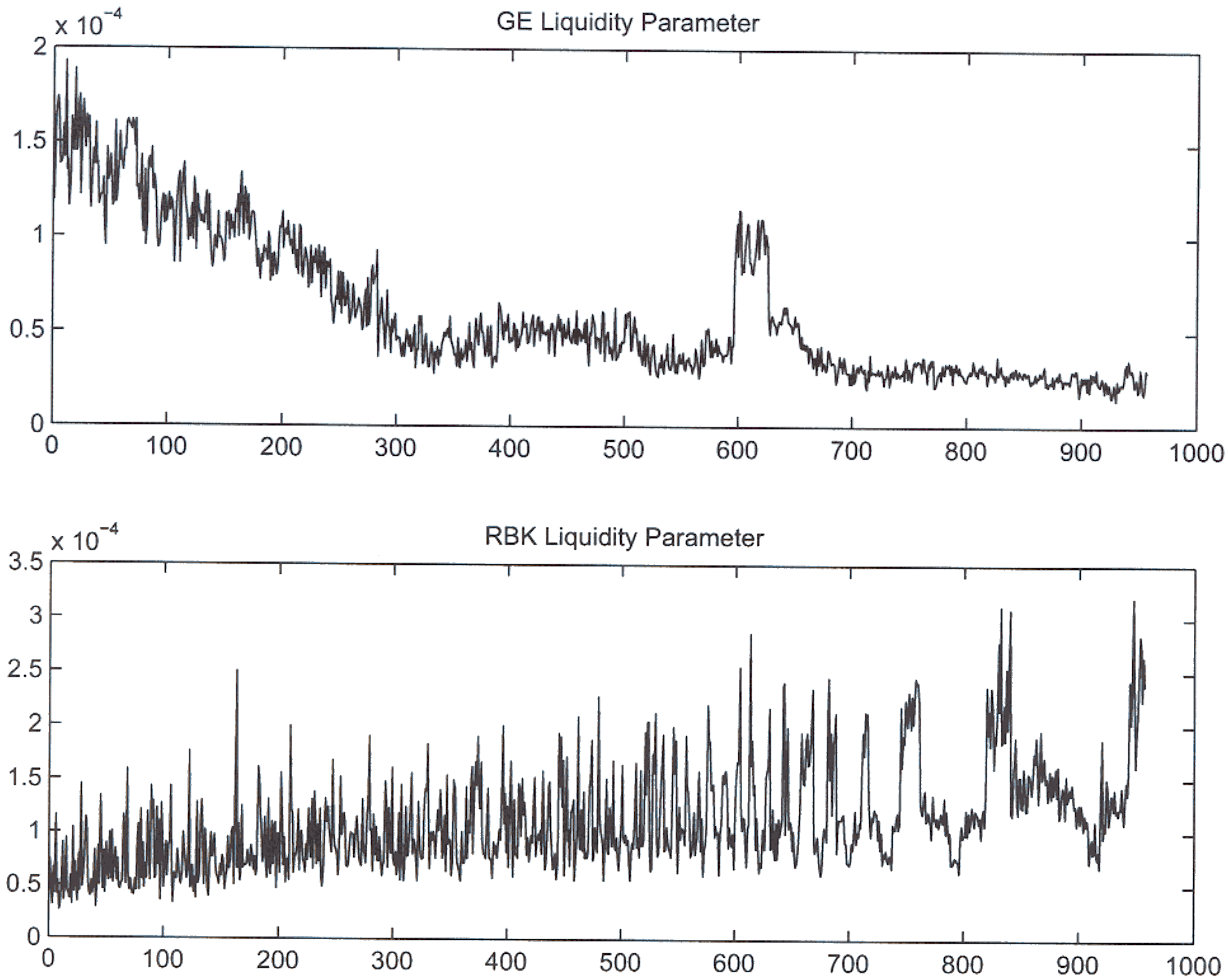

Figure 2: Plot of estimated α parameters each day of sample period from January 3, 1995 to December 31, 1998 based on equation (11) for GE and Reebok (RBK). Only days when the α estimates of both companies are significant at the 5% level are plotted with details in Table 1. The downward (upward) trend in the α estimates of GE and RBK may be partially attributed to the dramatic increase (less dramatic decrease) in their stock prices during the sample period. Overall, it appears that α and the stock price vary inversely, suggesting that market makers attempt to earn the same amount (in dollars) per transaction over time.

Table 1: Summary statistics for parameter estimates of α (in lots) and μ generated by regression model in equation (11) for each of the five firms. The second column records the average number of daily transactions used in the regression. A definite increase in the number of trades per day is observed for all five companies. Two columns detail the number of days for which the parameter estimates are significant at the 5% level relative to the 1,011 day sample period. The daily parameters estimates of α are almost always significant in contrast to the estimates of μ. A time series of daily α estimates for IBM, FDX, and BKS are plotted in Figure 1 with GE and RBK plotted in Figure 2. The last two columns record the average stock price and implied volatility of each firm during the sample period. These figures are the basis for subsequent tables. In particular, the stock price reported below equals the strike price for at-the-money options and is increased (decreased) by $5 for in-the-money (out-of-the-money) options.

Company Ticker	Mean n	Parameter	1st Percentile	Median	99th Percentile	Mean	Standard Deviation	5% Level Days	5% Level %	Stock Price	Volatility (annual)
GE	1,060	$\hat{\alpha} \times 10^{-4}$	0.20	0.44	1.63	0.59	0.38	1,011	100	$75	23.25%
		$\hat{\mu} \times 10^{-7}$	-33.96	-3.62	13.03	-4.34	9.08	24	2.37		
IBM	656	$\hat{\alpha} \times 10^{-4}$	0.06	0.17	0.43	0.19	0.08	998	98.71	$115	30.93%
		$\hat{\mu} \times 10^{-7}$	-26.22	-2.81	13.76	-3.71	7.93	31	3.07		
FDX	248	$\hat{\alpha} \times 10^{-4}$	0.15	0.53	1.32	0.56	0.21	971	96.04	$65	28.41%
		$\hat{\mu} \times 10^{-7}$	-13.12	-0.61	10.90	-0.66	4.83	50	4.95		
BKS	168	$\hat{\alpha} \times 10^{-4}$	0.34	1.18	2.84	1.28	0.50	962	95.15	$35	37.39%
		$\hat{\mu} \times 10^{-7}$	-30.46	-0.73	14.58	-1.41	7.56	46	4.55		
RBK	158	$\hat{\alpha} \times 10^{-4}$	0.11	0.99	2.50	1.08	5.00	957	94.66	$35	36.22%
		$\hat{\mu} \times 10^{-7}$	-23.26	-2.70	14.24	-3.09	7.01	25	2.47		

Table 2: Summary of Strategy 1. Strategy 1 consists of the binomial trading strategy. Two hedging frequencies are considered, daily and every two days. The liquidity cost is determined by the firm specific α parameter and the change in the option's delta between the hedging dates. Movements in the option portfolio's delta are a function of the underlying stock's volatility and price, as well as the option's time-to-maturity. The column which denotes "Liquidity & Rounding" extends the analysis by rounding off the transactions associated with rebalancing the hedge portfolio to the nearest integer valued lot size. For at-the-money options, the strike price equals the initial stock price recorded in Table 1. The stock price is then increased (decreased) by \$5 for in-the-money (out-of-the-money) options. The implied volatility of the option is also found in Table 1 while the riskfree interest rate is set to 5%.

Company Name	Option Moneyness	Hedging Frequency	Binomial Price	Liquidity Cost	Total Price	Percentage Increase	Liquidity & Rounding	Rounding Error	Percentage Increase
GE	In	$\delta = 2$ days	547.57	38.39	585.96	7.01	38.26	-0.63	7.00
		$\delta = 1$ day	547.23	38.65	585.87	7.06	38.54	-0.11	7.05
	At	$\delta = 2$ days	204.15	20.41	224.56	10.00	20.22	-0.19	9.97
		$\delta = 1$ day	199.11	22.71	221.83	11.41	22.58	-0.13	11.42
	Out	$\delta = 2$ days	39.63	5.31	44.94	13.40	5.24	-0.07	13.50
		$\delta = 1$ day	38.34	5.91	44.26	15.42	5.89	-0.02	15.74
IBM	In	$\delta = 2$ days	717.82	16.27	734.09	2.27	16.31	0.04	2.27
		$\delta = 1$ day	714.34	16.52	730.86	2.31	16.58	0.06	2.32
	At	$\delta = 2$ days	413.51	10.08	423.59	2.44	9.98	-0.10	2.41
		$\delta = 1$ day	403.31	11.22	414.53	2.78	11.15	-0.07	2.76
	Out	$\delta = 2$ days	202.08	4.82	206.90	2.39	4.82	0.00	2.39
		$\delta = 1$ day	199.00	5.10	204.10	2.56	5.06	-0.04	2.54
FDX	In	$\delta = 2$ days	555.45	31.64	587.09	5.70	31.64	0.00	5.71
		$\delta = 1$ day	556.04	31.53	587.57	5.67	31.74	0.21	5.72
	At	$\delta = 2$ days	216.18	16.90	233.08	7.82	17.03	0.13	7.93
		$\delta = 1$ day	210.85	18.79	229.64	8.91	18.93	0.14	9.04
	Out	$\delta = 2$ days	45.75	4.86	50.61	10.62	4.80	-0.06	10.68
		$\delta = 1$ day	45.18	5.12	50.30	11.33	5.10	-0.02	11.52
BKS	In	$\delta = 2$ days	520.16	45.89	566.04	8.82	46.17	0.28	8.89
		$\delta = 1$ day	521.17	45.46	566.63	8.72	45.21	-0.25	8.68
	At	$\delta = 2$ days	153.16	21.04	174.20	13.74	20.90	-0.14	13.74
		$\delta = 1$ day	149.39	23.36	172.74	15.63	23.30	-0.06	15.70
	Out	$\delta = 2$ days	11.78	2.63	14.41	22.35	2.56	-0.07	22.23
		$\delta = 1$ day	11.64	2.91	14.55	25.02	2.88	-0.03	25.70
RBK	In	$\delta = 2$ days	517.17	39.41	556.58	7.62	39.81	0.40	7.71
		$\delta = 1$ day	518.94	38.60	557.54	7.44	38.70	0.10	7.46
	At	$\delta = 2$ days	148.37	17.72	166.10	11.95	17.63	-0.09	11.97
		$\delta = 1$ day	144.72	19.68	164.40	13.60	19.66	-0.02	13.68
	Out	$\delta = 2$ days	10.49	1.95	12.44	18.60	1.88	-0.07	18.43
		$\delta = 1$ day	9.95	2.34	12.28	23.49	2.26	-0.08	23.25

Table 3: Summary of Strategy 2. As detailed in Subsection 5.2, strategy 2 consists of hedging the options at predetermined timepoints. Since a geometric Brownian motion, rather than a binomial price process is assumed, strategy 2 facilitates a study of the approximation error. The liquidity cost and approximation error are recorded below along with their percentage impact on the option price. For at-the-money options, the strike price equals the initial stock price recorded in Table 1. The stock price is then increased (decreased) by $5 for in-the-money (out-of-the-money) options. The implied volatility of the option is also found in Table 1 while the riskfree interest rate is set to 5%.

Company Name	Option Moneyness	Hedging Frequency	Black Scholes	Liquidity Mean	Approximation Mean	Mean % Increase	Liquidity Std. Dev.	Approximation Std. Dev.	Std. Dev of Increase (%)
GE	In	$\delta = 2$ days	546.56	37.96	4.18	7.71	5.24	32.83	5.87
		$\delta = 1$ day	546.56	38.46	2.11	7.42	5.78	23.92	4.32
	At	$\delta = 2$ days	200.78	20.22	6.75	13.43	5.38	44.95	21.70
		$\delta = 1$ day	200.78	20.84	3.47	12.11	6.00	31.99	15.57
	Out	$\delta = 2$ days	38.39	5.36	3.69	23.56	5.26	31.46	79.46
		$\delta = 1$ day	38.39	5.72	1.94	19.95	5.61	22.47	57.90
IBM	In	$\delta = 2$ days	715.01	14.71	13.31	3.92	2.70	85.29	11.84
		$\delta = 1$ day	715.01	15.15	6.07	2.97	3.08	61.55	8.53
	At	$\delta = 2$ days	409.50	10.05	12.83	5.59	2.65	90.51	21.91
		$\delta = 1$ day	409.50	10.48	6.95	4.26	2.98	66.20	16.02
	Out	$\delta = 2$ days	199.20	5.72	11.55	8.67	2.78	82.72	41.13
		$\delta = 1$ day	199.20	6.11	6.30	6.23	3.03	60.49	30.11
FDX	In	$\delta = 2$ days	555.16	30.93	4.84	6.44	4.42	35.62	6.28
		$\delta = 1$ day	555.16	31.39	2.19	6.05	4.84	25.80	4.59
	At	$\delta = 2$ days	212.61	16.69	7.13	11.20	4.47	47.27	21.65
		$\delta = 1$ day	212.61	17.26	3.60	9.81	4.88	33.66	15.56
	Out	$\delta = 2$ days	44.64	4.77	3.82	19.25	4.41	34.47	75.03
		$\delta = 1$ day	44.64	5.02	2.40	16.64	4.68	24.06	53.25
BKS	In	$\delta = 2$ days	520.96	44.92	2.53	9.11	4.58	20.34	3.83
		$\delta = 1$ day	520.96	45.31	0.91	8.87	5.19	14.46	2.78
	At	$\delta = 2$ days	150.64	20.84	4.86	17.06	5.47	33.78	21.58
		$\delta = 1$ day	150.64	21.60	2.39	15.93	6.12	24.20	15.74
	Out	$\delta = 2$ days	12.04	2.58	1.73	35.78	4.18	16.98	136.69
		$\delta = 1$ day	12.04	2.81	1.10	32.42	4.52	12.08	102.31
RBK	In	$\delta = 2$ days	518.68	38.33	2.13	7.80	3.89	19.32	3.63
		$\delta = 1$ day	518.68	38.59	1.08	7.65	4.22	13.73	2.64
	At	$\delta = 2$ days	145.93	17.59	4.87	15.39	4.71	32.96	21.76
		$\delta = 1$ day	145.93	18.18	2.72	14.32	5.18	23.25	15.63
	Out	$\delta = 2$ days	10.52	1.92	1.79	35.32	3.32	14.93	139.60
		$\delta = 1$ day	10.52	2.14	0.87	28.61	3.67	11.17	106.58

Table 4: Summary of Strategy 3. As detailed in Subsection 5.3, strategy 3 differs from strategy 2 as trading only occurs when the delta of the option's portfolio has changed by a minimum fixed amount, not according to fixed timepoints. Two thresholds are considered for initiating a transaction, 5 lots and 1 lot (or θ equal to 0.05 and 0.01 respectively). As in the previous table, the associated liquidity cost and approximation error, as well as their joint influence on the option's price, are examined. For at-the-money options, the strike price equals the initial stock price recorded in Table 1. The stock price is then increased (decreased) by $5 for in-the-money (out-of-the-money) options. The implied volatility of the option is also found in Table 1 while the riskfree interest rate is set to 5%.

Company Name	Option Moneyness	Hedging Frequency	Black Scholes	Liquidity Mean	Approximation Mean	Mean % Increase	Liquidity Std. Dev.	Approximation Std. Dev.	Std. Dev of Increase (%)
GE	In	$\theta = 5$ lots	546.56	39.50	0.07	7.24	7.59	11.55	2.62
		$\theta = 1$ lot	546.56	39.50	0.00	7.23	7.68	1.92	1.44
	At	$\theta = 5$ lots	200.78	22.52	0.13	11.28	8.42	11.60	7.26
		$\theta = 1$ lot	200.78	22.65	0.04	11.30	8.55	2.57	4.43
	Out	$\theta = 5$ lots	38.39	6.51	0.16	17.37	7.34	10.08	33.08
		$\theta = 1$ lot	38.39	6.64	0.02	17.34	7.55	1.83	20.24
IBM	In	$\theta = 5$ lots	715.01	15.77	0.25	2.24	4.14	27.69	3.93
		$\theta = 1$ lot	715.01	15.88	0.41	2.28	4.11	11.06	1.63
	At	$\theta = 5$ lots	409.50	11.13	0.03	2.72	4.07	27.24	6.71
		$\theta = 1$ lot	409.50	10.94	0.19	2.72	3.84	11.43	2.90
	Out	$\theta = 5$ lots	199.20	6.67	0.95	3.82	3.94	25.79	13.15
		$\theta = 1$ lot	199.20	6.75	0.04	3.41	4.08	10.43	5.54
FDX	In	$\theta = 5$ lots	555.16	32.16	-0.09	5.78	6.09	13.33	2.69
		$\theta = 1$ lot	555.16	32.17	0.14	5.82	6.27	4.67	1.40
	At	$\theta = 5$ lots	212.61	18.32	-0.15	8.54	6.27	13.74	7.11
		$\theta = 1$ lot	212.61	18.49	0.28	8.83	6.72	5.86	4.18
	Out	$\theta = 5$ lots	44.64	5.88	0.64	14.60	5.85	11.66	29.81
		$\theta = 1$ lot	44.64	5.69	0.17	13.13	5.81	4.28	15.92
BKS	In	$\theta = 5$ lots	520.96	45.83	-0.28	8.74	6.21	13.76	2.97
		$\theta = 1$ lot	520.96	45.80	0.06	8.80	6.48	2.61	1.34
	At	$\theta = 5$ lots	150.64	23.06	0.02	15.32	8.29	9.98	8.80
		$\theta = 1$ lot	150.64	23.04	0.05	15.33	8.06	4.16	5.90
	Out	$\theta = 5$ lots	12.04	3.16	-0.07	25.70	5.73	8.59	86.84
		$\theta = 1$ lot	12.04	3.44	0.03	28.88	6.01	2.25	52.94
RBK	In	$\theta = 5$ lots	518.68	38.85	-0.10	7.47	5.15	12.15	2.59
		$\theta = 1$ lot	518.68	39.21	0.07	7.57	5.46	2.49	1.16
	At	$\theta = 5$ lots	145.93	19.16	0.08	13.18	6.61	9.76	8.17
		$\theta = 1$ lot	145.93	19.44	0.14	13.42	6.76	4.03	5.19
	Out	$\theta = 5$ lots	10.52	2.49	0.38	27.24	4.60	7.08	79.83
		$\theta = 1$ lot	10.52	2.53	0.02	24.25	4.57	2.09	47.47