Valuing a CDO and nth to Default CDS

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Road Map

- We first consider a portfolio of bonds (or loans) where the principals and recovery rates are equal
- We calculate the probability distribution of the number of defaults) by time T
- We convert this to a probability distribution for the time of the n^{th} loss
- We use this to
  – Value n^{th} to default CDSs
  – Value tranches of CDOs (cash or synthetic)

Background

- Consider a portfolio of N bonds, each having the same principal
- The time to default of the i^{th} bond is t_i
- The cumulative probability distribution of t_i is Q_i
- The probability that the i^{th} bond will survive beyond time t is $S_i(t) = 1 - Q_i(t)$

Copula Default Correlation Model

(General One-factor Case)

- We model the default correlation in terms of variables x_i where

$$x_i = a_i M + \sqrt{1 - a_i^2} Z_i$$

The M and the Z_i have independent (zero mean, unit variance) distributions with the Z_i being identically distributed

- The simplest version of the model is where M and the Z_i are normal

The Correlation Model continued

- The a_i's define a one-factor correlation structure between the x_i's.
- We assume that x_i is mapped to the default time of the i^{th} loan on a "fractile-to-fractile" basis
- This means that if F_i is the unconditional cumulative distribution of x_i and we define a relationship between t and x so that $Q_i(t) = F_i(x)$ then

$$\text{Prob}(t_i < t) = \text{Prob}(x_i < x)$$

Why Use a Copula Model?

- It creates a tractable multivariate distribution while preserving the unconditional probability distributions of the t_i
- In the one-factor copula model it is natural to set the a_i*'s* equal to the correlation ρ_i of the stock price of the issuer of bond i with a well diversified market index.
- A calibration approach: Set a_i equal to $\lambda(t)\rho_i$ and choose $\lambda(t)$ so that the market prices of correlation-dependent products are matched as closely as possible at each time t

Conditional Distributions

- Define H as the cumulative probability density of the Z_i

$$\text{Prob}(x_i < x \mid M) = H\left[\frac{x - a_i M}{\sqrt{1 - a_i^2}}\right]$$

- When $x = F_i^{-1}[Q_i(t)]$, $\text{Prob}(t_i < t) = \text{Prob}(x_i < x)$. Hence

$$\text{Prob}(t_i < t \mid M) = H\left[\frac{F_i^{-1}[Q_i(t)] - a_i M}{\sqrt{1 - a_i^2}}\right]$$

- The conditional probability that the i^{th} bond will survive beyond time t is

$$S_i(t \mid M) = 1 - H\left\{\frac{F_i^{-1}[Q_i(t)] - a_i M}{\sqrt{1 - a_i^2}}\right\}$$

Default Probability Distribution Conditional on *M*

Define probability of k defaults by time T as $\pi_T(k)$

$$\pi_T(0\,|\,M) = \prod_{i=1}^{n} S_i(T\,|\,M)$$

$$\pi_T(1\,|\,M) = \pi_T(0\,|\,M)\sum_{i=1}^{n}\frac{1-S_i(T\,|\,M)}{S_i(T\,|\,M)}$$

Default Probability Distribution Conditional on M continued

Define:

$$w_i = \frac{1 - S_i(k \mid M)}{S_i(k \mid M)}$$

In general

$$\pi_T(k \mid M) = \pi_T(0 \mid M) \sum w_{p(1)} w_{p(2)} \ldots w_{p(k)}$$

where $\{p(1), p(2), \ldots p(k)\}$ is a set of k numbers from $\{1, 2, \ldots .N\}$ and the summation is taken over all the

$$\frac{N!}{(N-k)!k!}$$

different sets that can be chosen

Key Result (Newton-Girard)

Define

$$P_k = \sum w_{p(1)} w_{p(2)} \ldots . w_{p(k)} \quad \text{and} \quad S_k = \sum_{i=1}^{n} w_i^k$$

Then

$$P_1 = S_1$$

$$2P_2 = S_1 P_1 - S_2$$

$$3P_3 = S_1 P_2 - S_2 P_1 + S_3$$

$$\vdots$$

$$kP_k = S_1 P_{k-1} - S_2 P_{k-2} + S_3 P_{k-3} - \ldots (-1)^{k+1} S_k$$

Unconditional Default Probabilities

$$\pi_T(k) = \int_{-\infty}^{\infty} \pi_T(k \mid M)\, g(M)\, dM$$

where g is the probability density function for M

This can be calculated numerically very quickly

Extensions

- We can have several factors so that

$$x_i = a_{i1}M_1 + a_{i2}M_2 + \ldots a_{im}M_m + Z_i\sqrt{1 - a_{i1}^2 - a_{i2}^2 - \ldots - a_{im}^2}$$

- The math is similar except that we now have to integrate over m independent identical distributions to get the unconditional $\pi_T(k)$
- It is natural to choose M and the Z_i to be normally distributed but we do not have to do this. There are many variations on the standard Gaussian copula model

Probability Distribution of Time of n^{th} Default

- Define $P(n,T)$ as the probability of at least n defaults by time T

$$P(n,T) = \sum_{k=n}^{N} \pi_T(k) = 1 - \sum_{i=0}^{n-1} \pi_T(k)$$

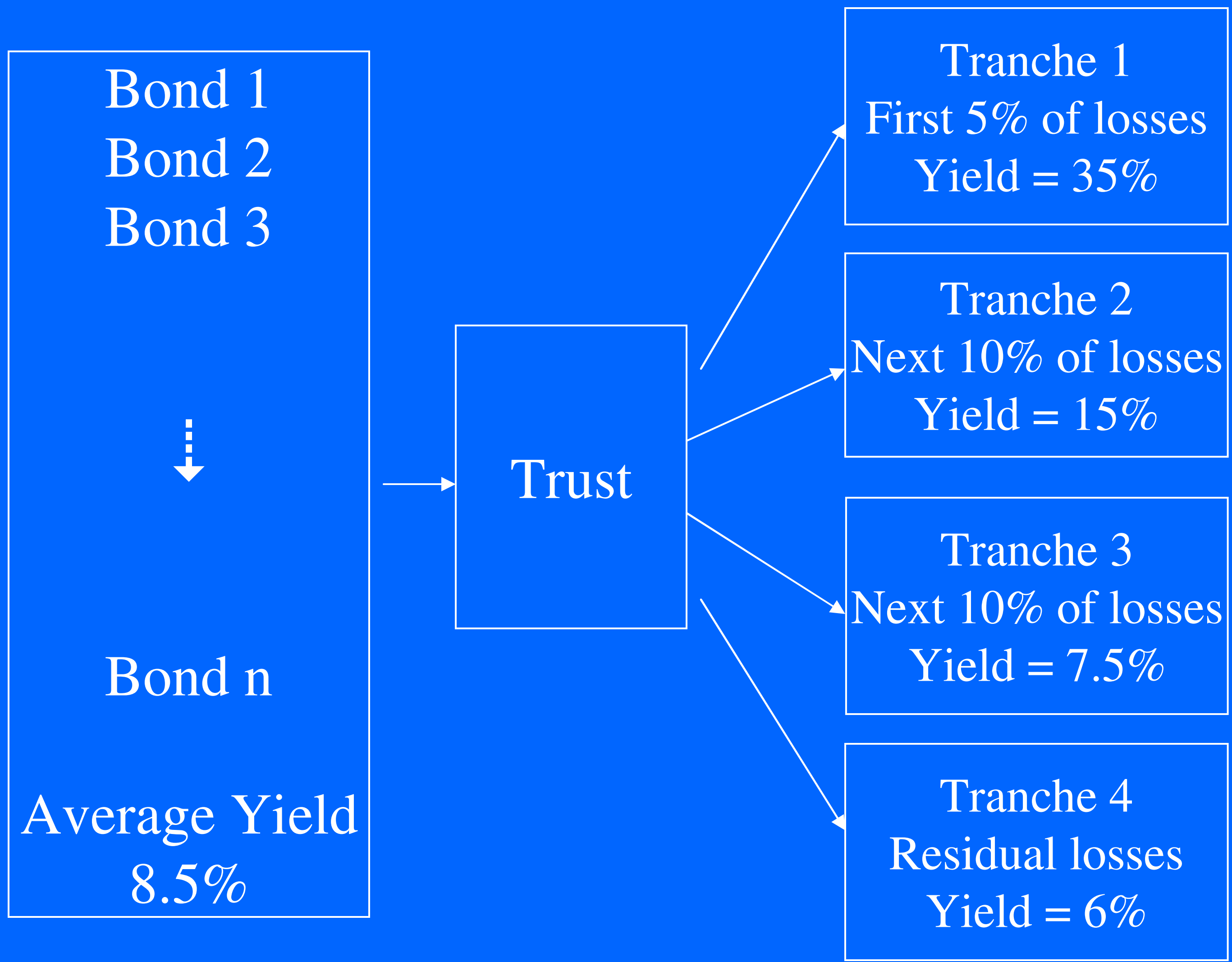
- The probability that the n^{th} default will happen between times T_1 and T_2 is

$$P(n,T_2) - P(n,T_1)$$

$$P(k,T) = \sum_{i=k}^{n} \pi_T(i) = 1$$

- Once this has been calculated an n^{th} to default CDS can be valued similarly to a regular CDS.

Cash CDO


Bond 1
Bond 2
Bond 3
Bond n
Average Yield
8.5%
Trust
Tranche 1
First 5% of losses
Yield = 35%
Tranche 2
Next 10% of losses
Yield = 15%
Tranche 3
Next 10% of losses
Yield = 7.5%
Tranche 4
Residual losses
Yield = 6%

Types of CDOs

- Cash CDO: Rules for determining how losses from a portfolio of bonds are distributed to tranches
- Synthetic CDO: Rules for determining how losses from portfolio of CDSs are distributed to tranches
- Percent of defaults CDO: Tranche responsible for between X_1% and X_2% of defaults
- Percent of losses CDO: Tranche responsible for between X_1% and X_2% of losses

In all cases promised yield is earned on remaining principal

Simplest Assumption

- All bonds/reference entities have same principal
- All recovery rates are the same
- This allows the losses from a tranche to be characterized as the payoffs on a portfolio of n^{th} to default CDSs
- The expected income to the tranche on a payment date, τ, can be calculated from the $\pi_\tau(k)$

Example

- Suppose that there are 100 bonds in a portfolio each with a principal of $1 million, the recovery rate is 40%, and a tranche is responsible for losses between 5% and 15%.
- The tranche is responsible for 66.67% of the 9th loss, and all losses from the 10th loss to the 25th loss, inclusive

Example continued

- Suppose there is income to the tranche holders of r percent at time τ
- The expected income is

$$10r[1-P(\tau,9)]+(10-0.6667\times0.6)r\pi_{\tau}(9)$$
$$+(10-1.6667\times0.6)r\pi_{\tau}(10)+\ldots+(10-15.6667\times0.6)r\pi_{\tau}(24)$$

Cash vs Synthetic Percent of Losses CDOs

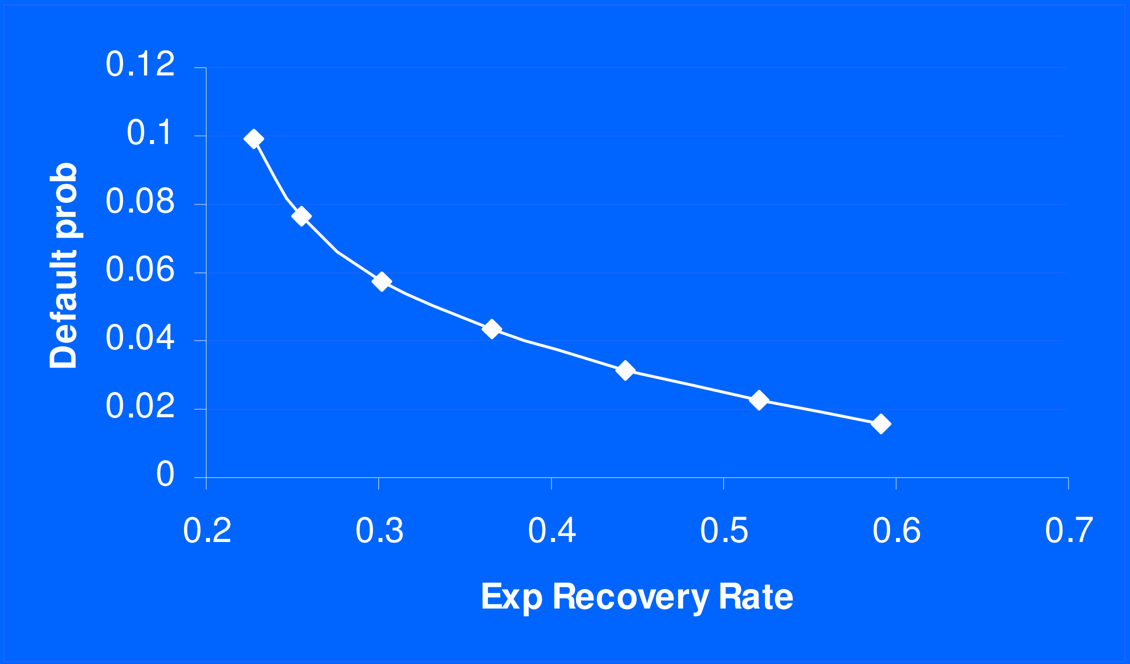
- The theoretical spread over the risk-free rate that should be earned on a new cash CDO is equal to the spread for corresponding synthetic CDO if interest rates are constant.
- Value of a cash CDO at any time is same as value of corresponding synthetic CDO plus remaining principal in tranche

Stochastic Recovery Rates

- If recovery rate is same for all companies but stochastic we repeat calculations for several different recovery rates and take a weighted average
- We can make the recovery rate correlated with value of factor

Relation Between PD and E(RR)

Ave Def Prob=0.041, Mean of RR=0.41, SD of RR=0.25
Correlation in Copula Model =0.5



When Principals and/or Recovery Rates are not the Same

- Assume there are several groups of companies and loss from a default is the same for each company within a group
- Define $k_j(T)$ as the number of companies that default from group j by time T
- Our approach can be used to calculate the conditional probability distribution of $k_j(T)$ and therefore the conditional loss distribution from group j
- The total conditional loss distribution can be calculated by convolution

Alternative Approaches

- Use Fast Fourier Transform
  - Fourier transform of sum of independent distributions is the product of the Fourier transform
- Use other iterative procedures to build up the conditional loss distribution

Test Results 1

- As default probabilities rise the cost of protection for k^{th} default rises at a decreasing rate when k is small and rises at an increasing rate when k is large
- Increasing correlations reduces the cost of protection for kth default when k is small and increases it when k is large. In the limit when correlations are perfect the cost of protection for the kth default is the same for all k

Test Results 2

- Using heavy tails for M increases the chance of an extreme event where several companies default early. The cost of protection for k^{th} default decreases when k is small and increases when k is large
- Using heavy tails for Z decreases the chance of an extreme event where several companies default early. The cost of protection for k^{th} default increases when k is small and decreases when k is large
- Using heavy tails for M and Z increases cost of buying protection against k^{th} default for low k and high k and reduces it for intermediate values of k

Test Results 3

- The impact of moving from all firms having the same default probability to firms having different default probabilities, while keeping average default probabilities the same, is small when correlation is zero, but larger when correlation is non-zero
- The effect of building dispersion into the pairwise correlations is modest when correlations are independent of default probabilities but can be quite large when there is dependence

Test Results 4

Moving from one to two factors increases the cost of buying protection against k^{th} default for low k and high k and reduces it for intermediate values of k

Test Result 5

A correlation between recovery rate and probability of default can dramatically increase spread for senior tranches

From %	To %	Const RR	SD of RR=0.2	PD/RR corr=0.5
0	3	1334	1304	1343
0	4	1091	1074	1125
2	5	526	537	610
3	6	366	374	452
6	10	126	130	195
7	10	107	110	173
10	15	40	42	84
10	20	26	27	59
10	30	13	14	32
15	30	4	5	15
10	100	3	3	7