

Early Exercise Boundary
Analytical and Numerical Approximations

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OUTLINE

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1. Introduction/Background

$$p_t + \frac{\sigma^2}{2} S \frac{\partial^2 p}{\partial S^2} + rS \frac{\partial p}{\partial S} - rp = 0 \quad S > S_f(t), \quad 0 < t < T$$

$$\left. \begin{array}{l} p(S, t) = E - S \\ \frac{\partial p}{\partial S}(S, t) = -1 \end{array} \right\} S = S_f(t), \quad 0 < t < T$$

$$p(S, t) \rightarrow 0 \quad \text{as } S \rightarrow \infty, \quad 0 < t < T$$

$$p(S, T) = \max(E - S, 0), \quad S_f(T) = E$$

Difficulties

1. No analytical solution for $p(S, t), S_f(t)$ - Existence, uniqueness and convexity of the boundary proved (Chen, Chadam, Jiang & Zheng, 2004).
2. $S_f(t) \rightarrow -\infty$ as $T \rightarrow T_-$ (van Moerbke, 1976).

Previous Results:

Aitshlia & Lai;

Broadie & Detemple;

Carr, Jarrow & Myneni;

Geske & Johnson;

Huang, Subrahmanyam & Yu;

Jacka, Jaillet, Lamberton & Lapeyre;

Karatzas;

Kim;

Parkinson;

Salopek; etc.

Recent Advances:

$$\tau = \frac{\partial^2}{2}(T - t), \quad k = \frac{2r}{\partial^2}, \quad S_f(\tau) = E e^{-2\sqrt{\tau}} \sqrt{\delta(\tau)}$$

Barles, Burdeau, Romano & Samsoen (1995) - BBRs

$$S_f(t) \sim E \left(1 - \sigma \sqrt{T-t} \sqrt{1 \ln(T-t)} \right), \quad t \sim T$$

($\Leftarrow \alpha(\tau) = -\ln \sqrt{c\tau}$, c arbitrary)

Barone-Adesi & Whaley (1987); MacMillan (1997) - BWM

$$\sqrt{\pi} h(\tau) = \int_{\sqrt{\alpha(\tau)}}^{\infty} e^{-[z - \frac{(k+1)}{2}\sqrt{\tau}]^2} \left\{ (1 + \eta(\tau)) e^{-2\sqrt{\alpha(\tau)}\sqrt{\tau}} - e^{-2z\sqrt{\tau}} \right\} \alpha z$$

$$\text{with } h(\tau) = 1 - e^{-k\tau}, \quad \eta(\tau) = \sqrt{h(\tau)} \left[k + \frac{(k-1)^2}{4} h(\tau) + \frac{(k-1)}{2} \sqrt{h(\tau)} \right]^{-1}$$

Kuske & Keller (1998) - KK

$$\sqrt{\tau} \alpha e^\alpha = 1/\sqrt{9\pi k^2}$$

Bunch & Johnson (2000) - BJ

$$\sqrt{\alpha} e^{\alpha - (k-1)\sqrt{\tau}\sqrt{\alpha}} = \sqrt{b} e^{(b-1)(k+1)^{2/4}} (4k^2 \tau)^{-1/2}$$

$$\text{with } b = 1 - k^2[(1+k)^2(2 + (1+k)^2\tau)]^{-1}$$

Behaviour Near Expiry ($t \sim T$, $\tau \sim 0$):

$$S_f(\tau) = E e^{-2\sqrt{\tau}\sqrt{\alpha}}, \quad \tau = \frac{\sigma^2}{2}(T-t), \quad k = \frac{2r}{\sigma^2}$$

$$(BBRS) \quad \alpha(\tau) \sim -\ln\sqrt{c\tau}, \quad c \text{ arbitrary}$$

$$(BWM) \quad \sqrt{\tau} \sqrt{\alpha} e^\alpha \sim 1/\sqrt{4\pi k^2}$$

$$(KK) \quad \sqrt{\tau}\alpha e^\alpha \sim 1/\sqrt{9\pi k^2}$$

$$(BJ) \quad \sqrt{\tau} \sqrt{\alpha} e^\alpha \sim \left[\left(1 - \frac{1}{2} \left(\frac{k}{1+k} \right)^2 \right) / 4k^2 \right]^{-1/2}$$

Stamicar, Sevocic & Chadam (199); Chen, Chadam & Stamicar (2000)
- CCSS

$$(CCSS) \quad \sqrt{\tau} e^\alpha \sim 1/\sqrt{4\pi k^2} \\ \Rightarrow \alpha(\tau) \sim -\ln\left(\sqrt{4\pi k^2 \tau}\right) = -\frac{\xi}{2}, \quad \xi = \ln(4\pi k^2 \tau)$$

$$(BBRS) \Leftrightarrow \alpha(\tau) \sim -\frac{\xi}{2} + c, \quad c \text{ unspecified}$$

$$(BWM) \Rightarrow \alpha(\tau) \sim -\frac{\xi}{2} - \frac{1}{2} \ln\left(-\frac{\xi}{2}\right)$$

$$(KK) \Rightarrow \alpha(\tau) \sim -\frac{\xi}{2} + \ln\left(\frac{3}{2}\right) - \ln\left(-\frac{\xi}{2} + \ln\left(\frac{3}{2}\right)\right)$$

$$(BJ) \Rightarrow \alpha(\tau) \sim -\frac{\xi}{2} + \ln\left(\kappa\right) - \frac{1}{2} \ln\left(-\frac{\xi}{2} + \ln(\kappa)\right)$$

2. Free Boundary/Green's Function Method

$$\tau = \frac{\sigma^2}{2}(T - t), \quad x = \ln(S/E), \quad P_{\text{new}} = P/E$$

$$S(\tau) = \ln(s_f/E) \text{ (i.e., } S(t) = -2\sqrt{\tau} \sqrt{\alpha(\tau)}$$

$$\begin{cases} p_\tau - \{p_{xx} + (k-1)p_x - kp\} = kH(S(\tau) - x) \\ p(x, 0) = \max(1 - e^x, 0). \end{cases}$$

$$p(x, \tau) = \int_{-\infty}^{\infty} p(y, 0) \Gamma(x - y, \tau) dy + k \int_0^\tau \int_{-\infty}^{S(u)} -\infty \Gamma(x - y, \tau - u) dy du$$

in terms of the fundamental solution

$$\Gamma(x, \tau) = e^{-k\tau} F(x + (k-1), \tau, \tau)$$

$$F(z, \tau) = \frac{1}{2\sqrt{\pi\tau}} e^{-z^2/4\tau}$$

$$p(x, \tau) = \int_{-\infty}^{S(0)=0} (1 - e^y) \Gamma(x - y, \tau) dy + k \int_0^\tau \int_{-\infty}^{S(u)} \Gamma(x - y, \tau - u) dy du$$

$$p_\tau(x, \tau) = \Gamma(x, \tau) + k \int_0^\tau \Gamma(x - S(u), \tau - u) \dot{S}(u) du.$$

$$p_\tau(S(\tau), \tau) = 0$$

$$\Gamma(s(\tau), \tau) = -k \int_0^\tau (S(\tau) - S(u), \tau - u) \dot{S}(u) du.$$

$$\Gamma(S(\tau) - S(u), \tau - u) = F(S(\tau) - S(u), \tau - u)[1 + O(\tau)], \text{ small } 0 < u < \tau$$

With $\eta = (S(\tau) - S(u))/2\sqrt{\tau - u}$, the rhs for small τ

$$\begin{aligned}
 & \sim -k \int_0^{S(\tau)/2\sqrt{\tau}(\rightarrow -\infty)} \underbrace{\left[1 - \frac{S(\tau) - S(u)}{2\dot{S}(u)(\tau - u)} \right]^{-1}}_{\longrightarrow \frac{1}{2} \text{ uniformly in } u} \frac{e^{-\eta^2} \sqrt{\pi}}{2\sqrt{\pi\tau}} d\eta. \\
 \Rightarrow \quad \Gamma(S(\tau), \tau) & \simeq \frac{e^{-S(\tau)^2/4t}}{2\sqrt{\pi\tau}} \sim k \\
 \Rightarrow \quad S(\tau) & \sim -2\sqrt{\tau} \sqrt{-\ln \sqrt{4\pi k^2 t}} \\
 \text{i.e., } \alpha(\tau) & \sim -\frac{\ln(4\pi k^2(\tau))}{2} = -\frac{\xi}{2}, \quad \tau \sim 0
 \end{aligned}$$

Similar approach developed independently by Goodman & ().

3. Analytical Approximations

$$s_f(\tau) = E e^{-2\sqrt{\tau}} \sqrt{\alpha(\tau)} = E e^{S(\tau)}$$

$$\Gamma(S(\tau), \tau) = -k \int_0^\tau \Gamma(S(\tau) - S(u); \tau - u) \dot{S}(u) du$$

$$\alpha(\tau) \sim -\frac{\ln(4\pi k^2 \tau)}{2} = -\frac{\xi}{2}, \quad \tau \sim 0$$

$$\alpha(\tau) = -\frac{\xi}{2} - \frac{1}{\xi} + \frac{1}{2\xi^2} + \frac{17}{3\xi^3} - \frac{51}{4\xi^4} - \frac{1148}{15\xi^5} + \frac{398}{\xi^6} + \cdots + t < \frac{1}{4\pi k^2}$$

$$\alpha(\tau) = -\frac{\xi}{2} - \frac{1}{\xi - a} + \frac{(1+2a)}{2(\xi - a)^2} + \frac{17/3 - a - a^2}{(\xi - a)^3} + \cdots, \quad t < e^a / 4\pi k^2$$

$$-\frac{\xi}{2} = \alpha + \ln \left[1 - \frac{1/2}{\alpha + b} - \frac{b/2}{(\alpha + b)^2} + \frac{(1 - b^2)}{2(\alpha + b)^3} + \cdots \right], \quad \alpha \rightarrow \infty$$

bigskip Truncating at third term by taking $b = 1$.

$$\tau e^\alpha \left[1 - \frac{1}{2(\alpha + 1)} - \frac{1}{2(\alpha + 1)^2} \right] = 1/\sqrt{4\pi k^2}$$

Can also interpolate with Merton's infinite horizon solution.

4. ODE/IODE Approximation

$$\begin{aligned}\dot{S}(\tau) &= \frac{S(\tau)}{2k\tau} \Gamma(S(\tau), \tau) \quad [1 = m(t)] \\ m(\tau) &= k \int_0^\tau \left[\frac{S(\tau) - S(u)}{\tau - u} \frac{2\tau}{S(\tau)} - 1 \right] \frac{\Gamma(S(\tau) - S(u), t - u)}{\Gamma(S(\tau), \tau)} \dot{S}(u) du. \\ s(0) &= 0.\end{aligned}$$

Actually solve an IODE $\frac{d\alpha}{d\xi} = \dots$ subject to

$$\alpha \rightarrow -\frac{\xi}{2} \text{ as } \xi = \ln(4\pi k^2 \tau) \rightarrow -\infty$$

Rigorous proof of convergence - Chen & Chadam (2002).