# Super-Brownian Motion, Critical Spatial Stochastic Systems and Measure-Valued Diffusions

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## 1. Brownian Motion:

Start at 0 in $\mathbb{Z}^d$, take ind't random steps $(W_i)$ to neighbours with equal probability.
$S_n = \sum_{i=1}^n W_i =$ position at time $n$.

Rescale time by $N$, space by $1/\sqrt{N}$:
$B^{(N)}(t) = S_{[Nt]}/\sqrt{N}$, $t \geq 0$.

**Theorem 1.1** (CLT, Demoivre 1718,...,
Lindeberg 1922)
$lim_{N\to\infty} P(B^{(N)}(1) \in I) = \int_I e^{-d|x|^2/2}(2\pi/d)^{-d/2}dx.$

**Theorem 1.2** (FCLT, Donsker 1951)
$lim_{N\to\infty} P(B^{(N)} \in A) = P(d^{-1/2}B \in A)$
"for any" set $A$ of paths from $\mathbb{R}_+$ to $\mathbb{R}^d$.

$B$ is standard Brownian motion (Wiener 1923).

## Universality of B

· $\{W_i\} \in \mathbb{R}^d$ any repeated independent random quantities with mean 0 and covariance $\sigma^2 I_{d\times d}$ leads to $\sigma B$ in limit. $S_n$ could be a fair game.

· Instead of $t = i/N$ may take steps "at rate $N$", i.e. at random times $T_i$, $P(T_i - T_{i-1} > t) = e^{-Nt}$ (mean $1/N$ exponential inter-step times).

· $B$ is a stochastic model for particle suspended in liquid (Brown 1828), stock market fluctuations (Bachelier 1900), as building block for more realistic stochastic models (Itô 1946)...

Why Normal? If $Z_1$, $Z_2$ and $Z$ are independent copies of limit in CLT, then

$$\sqrt{t_1}\frac{S_{Nt_1}}{\sqrt{Nt_1}} + \sqrt{t_2}\frac{S_{N(t_1+t_2)} - S_{Nt_1}}{\sqrt{Nt_2}} = \sqrt{t_1+t_2}\frac{S_{N(t_1+t_2)}}{\sqrt{N(t_1+t_2)}}$$

$\sqrt{t_1}Z_1 + \sqrt{t_2}Z_2 \equiv \sqrt{t_1+t_2}Z$ ( same distribution)
Normal distributions (mean 0) are only such distributions.

## 2. Super-Brownian Motion (SBM)

$b > 0$ (branching rate), $\sigma^2 > 0$ (diffusion rate), $g \in \mathbb{R}$ (growth rate).

$W \in \mathbb{R}^d$ random displacement with mean 0, covariance $\frac{\sigma^2}{b} I_{d \times d}$.

Start $O(N)$ particles in $\mathbb{R}^d$.
Particles behave independently.
With rate $bN$ particle at x dies.
With rate $bN + g$ particle at x gives birth to particle at x$+W/\sqrt{N}$.

Define a finite measure on $\mathbb{R}^d$, $X_t^N$, by

$$X_t^N(A) = \frac{1}{N} \#(\text{particles in } A \text{ at time } t).$$

**Theorem 2.1** (S. Watanabe 68). If $X_0^N \to X_0$, $\lim_{N\to\infty} P(X^N \in A) = P_{X_0}(X \in A)$ "for any" set $A$ of measure-valued paths.
$X$ is a continuous measure-valued Markov process whose law $P_{X_0}$ depends only on $(X_0, b, \sigma^2, g)$.

Call $X$ super-Brownian motion.

Set $g = 0$. If $X_t^{N,i}$ describes descendants of $x_i$ at $t = 0$, then $X_t^N$ decomposes into a sum of $NX_0^N(R^d)$ ind't clusters:

(*) $\qquad X_t^N = \sum_{x_i} X_t^{N,i}$.

Sad fact: Critical branching processes die out. $P(X_t^{N,i} \neq 0) \sim (Ntb)^{-1}$ as $N \to \infty$.

(*) has $NX_0^N(\mathbb{R}^d)$ summands each non-zero with prob. $\sim (Ntb)^{-1}$. Let $N \to \infty$ in (*):

$$X_t = \sum_{j=1}^{M(t)} X_t^j,$$

where $(X_t^j)$ are independent clusters descending from a single ancestor at $t = 0$, and $M(t)$ has mean $\frac{X_0(R^d)}{tb}$ (Poisson distribution).

**Additive Property**: Run 2 independent SBM's $X^1$, $X^2$, then clearly $X^1 + X^2$ is a SBM starting at $X_0^1 + X_0^2$.

## 3A Properties of Brownian Motion

1. PDE. $u(t,x) = E_x(\phi(B_t))$ solves $\frac{\partial u}{\partial t} = \frac{\Delta}{2} u$, $u_0 = \phi$.

2. Longterm Behaviour.
$d \leq 2$. $B_t \in G$ infinitely often as $t \to \infty$ for any open set $G$ a.s. (neighbourhood recurrence).

$d \geq 3$. $\lim_{t\to\infty} \|B(t)\| = \infty$ a.s. (transience).

3. Local Behaviour (Ciesielski-Taylor (1962))

Let

$$\phi_d(r) = \begin{cases} r^2 \log(1/r) \log\log\log(1/r) & d = 2 \\ r^2 \log\log 1/r & d \geq 3. \end{cases}$$

$\phi_d - m(\{B_s : s \leq t\}) = c_d t$ for all $t \geq 0$ a.s. So the range of Brownian motion is a random set of Hausdorff dimension 2.

## 3B. Properties of SBM

1. PDE. $E_{X_0}(e^{-\langle X_t,\phi\rangle}) = e^{-\langle X_0, v_t\rangle}$, where $\frac{\partial v}{\partial t} = \frac{\sigma^2 \Delta v}{2} - bv^2 + gv,\ v_0 = \phi \geq 0$.
(Dynkin, Le Gall, Mselati; Iscoe,...)

2. Longterm Behaviour.
$X_0$ finite. $X$ becomes extinct in finite time almost surely iff $g \leq 0$ (eg. use PDE).
$g = 0$, $X_0(dx) = m dx$, Dawson (1977) showed:
(a) For $d \leq 2$, for any $R > 0$, $X_t(|x| \leq R)$ becomes neglible as $t \to \infty$.
(b) For $d \geq 3$, $\lim_{t\to\infty} P(X_t \in A) = P(X_\infty \in A)$ where $E(X_\infty(A)) = mLeb(A)$ and $X_\infty$ are (the only extremal) equilibrium distributions for $X$.

3. Local Behaviour.
$S(X_t) =$ closed support of $X_t$. If $d \geq 2$
$X_t(A) = C_d \frac{b}{\sigma^2} \phi_d - m(A \cap S(X_t))\ \forall A$ a.s. $t > 0$.
(Dawson-Hochberg 79, P. 90, Le Gall-P. 95)
$S(X_t)$ is a Leb. null set of Hausdorff dimension 2 for all $t > 0$ a.s.

If $d = 1$, $X_t(dx) = X(t,x)dx$, where $X(t,x)$ is the unique solution of a stochastic pde.

## 4. Voter Model

Each site in $\mathbb{Z}^d$ has type 0 or 1. $\xi_t(x) = 0$ or 1. With rate 1, the type at x choses a neighbour at random and imposes its type on it.

Rescale space and time:
$\xi_t^N(x) = \xi_{tN}(x\sqrt{N})$, $x \in \mathbb{Z}^d/\sqrt{N}$. $V_t^N(A) = m_N^{-1} \sum_{x \in A} \xi_t^N(x)$ is empirical distribution of 1's.

**Notation**

$$m_N = \begin{cases} N, & d \geq 3 \\ N/\log N & d = 2 \end{cases}$$

$(S_n)$ is a nearest neighbour rw.

$$p_{\text{esc}} = \begin{cases} P_0(S_n \neq 0\ \forall n \geq 1) & d \geq 3 \\ 2\pi\sigma^2 = \pi & d = 2 \end{cases}$$

**Theorem 4.1** (Cox, Durrett, P. 00) Assume $d \geq 2$, $V_0^N \to X_0$. Then
$\lim_{N\to\infty} P(V^N \in A) = P_{X_0}(X \in A)$ "$\forall$" $A$, where $X$ is SBM with $g = 0$, $\sigma^2 = 1/d$ and $b = p_{\text{esc}}$.

**Remark** Changing local dynamics (eg. new definition of "neighbour") leads to a new $S_n$ and hence a new $p_{\text{esc}}$–local dynamics affect limit only through value of $p_{\text{esc}}$.

**Proof.**

Reinterpret dynamics: $\xi_t(x) = 1 \iff$ particle at $x$ $\xi_t(x) = 0 \iff$ no particle at $x$.

$f_0^N(t,x) = \{\text{no. of neighbouring 0's to } x\}/2d$

Particle at $x$ dies with rate $Nf_0^N(t,x)$, and with rate $Nf_0^N(t,x)$ produces child at $y$ chosen at random from the neighbouring 0 sites.

Similar to branching random walk with $g = 0$, $\sigma^2 = 1/d$ and a random $b = f_0^N(t,x)$.

Proof shows that if $\xi_t(x) = 1$, $f_0^N(t,x) \sim p_{\text{esc}}$ on average by by using a "dual coalescing random walk" to find conditional third moments. Dual RW traces back your history in time.

## An Application

Let $\xi_t$ be voter model starting from a single 1 at $x = 0$. $S(\xi_t) = \{x : \xi_t(x) = 1\}$. Take $d \geq 2$.

Q. (Bramson, Griffeath 81).
Conditional on $\xi_t \neq 0$, what is asymptotic shape of $S(\xi_t)$ as $t \to \infty$?

Set $t = N$. $S(\xi_N)/\sqrt{N} = S(V_1^N)$ and so expect
$P(S(\xi_N)/\sqrt{N} \in \cdot|\xi_N \neq 0)$
$\sim P_{\delta_0/N}(S(X_1) \in \cdot|X_1 \neq 0)$
LHS is the law of a single cluster of SBM.

**Theorem 4.2.**(Bramson, Cox, Le Gall 01)
For $d \geq 2$ and "for all" sets of sets $A$
$\lim_{t\to\infty} P(S(\xi_t)/\sqrt{t} \in A|\xi_t \neq 0)$
$= \mathrm{N}_0(S(X_1) \in A|X_1 \neq 0)$ (law of a single cluster conditioned on non-extinction).

## 5. Other Limit Theorems

(Slade, Notices AMS '02)

(a) Interacting Particle systems.

(i) Rescaled Contact Process (stochastic model for spread of disease) at, or near criticality, will converge to super-Brownian motion with non-trivial parameters.

Durrett-P. (99), long range, $d > 1$.

van der Hofstad-Sakai (04), medium range, $d > 4$.

(ii) Near critical rescaled stochastic Lotka Volterra models of Neuhauser and Pacala converge to super-Brownian motion with non-trivial drift and branching rates. $d > 2$. (Cox-P. 03).

(b) Lattice Trees. A connected set of neighbouring (range $L$) bonds in $\mathbb{Z}^d$ containing 0 with no cycles. Pick a lattice tree with $N^2$ vertices at random. $I^N$ assigns mass $N^{-2}$ to each vertex scaled by $1/\sqrt{N}$.

(Derbez-Slade 98) (conj. Aldous 93). For $d > 8$ and $L > L_0$ $P(I^N \in \cdot)$ approaches $\mathbf{N}_0(\int_0^\infty X_s ds \in \cdot | \int_0^\infty X_s(\mathbb{R}^d) ds = 1)$, i.e., the law of an integrated cluster conditioned on total mass 1, Integrated Super Excursion (ISE). Here $X$ is SBM $(b(L,d), \sigma^2(L,d))$

$\Lambda = \sum_i N X^N_{i/N}(\mathbb{R}^d) =$ total progeny of br. process
$ISE = \lim_N P(N^{-2} \sum_i \sum_{``x \in S(X^N_{i/N})"} \delta_x | \Lambda = N^2)$.

Proof uses lace expansion (Brydges-Spencer, Madras-Slade). $d > 8$ is needed since $S(\int_0^\infty X_s\, ds)$ has dimension 4 and so will be a tree iff $d \geq 8$. Get strong non-local interactions for $d < 8$ and logarithmic corrections if $d = 8$.

(c) Critical Percolation in $\mathbb{Z}^d$. $d > 6$, $p = p_c$. Assign mass $N^{-2}$ to each vertex in cluster containing 0 conditioned to have size $N^2$ and rescaled by $N^{-1/2}$.

Conj. (Hara, Slade 98): $X^N$ converges to ISE.

Theorem (Hara, Slade 00): $L > L_0$. First and second moments converge.

(d) Fleming-Viot Processes

Modified Moran particle system.

$\mathbb{R}^d$ is space of allele types. Migration is now mutation of type of offspring from that of parent. $N$ is *fixed* population size. Branching becomes resampling from gene pool: at $t = i/N$ each particle (j) at $x$ is replaced by $k_j$ offspring of at $x + W_m/\sqrt{N}$, $m = 1, \ldots, k_j$, where $(k_1, \ldots, k_N)$ is multinomial $(N; 1/N, \ldots, 1/N)$.

Then the empirical measure of types $F_t^N$ is a random probability on types which converges to $F_t$, the Fleming-Viot process.

**Theorem 6.1** (Etheridge-March 91)

$$P(F \in \cdot) = \lim_n P_{F_0}(X \in \cdot | \sup_{t \le n} |X_t(\mathbb{R}^d) - 1| \le 1/n)$$

Here $X$ is SBM ($g = 0$, $\sigma^2$, $b = 1$).

**"Proof".** Consider discrete time branching random walk in which at $t = 1/N$ a particle is replaced by a Poisson (1) number of offspring displaced from it parent by $W_m/\sqrt{N}$. If $X_t^N$ assigns mass $1/N$ to each particle at time $t$, then $X^N \to X$, SBM as before ($g = 0,\ b = 1,\ \sigma^2$).
Easy Fact: $Z_1, \ldots, Z_N$ independent Poisson (1),

$$P((Z_1, \ldots, Z_N) \in \cdot | \sum_i Z_i = N)$$
$$\sim \text{multinomial}(N : 1/N, \ldots, 1/N).$$

This implies

$$P(X^N|_{[0,T]} \in \cdot | X_t^N(\mathbb{R}^d) = 1 \forall t \le T)$$
$$= P(F^N|_{[0,T]} \in \cdot).$$

Let $N \to \infty$ and hope interchange of limits on left side is OK. $\square$

## 7. A Family of Stochastic pde's

SBM $X$, "$X_t(dx) = X(t,x)dx$" where $X(t,x)$ is the unique solution of the stochastic pde (SPDE) $\frac{\partial X}{\partial t} = \frac{\sigma^2 \Delta X}{2} + gX + \sqrt{2bX}\dot{W}$,

Here $\dot{W}(s,x)dsdx$ are iid Normal with mean 0 and variance $dsdx$ (space time white noise).

$d = 1$ (Reimers, Konno-Shiga 1988)
$X(t,x)$ exists, is continuous and unique.

$d > 1$ $X_t(dx) \perp dx$. Solutions to (SPDE) exist and are unique when interpreted in a generalized sense.

Why $\sqrt{\ }$? By Additive Property we need
$\sqrt{2bX^1}\dot{W}_1 + \sqrt{2bX^2}\dot{W}_2 \equiv \sqrt{2(b(X^1 + X^2)}\dot{W}$
which is true since $\sqrt{c_1}Z_1 + \sqrt{c_2}Z_2 \equiv \sqrt{c_1 + c_2}Z$ for independent normals.

In previous limit theorems, local interactions and LLN led to constant (non-obvious) parameters in limit. For many models truly interactive models arise:

- attraction/repulsion of particles
- competing species/predator prey models
- symbiotic/diploid branching

(Itô '46) $dY_t = d(Y_t)dt + \sigma(Y_t)dB_t$

( Dawson '87) Use of SBM as a building block for interactive population models.

(ISPDE)

$\frac{\partial X}{\partial t} = (A^*_{X_t} + g(x, X_t))X_t(x) + \sqrt{2b(x, X_t)X_t(x)}\dot{W}$

$A_{X_t}\phi(x) = \frac{1}{2}\sum_{1\le i,j\le d} a_{ij}(x, X_t)\phi_{ij} + \sum_{i=1}^d d_i(x, X_t)\phi_i$

$a_{ij} = (\sigma\sigma^*)_{ij}$.

Question: Does (ISPDE) characterize a unique measure-valued process? Existence via weak limits of natural branching particle systems. Uniqueness open–infinite dimensional, non-Lipschitz degenerate coefficients, generalized solutions (with nonlinearity).

Dawson '78:Solutions with $g$ are absolutely continuous wrt solutions with $g = 0$. Set $g = 0$.

P. '95, Donnelly Kurtz '99: Strong equation, genealogical information, exchangeable particles systems and historical processes to construct and characterize solutions with $b$ constant and $\sigma$, $d$ Lipschitz. Gave tools for further study: stochastic calc., lookdown system.

$$\text{(SE)}\ Y_t(\omega, y) = y_0 + \int_0^t \sigma(Y_s, X_s) dy_s + \int_0^t d(Y_s, X_s) ds,$$
$$X_t(\cdot) = \int 1(Y_t(\omega, y) \in \cdot) H_t(dy).$$

$\omega$ selects random tree; $y$ selects branch on tree.

**Theorem 7.1.** $\sigma$, $b$ Lipschitz, $b$ constant, $g = 0$. (a) $\exists$ a unique solution to (SE) which is a function of $H$ and is strong Markov.

(b) $X$ satisfies (ISPDE) iff $X$ satisfies (SE) and so (ISPDE) has a unique solution.

(ISPDE) $\frac{\partial X}{\partial t} = A^*_{X_t} X_t(x) + \sqrt{2b(x, X_t) X_t(x)} \dot{W}$
Uniqueness for $b(x, X_t)$ is still open.

Finite dimensional problem: Replace $x \in \mathbb{R}^d$ by $i \in \{1, \ldots, k\}$ and $X_t \in M_f(\mathbb{R}^d)$ by $X_t \in \mathbb{R}^k_+$.
$A_{X_t}\phi(i) = \sum_{j=1}^k q_{ij}(X_t)\phi(j)$; $i \to j$ at rate $q_{ij}(X_t)$.
(ISPDE) becomes:
(SDE) $dX^i_t = \sum_{j=1}^k X^j_t q_{ji}(X_t) dt + \sqrt{b_i(X_t) X^i_t} dB^i_t$,
$B^i$ independent Brownian motions, $i = 1, \ldots, k$.
More generally consider:
(GSDE) $dX^i_t = d_i(X_t)dt + \sqrt{b_i(X_t) X^i_t} dB^i_t$.

Assume: (i) $b_i > 0$, continuous.
(ii) $d_i \geq 0$ on $\{x_i = 0\}$, continuous.

Problem: Degenerate diffusion so Stroock-Varadhan (1969) won't apply. Non-Lipschitz diffusion so Itô (1951) won't apply.

**Counter-example:** Uniqueness may fail in (GSDE).
$X_t = \int_0^t \sqrt{2X_s} dB_s + \int_0^t C(1 + \log^+(\frac{1}{X_s}))^{-1} ds$
If $C > 1$, 0 is a regular boundary point and so there are solutions s.t. $\int_0^\infty 1(X_s = 0) ds = 0$ and $X \equiv 0$.

**Thm. 7.2.**(a) (Athreya, Barlow, Bass, P. 02) Assume (*) $d_i(x) > 0$ on $\{x_i = 0\}$.
There is a unique (in law) solution to (GSDE).

(b) (Bass, P. 03) If $d_i, b_i$ are locally Hölder continous then there is a unique solution to (GSDE).

**Remark.** (*) will fail for many examples of interest such as nearest neighbour random walk migration, $q_{ij} = 1(|i-j| = 1)$ for $i \neq j$. Hence interest in (b).

Proof. Stroock-Varadhan perturbation.
$\mathcal{L}^0\phi = \sum_i b_i^0 x_i \phi_{ii} + d_i^0 \phi_i$, $R_\lambda^0 f = (\lambda - \mathcal{L}^0)^{-1} f$.
Must show $\frac{\partial^2 R_\lambda^0 f}{\partial x_i \partial x_j}$ is bounded on an Banach space $B$. For (a) $B = L^2(\mu)$. For (b) $B$ is weighted Hölder space.

**Remark.** Constants in bound for (b) do not depend on $k$ and so results extend to infinite dimensional sde's. Some hope for (ISPDE)?