

# §1. The inviscid Limit

I start with 2D NSE:

$$\begin{aligned} \dot{u} - \nu \Delta u + (u \cdot \nabla) u + \nabla p &= \sqrt{\nu} \eta(t, x), \quad 0 < \nu \leq 1, \\ \operatorname{div} u &= 0, \quad (\text{NSE}), \\ u = u(t, x), \quad x \in \mathbb{T}^2 &= \mathbb{R}^2 / 2\pi \mathbb{Z}^2, \\ \int u dx &= \int \eta dx = 0. \end{aligned}$$

$$H = \{u(x) \in L_2(\mathbb{T}^2; \mathbb{R}^2) \mid \operatorname{div} u = 0, \int u dx = 0\}$$

$\|\cdot\|$  - the  $L_2$ -norm in  $H$ ;  $(\cdot, \cdot)$  - scalar product  
 $u(t, x)$  will be also interpreted as  
a curve  $u(t) \in H$ .

$\{e_s, s \in \mathbb{Z}^2 \setminus 0\}$  - 'usual' basis  
in  $H$ :

$$e_s(x) = c_s s^\perp \cos s \cdot x, \quad e_{-s} = c_s s^\perp \sin s \cdot x$$

$$s^\perp = \begin{pmatrix} -s_2 \\ s_1 \end{pmatrix} \quad \text{for } s = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \in \mathbb{Z}^2.$$

$\eta(t, x)$  is white in time, smooth  
in  $x$ :

$$\eta = \frac{d}{dt} \zeta(t, x), \quad \zeta = \sum_{s \in \mathbb{Z}^2 \setminus 0} b_s \beta_s(t) e_s(x)$$

$\{b_s\}$  - constants,  $b_s \approx |s|^{-m}$   $\forall m$   
 $\{\beta_s(t)\}$  - indep. standard Wiener  
processes.

$$B_j = \sum_s |s|^{2j} b_s^2 < \infty.$$

- (NSE) defines a Markov process in  $H$

Let  $u_\nu(t, x)$  be a stat. in time solution

$\mathcal{D} u_\nu(t) = \mu_\nu$  — stat. Borel measure in  $H$ .

Apply Ito's formula to  $|u_\nu(t)|^2$ , take the expectation:

$$\frac{d}{dt} \mathbb{E} |u_\nu(t)|^2 = 2\nu \mathbb{E} (\Delta u_\nu, u_\nu) + B_0$$

$\parallel$   
0

so,

$$\mathbb{E} |\mathcal{D} u_\nu(t)|^2 = \frac{1}{2} B_0 \quad \forall t, \forall \nu > 0$$

$$(B_0 = \sum_s b_s^2).$$

Similar, applying Ito to  $|\mathcal{D} u_\nu(t)|^2$ :

$$\mathbb{E} |\Delta u_\nu(t)|^2 = \frac{1}{2} B_1 \quad \forall t, \forall \nu$$

$$(B_1 = \sum_s |s|^2 b_s^2).$$

Reynolds number of  $u_\nu(t, x)$ :

$$R(u_\nu) = \frac{1 \cdot (\mathbb{E} |u_\nu|^2)^{1/2}}{\nu} \sim \nu^{-1}$$

due to (1), (2)

# The Limit

$$\dot{u} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = \sqrt{\nu} \eta, \operatorname{div} u = 0 \quad (\text{NSE})$$

$$\mathbb{E} |\nabla u_2(t)|^2 = \frac{1}{2} B_0, \quad \mathbb{E} |\Delta u_2(t)|^2 = \frac{1}{2} B_1$$

Thm 1 (SK "The Eulerian Limit...", to appear in JSPH) vol. 115 (2001)

Any sequence  $\tilde{v}_j \rightarrow 0$  contains a subsequence  $v_j \rightarrow 0$  s.t.

$$u_{v_j}(\cdot) \rightarrow u(\cdot) \quad \text{in distr. in } C([\Sigma, \infty), H),$$

where  $u(t)$  is a stat. process, and

1)  $\mathbb{E} |\nabla u(t)|^2 = \frac{1}{2} B_0,$

$$\frac{1}{2} B_0 \leq \mathbb{E} |\Delta u(t)|^2 \leq \frac{1}{2} B_1,$$

$$\frac{B_0^2}{2 B_1} \leq \mathbb{E} |u(t)|^2 \leq \frac{1}{2} B_0$$

2) Each realisation of  $u(t, \omega)$  satisfies

$$\dot{u} + (u \cdot \nabla) u + \nabla p = 0, \operatorname{div} u = 0 \quad (E)$$

and

$$|u(t)|^2 = C_1, \quad |\nabla u(t)|^2 = C_2$$

$C_1, C_2$  - random const.

3)  $\mu_0 = \mathcal{D}(u(t))$  - inv. measure for (E), s.t.

$$\mu_0(H^2) = 1 \quad (H^2 - \text{Sobolev space})$$

Maybe Thm 1 is an artifact  
due to the 'wrong scaling'?

Try another one:

$$\dot{u} - \nu \Delta u + (u \cdot D) u + D p = \nu^\alpha n(t, x), \quad (*)$$

$\operatorname{div} u = 0$

Let  $\tilde{\mu}_\nu$  be a stationary measure  
for  $(*)$ .

**Proposition.** Consider any sequence  
 $\nu_j \rightarrow 0$ . Then

1) if  $\alpha < \frac{1}{2}$ , then

$$\lim_{\nu_j \rightarrow 0} \tilde{\mu}_{\nu_j}$$

does not exist.

2) If  $\alpha > \frac{1}{2}$ , then the limit is the  
S-measure at  $D \in H$ .

## §2. The double limit

$$\dot{u} - \nu \Delta u + (u \cdot D) u + D p = \sqrt{\nu} n, \quad \operatorname{div} u = 0 \quad (\text{NSE})$$

$$n = \frac{d}{dt} \mathcal{Z}(t, x), \quad \mathcal{Z} = \sum b_s \beta_s(t) \varphi_s(x)$$

Assume that

$$b_s \neq 0 \quad \forall s \quad (**)$$

Then (NSE) has a unique stat. measure  $\mu_\nu$ . Fix any  $u_0 \in H$ . Denote by  $u = u(t; \nu, u_0)$  solution, equal  $u_0$  at  $t=0$ .

Then

$$D u(t) \xrightarrow[t \rightarrow \infty]{} \mu_\nu \quad \forall \nu$$

For the case (\*\*), see

SK, A. Shirikyan 'JMPA 81 (2002), 567-602'.

Due to Thm 1 we have:

$$D u(t; \nu, u_0) \xrightarrow[t \rightarrow \infty]{} \mu_\nu$$

$\downarrow$   
 $\{\nu_j\} \ni \nu \rightarrow 0$   
 $\mu_0 -$

The two limits do not commute.  
In stat. hydrodynamics the limit are taken exactly in this order.

Distributions in the space of trajectories also converge:

$$\mathcal{D} u(T+t) \xrightarrow[T \rightarrow \infty]{} \mathcal{D} u_\nu(t)$$

$\downarrow$

$$\nu = \nu_j \rightarrow 0$$

$$\mathcal{D} u(t)$$

Convergences hold in the space of measures in  $C(\Sigma_0, \omega), H$

Problem 1. Does the measure  $\mu_0$  depend on the sequence  $\{\omega_j\}$ ?

Problem 2. How to find properties of  $\mu_0$ ?

- For finite-dimen. diffusion processes, the two problems are well understood in the 2-dimensional case  
(M. Freidlin and others) \*)

Space - homogeneous forces.

The force, applied to (NSE), is

$$\sqrt{2} \frac{d}{dt} \mathcal{S}(t, x), \quad \mathcal{S} = \sum b_s \beta_s(t) e_s(x)$$

Assume that

$$b_s = b_{-s} \neq 0 \quad \forall s \quad (*)$$

Then  $\mathcal{S}(t, x)$  is homogen. in  $x$ . Let  $u_s(t, x)$  be a stat. in  $t$  solution

Lemma.  $u_s(t, x)$  is homogen. in  $x$

Let  $u(t, x)$  be a limiting process as in Thm 1.

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\*) Freidlin, Wentzell, Stochastics and Dynamics' 3:2 (2003), 393-408

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Proposition. If (\*) holds, then the process  $u(t, x)$  is stationary in  $t$ , and in  $x$ .

For each  $(t, x)$  we have:

1)  $E u(t, x) = 0$ , and  $D u(t, x) = D - u(t, x)$ ;

2)  $E |Du(t, x)|^2 = B_0 / 8\pi^2$ ;

3)  $\frac{B_0}{8\pi^2} \leq E |\Delta u(t, x)|^2 \leq \frac{B_1}{8\pi^2}$ ,

and similar estimates for  $E |u(t, x)|^2$ .

$$(B_0 = \sum b_s^2, \quad B_1 = \sum |s|^2 b_s^2)$$

Explicit Relations for Distribution  
of Solutions  $u$ ,

SK, D. Penrose 'A family of balance  
relations for 2D NSSE, with  
random forcing'. Preprint, 2004.

As before, assume that

$$b_s = b_{-s} \neq 0 \quad \forall s \quad (*)$$

$u_\nu(t, x)$  — solution, stat. in time  
(and in  $x$ ). Denote

$$\begin{aligned}\omega_\nu(t) &= \omega_\nu(t, x) = 2\nu t \quad u_\nu(t, x) = \\ &= \frac{\partial u_\nu^2}{\partial x_1} - \frac{\partial u_\nu^1}{\partial x_2}\end{aligned}$$

**Thm 2.** Let  $g(z)$  be any contin.  
function s.t.

$$|g(z)| \leq C_1 (1 + |z|)^{C_2} \quad \forall z.$$

Then  $\forall x \in \mathbb{T}^2, \forall t > 0, \forall \nu > 0$

$$\begin{aligned}|\mathbb{E} g(\omega_\nu(t, x))| D_x \omega_\nu(t, x) |^2 &= \\ &= \frac{1}{2} B_1 |\mathbb{E} g(\omega_\nu(t, x))|^2\end{aligned} \tag{1}$$

$$(B_1 = \sum |s|^2 B_s^2).$$

**Rmk.** I cannot pass in (1) to a limit  
as  $\nu \rightarrow 0$ .

Let us fix any  $t > 0$ .

Take any  $x \in \mathbb{T}^2$

Abbrev.  $w(\omega) = w_\nu(t, x)$ .

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**Probabil. Corollary.** Denote

By  $\mathcal{F}_{w(x)}$  the  $G$ -algebra, generated  
by the r.v.  $w(\omega)$ . Then

$$|\mathbb{E}[|Dw(x)|^2 | \mathcal{F}_{w(x)}] = \frac{1}{2} B_1, \quad (2)$$

$\forall t, x, \nu$ .

Rmk. (1)  $\Leftrightarrow$  (2)

Denote by  $\mathcal{M}_{B_1}$  the class of  
homogeneous Borel measures on  
 $C^1(\mathbb{T}^2)$ , satisfying (2).

This is a closed convex subset  
of  $\infty$  codim in the space of  
homogeneous probability measu-  
res on  $C^1$ .

The NSE:

$$i - \nu \Delta u + (u \cdot \nabla) u + \nabla p = \sqrt{\nu} n, \quad \text{NSE}$$
$$\operatorname{div} u = 0,$$

$$n = \frac{d}{dt} \sum B_s \beta_s(t) e_s(x),$$
$$B_s = B_{-s} \neq 0 \quad \forall s$$

The numbers  $\{B_s\}$  fast converge to 0, and satisfy one relat.

$$\sum |s|^2 B_s^2 = B_1 - \text{fixed const.}$$

~~Let  $\text{dist}$  be the Prokhorov distance in the space of measures in  $C^1$  =  $C(X)$ .~~

**Corollary.** Let  $u(t)$  be any solution of NSE,  $w(t) = \operatorname{rot} u(t)$  and  $D w(t)$  — distribution of  $w(t)$  [i.e. a measure in  $C^1$ ].

Then  $\exists \Omega \in \mathcal{M}_{B_1}$  s.t.

$$D w(t) \xrightarrow[t \rightarrow \infty]{} \Omega \in \mathcal{M}_{B_1}$$

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so,  $\mathcal{M} = \bigcup_{B_1} \mathcal{M}_{B_1}$  attracts distrib.  
of all solutions of all equations  
(NSE)

# The Inviscid Limit: Pro & Con

$u_{\text{lt}}, x)$  — the limiting process in Thm 1,  
 $D u(t) = \mu_0$  — measure in  $H$ .

P20 i) If  $u_{\text{lt}}(t; u_0, v)$  is any solution of NSE, then

$$\lim_{v_j \rightarrow 0} \lim_{t \rightarrow \infty} D u_{\text{lt}}(t; u_0, v) = \mu_0$$

ii) Thm 1 agrees well with Onsager's claim (1949) that 2D Euler eq. describes the 2D turbulence

iii) Take any other scaling of the equation:

$$u - v \Delta u + (u \cdot \nabla) u + \nabla p = v^\alpha \eta, \quad \alpha \neq \frac{1}{2}$$

Let  $\tilde{\mu}_v$  be the stat. measure for any sequence  $v_j \rightarrow 0$ ,

Proposition.  $\lim_{v_j \rightarrow 0} \mu_{v_j}$

does not exist if  $\alpha < \frac{1}{2}$ , and is the  $S$ -measure at  $D \cap H$  if  $\alpha > \frac{1}{2}$ .

So  $\alpha = \frac{1}{2}$  is the only scaling when a nontrivial inviscid limit exists

Con. Is the result 'physically correct'?

Consider the physically correct force:

$$\dot{u} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = \sqrt{\nu} \frac{d}{dt} f(t, x) \quad (3)$$

The force  $\frac{d}{dt} f(t, x)$  is

- smooth in  $t, x$
- stationary in  $t, x$
- correlation in  $t$  decays fast

For the Inviscid Limit as in Thm 1 to be a physically correct result, it should also apply to (3). Does it?

Introduce <sup>slow</sup> fast time  $\tilde{\tau}$ ,

$$\gamma = \nu \tilde{\tau}.$$

Then

$$\frac{\partial u}{\partial \tilde{\tau}} - \Delta u + \nu^{-1} (u \cdot \nabla) u + \nabla p' = \quad (4)$$

$$= \frac{d}{d\tilde{\tau}} (\sqrt{\nu} f(\nu^{-1} \tilde{\tau}, x)) =: \frac{d}{d\tilde{\tau}} \zeta_{\nu} (\tilde{\tau}, x)$$

$$\zeta_{\nu} (\cdot) \xrightarrow{\nu \rightarrow 0} \zeta (\cdot) \text{ - Donsker}$$

If in (4)  $\zeta_{\nu} = \zeta$ , then

$$\lim_{\nu \downarrow 0} \lim_{\tilde{\tau} \rightarrow \infty} \mathcal{D} u(\tilde{\tau}) = M_0.$$

$$\nu \downarrow 0 \quad \tilde{\tau} \rightarrow \infty$$

? Can we in (4) replace  $\zeta_{\nu}$  by  $\zeta$  ??  
This is far from obvious !!

## 2D versus 3D

Consider the 3D case:

$$\dot{u} - \nu \Delta u + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \nu^\alpha \eta \quad (5)$$

$$\operatorname{div} \mathbf{u} = 0, \quad \mathbf{x} \in \overline{\mathbb{T}}^3.$$

Let  $\tilde{\mu}_\nu$  be a stat. measure.

'Proposition'. If Kolmogorov's theory of turbulence applies to (5) and  $\lim_{\nu \rightarrow 0} \tilde{\mu}_\nu$  exists,

then  $\alpha = D$ .

! Note the difference!

### § 3. Other equations.

Thm 1 applies to damped - driven Hamiltonian PDE (HPDE), if the force is  $\sim \sqrt{\text{viscosity}}$ :

$$\text{HPDE} \quad -\nu \Delta u = \sqrt{\nu} \eta(t, x)$$

white in time, smooth in  $x$

Provided That the HPDE has two "good" integral of motion

#### Examples

A) Navier - Stokes

$$\underbrace{(u + (u \cdot \nabla) u + \nabla p)}_{\text{Euler eq}} - \nu \Delta u = \sqrt{\nu} \eta(t, x)$$

Euler eq - an HPDE, the two  
integrals - energy & enstrophy

B) CGL equation:

$$\underbrace{(u - i \Delta u + i |\lambda| u^2 u)}_{x \in D \subset \mathbb{R}^n, n \leq 3, u|_{\partial D} = 0} - \nu \Delta u = \sqrt{\nu} \eta(t, x) \quad (\text{CGL})$$

Let  $u_\nu(t, \omega)$  be a stat. in time solution of (CGL).

Thm 3.  $u_\nu(\cdot) \rightarrow u(\cdot)$ ,

where

$$u - i \Delta u + i |\lambda| u^2 u = 0, \quad (\text{NLS})$$

and  $M_0 = \mathbb{D} u(t)$  is an invar. measure.

see

SK, A. Shirikyan "Randomly forced  
 CGL equation..."  
 preprint 2003

Problem 1. Does  $\mu_0$  depend on  $\{\omega_j\}$ ?

Problem 2. How to find  $\mu_0$ ?

## §4. The inviscid limit and statistical mechanics

Galerkin approximation for NSSE:

$$\dot{u} - \nu \Delta u + P_N(u \cdot \nabla) u + \nabla p = \sqrt{\nu} \eta(t, x), \quad (\text{NSSE}_N) \\ \operatorname{div} u = 0$$

Here  $P_N: H \rightarrow H_{(N)} = \operatorname{span}\{e_s, |s| \leq N\}$

Thm 1 applies to  $(\text{NSSE}_N)$ , uniformly  
 in  $N$ ?

So, if  $u_N$  is a stat. solution of  
 $(\text{NSSE}_N)$ , then

$$\mathcal{D} u_{N_j}(\cdot) \longrightarrow \mathcal{D} u(\cdot),$$

where

$$\dot{u} + P_N(u \cdot \nabla) u + \nabla p = 0, \quad \operatorname{div} u = 0 \quad (E_N)$$

$\mathcal{D}(u(t)) = \mu_0$  — invar. measure  
 for  $(E_N)$ . How to find  $\mu_0$ ?

$(E_N)$  is a finite-dimen. Hamilt. system with two integrals:

$$E(u) = \|u\|^2, \quad \Omega(u) = \|Du\|^2$$

energy                            enstrophy

- The Lebesgue measure  $\ell(da)$  is invariant for  $(E_n)$ .
  - Hence, the microcanonical measure

$$E_{C_1, C_2}(du) = S_{E=C_1, \Sigma=C_2} E(du)$$

also is invar. (for any  $c_1, c_2$ ).

Conjecture.  $\mu_0$  is a superposition of the measures  $\rho_{C_1, C_2}$ . That is,  
 $\exists$  a measure  $\sigma$  on  $\mathbb{R}^2$  s.t.

$$\mu_0(F) = \iint_{\mathbb{R}^2} \rho_{c_1, c_2}(F) g(dc_1, dc_2)$$

Thm 4. This conjecture is wrong.

Proof. By Thm 1,  $\int \|u\|_2^2 \mu_0(du) \leq C$   
 unif. in  $N$ . But

$$\int \|u_k\|_2^2 \, P_{C_1, C_2}(du) \xrightarrow[N \rightarrow \infty]{} \infty$$

(Andrei Birinck, 2003).