

# Lecture 2: The uniqueness Theorem & Related Results

## Lecture 3: The inviscid limit ( $\nu \rightarrow 0$ )

- results on random forces with only a few modes excited

I shall mostly talk about the  
2D RSE, forced by random  
kicks:

$$\dot{u} + \nu A u + B(u) = \eta(t) \quad (\text{RSE})$$

$$u(t), \eta(t) \in H,$$

$$H = \{ u(x) \in L_2 \mid \operatorname{div} u = 0, \int u dx = 0 \},$$

$$\eta^{\omega} = \sum_{j \in \mathbb{Z}} \eta_j^{\omega} \delta(t-j)$$

$\eta_j^{\omega}$  — kick number  $j$

The kicks  $\dots \eta_{-1}, \eta_0, \eta_1, \dots$  are i.i.d. random variables in  $H$ ,

$$\eta_j^{\omega} = \sum B_s \xi_s^{j, \omega} e_s(x)$$

$$\{e_s(x), s \in \mathbb{Z}^2 \setminus D\} — \text{trig. basis in } H$$

$\{\xi_s^{j, \omega}\}$  — independent random variables (r.v); distrib. is independent of  $j$ .

Say each  $\xi_s^{j, \omega}$  is uniformly distributed on  $[-1, 1]$

$$u(n) = S(u(n-1)) + \eta_n^{\omega} \quad (1)$$

$S: H \rightarrow H$  — time-one shift for free NSE,

$$u(n) = u^{\omega}(n) \in H.$$

Rotation.  $u(n; u_0)$ ,  $n = 0, 1, 2, \dots$  — the (random) trajectory of  $\{z\}$ , equal  $u_0$  at  $n=0$ .

(1) defines a Markov chain in  $H$ :

$$P(\rho, u_0, \cdot) = D(u(\rho; u_0))$$

$(D(\xi))$  — the distribut. of a r.v.  $\xi$ ,

We have the two semigroups:

$$P_{n+1}: C_b(H) \rightarrow C_b(H), \quad n = 0, 1, \dots,$$

$$P_n^*: P \rightarrow P, \quad n = 0, 1, \dots$$

$P$  — space of probability Borel measures on  $H$ .

To find  $P_n^*(\mu)$ ,  $\mu \in P$ , we have:

- find a r.v.  $u_0$  in  $H$  s.t.  $D(u_0) = \mu$ ,
- set  $P_n^*(\mu) = D(u(n; u_0))$

$$P_n^* = (P_1^*)^n$$

## §1. Around Thm 1.

Thm 1. If  $B_s \neq 0$  &  $1/s \leq N_2$ , then  
 $\exists!$  measure  $\mu \in \mathcal{P}$  s.t.  $P_n^* \mu = \mu$   
 $\forall n$ . Moreover,

$$D(u(n; u_0)) \xrightarrow[n \rightarrow \infty]{} \mu \quad \forall u_0 \in H. \quad (2)$$

More generally, (2) holds if  $u_0$  is random s.t.  $\mathbb{E} |u_0|^2 < \infty$ .

Discussion of the proof, following

[1] SK, Ashurikyan [CMP 221 (2001),  
351 - 366]

[2] SK "On exponential convergence..."  
in Amer. Math. Soc. Transl. (2),  
Vol. 206 (2002), 161 - 176

[3] N. Masmoudi, L-S Young, [CMP 227  
(2002), 461 - 481]

# 1<sup>o</sup> The Kantorovich distance in $\mathcal{P}$

( $\mathcal{P}$  - set of prob. Borel measures on  $H$ )

$$K = \{f: H \rightarrow \mathbb{R} \mid |f| \leq 1, \text{Lip } f \leq 1\},$$

for  $\mu_1, \mu_2 \in \mathcal{P}$ , set

$$\|\mu_1 - \mu_2\|_L^* = \sup_{f \in K} |\langle f, \mu_1 \rangle - \langle f, \mu_2 \rangle|. \quad (3)$$

Thm (Kantorovich).  $\mathcal{P}$ , given the distance (3), becomes a complete metric space. The convergence w.r.t. this distance is the weak convergence of measures.

Denote  $\mu(n; u_0) = D(u(n; u_0))$ . Then

$$\mu(0; u_0) = s_{u_0}, \text{ and}$$

$$\mu(n; u_0) = P_1^* \mu(n-1; u_0) = \dots = P_n^* s_{u_0}.$$

We need to prove that  $\exists \mu$  s.t.

$$\|\mu(n; u_0) - \mu\|_L^* \xrightarrow{n \rightarrow \infty} 0 \quad \forall u_0. \quad (4)$$

Lemma. If

$$\|\mu(n; u_1) - \mu(n; u_2)\|_L^* \xrightarrow{n \rightarrow \infty} 0 \quad (5)$$

$\forall u_1, u_2 \in H$ , then (4) holds with some  $\mu \in \mathcal{P}$ .

## 2<sup>o</sup>. Coupling.

Def. A coupling for  $\mu_1, \mu_2 \in \mathcal{P}$  is a pair of random variables  $U_1, U_2 \in \mathcal{H}$ , defined on SOME probability space  $\Omega$ , s.t.  $D(U_1) = \mu_1, D(U_2) = \mu_2$ .

Intuitively, measures  $\mu_1, \mu_2$  are close, if we can construct a coupling  $U_1, U_2$ , s.t. these two random variables are close.

### Abbreviate

$$\mu_1(n) = \mu(n; u_1), \mu_2(n) = \mu(n; u_2)$$

$$? \quad \| \mu_1(n) - \mu_2(n) \|_L^* \xrightarrow[n \rightarrow \infty]{} 0 ?$$

## 3<sup>o</sup>. Special coupling for $\mu_1(n), \mu_2(n)$

Assume that we already have 'good' coupling  $(V_n, W_n)$  for  $(\mu_1(n), \mu_2(n))$ .

$$D(V_n) = \mu_1(n), D(W_n) = \mu_2(n)$$

- Since the kicks  $\eta_1, \eta_2, \dots$  are independent, we may assume that

$$\eta_n = \eta_n(w_n), \quad w_n \in \Omega_n,$$

where  $\Omega_1, \Omega_2, \dots$  — different probability spaces.

The coupling  $(V_n, W_n)$  is defined on some other probability space  $\Omega$ .

We have:

$$\mu_1(n+1) = P_1^*(\mu_1(n)) = D \underbrace{(S(V_n(w)) + \eta_{n+1}(w_{n+1}))}_{V_{n+1}^\circ}$$

$$\mu_2(n+1) = D \underbrace{(S(W_n(w)) + \eta_{n+1}(w_{n+1}))}_{W_{n+1}^\circ}.$$

The r.v.  $V_{n+1}^\circ$  and  $W_{n+1}^\circ$  are defined on the probab. space  $\Omega \times \Omega_{n+1}$ .

This is a coupling for  $(\mu_1(n+1), \mu_2(n+1))$ .

Main Lemma. The coupling  $(V_{n+1}^\circ, W_{n+1}^\circ)$  can be replaced by a 'better' coupling  $(V_{n+1}, W_{n+1})(w, w_{n+1})$  with the following property:

$$\text{Denote } d_n(w) = \|V_n(w) - W_n(w)\|.$$

The  $\exists C$  s.t.  $\forall$  fixed  $w$  we have:

$$P^{w_{n+1}}(\|V_{n+1}(w, w_{n+1}) - W_{n+1}(w, w_{n+1})\| \geq \frac{1}{2} d_n) \leq C d_n.$$

"with high probability,  $V_{n+1}$  and  $W_{n+1}$  are twice closer than  $V_n$  and  $W_n$ , provided that  $V_n$  and  $W_n$  are close enough."

Claim. Main Lemma  $\Rightarrow$  Theorem

For proofs of Main Lemma and Claim, see [1, 2].

## §2. Ergodicity

$$u(n) = S(u(n-1)) + \eta_n, \quad u(n) \in H. \quad (1)$$

Let  $\mu$  be a stationary measure. Then  
 $\exists$  a stationary process  $(U(n), n \in \mathbb{Z})$ ,  
 $U(n) \in H$ , which satisfies (1)  $\forall n$ ,  
and

$$\mathcal{D}(U(\cdot)) = \mu \quad \forall \omega.$$

Denote

$$H^{\mathbb{Z}} = \left\{ (\dots, u(-1), u(0), u(1), \dots), \quad u(j) \in H \right\}$$

$\nwarrow$   
space of infinite trajectories in  $H$ .

Then  $U(\cdot) \in H^{\mathbb{Z}}$   $\forall \omega$ ; so

$m := \mathcal{D}(U)$  is a measure in  $H^{\mathbb{Z}}$ .

Consider the shift map:

$$\Theta: H^{\mathbb{Z}} \rightarrow H^{\mathbb{Z}}, \quad U(\cdot) \mapsto U(\cdot+1).$$

Then  $\Theta_*(m) = m$  since  $U$  is a stat. process.

Def. Stationary measure  $\mu$  is called ergodic, if  $(\Theta, m)$  is an ergodic process.

It is known that

|| If a stationary measure is unique,  
then it is ergodic

See in

[DZ] Da Prato, Zabczyk "Ergodicity  
for  $\infty$ -dimensional systems",  
CUP

• Leading Lyapunov exponent

For a random trajectory  $\{u^\omega(n)\}$  define

$$\phi(n, \omega) = dS(u(n)) \cdot \dots \cdot dS(u(1)) dS(u(0))$$

In terms of the dyn. syst.  $\{\Theta^\omega\}$ ,  
we set  $A(\omega) = dS(\Theta^\omega(0))$ . Then

$$\phi(n, \omega) = A(\Theta^{\omega, n-1}\omega) \cdot \dots \cdot A(\omega), \quad n \geq 1$$

~~Excess presentations the probability  
space is  $H^\omega$  with the measure  $m$ .~~

def  $\lambda_0(\omega) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln |\phi(n, \omega)|$ .

Since  $(\Theta, m)$  is ergodic, then:

|| Kingman's Subadditive Ergodic  
Theory  $\Rightarrow$  for  $\mu$ -a.a.  $u(0)$ ,  
the limit  $\lambda_0 \geq -\infty$  exists a.s.,  
and is a constant, independent of  $u_0$   
deterministic

Def(?) The flow, described by (1) is turbulent if  $\lambda_0 > 0$ , and is laminar if  $\lambda_0 < 0$ .

Recall that (1) corresponds to the kick-forced NSE

$$\dot{u} + \gamma A u + B u = \eta(t).$$

Statement (easy).  $\lambda_0 \rightarrow -\infty$  as  $\gamma \rightarrow \infty$ .

Problem.  $\lambda_0 \geq 0$ , when  $\gamma$  is small enough.

### § 3. Random attractors.

$$u(n) = S(u(n-1)) + \eta_n^\omega, \quad u(n) \in H. \quad (1)$$

$\omega \in (\Omega, \mathcal{F}, P)$

Since  $\dots, n_{-1}, n_0, n_1, \dots$  is a stationary sequence, then  $\exists$  measure-preserving automorphisms

$$\theta^j : \Omega \rightarrow \Omega, \quad j \in \mathbb{Z},$$

s.t.

$$n_j(\omega) = n_0(\theta^j(\omega)) \quad \forall \omega, \forall j$$

Rmk: This is true up to some equivalence

rotation. For a fixed  $\omega \in \Omega$ , denote

by  $S_{t_1}^{t_2}(\omega)$ ,  $t_2 > t_1$ , flow-maps for (1),  

$$u(t_1) \xrightarrow{\quad} u(t_2).$$

Let  $\mu$  be a stationary measure for (1).

Thm (Ledrappier, 1984/86). For

a.a.  $\omega$ , the limit

$$\lim_{T \rightarrow \infty} S_{-T}^0(\omega)_* \mu = \mu_\omega \in \mathcal{P}$$

exists. Moreover,

$$\mu = \int_{\Omega} \mu_\omega P(d\omega)$$

Def.  $\{\mu_w\}$  as in the Thm is called Markov desintegration of  $\mu$ .

Denote  $K_w = \text{supp } \mu_w \subset H$ .

Then  $K_w \subset H$  for a.e.  $w$ .

As before,  $u^w(t; u_0)$  - solution of  $(*)$ , equal  $u_0$  at  $t=0$ .

For the result below see  
[SK, A. Shirikyan] Funct. Anal. Appl.,  
38 (2004), 28-37.

Theorem 2. i)  $\forall u_0 \in H$ ,

$$\text{dist}_H(u^w(t; u_0), K_{\theta^t w}) \xrightarrow[t \rightarrow \infty]{\text{in probab.}} 0,$$

2)  $K_w$  is the minimal set s.t. 3)  
holds  $\forall u_0$ .

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That is,  $K_w = \text{supp } \mu_w$  is a random point-attractor for convergence in probability.

Cf. works by H. Crauel, A. Debussche,  
F. Flandoli, Chepyzhov - Vishik

Corollary.  $\exists C$  s.t.

$$\dim_H K_w \leq C \quad \text{a.s.}$$

## Remarks.

- 1) Our results show that the random attractor  $K_\omega$  ( $= \text{supp } \mu_\omega$ ) carries the natural measure  $\mu_\omega$ .
- 2) Theorem 2 holds true for the white-forced NSE:

$$\dot{u} + \nu Au + Bu = f(x) + \frac{\partial}{\partial t} \sum B_s \beta_s(t) R_s(x),$$

•  $B_s \neq 0 \quad \forall s \leq N_2,$

• If  $f(x) \neq 0$  then  $B_s \neq 0 \quad \forall s.$

- 3) Consider

$$\dot{u} + \nu Au + Bu = f(x) + \varepsilon \frac{\partial}{\partial t} \sum B_s \beta_s(t) R_s(x)$$

For  $\varepsilon = 0$  the eq. has an attractor  
for  $\varepsilon > 0$  it has an invariant  
measure  $\mu_\varepsilon$

? Relations Between  $\mu_\varepsilon$  and  $K$  ?

E.g. :

$$\mu_\varepsilon(K+s) \xrightarrow[\varepsilon \rightarrow 0]{} \emptyset \quad \forall s > 0.$$

## § 4. High-frequency kicks.

Consider (nSE)

$$\ddot{u} + 2Au + B(u) = n(t),$$

where

$$n = n_\varepsilon(t, x) = \sqrt{\varepsilon} \sum n_k^\omega(\omega) S(t - \varepsilon k),$$

$$y_k^{\omega} = \sum b_s \xi_s^k e_s(x),$$

$\xi_s^k$  - independent r.v., stationary  
in  $k$ , s.t.

$$\mathbb{E} \xi_s^k = 0, \quad \mathbb{E} (\xi_s^k)^2 = 1.$$

Assume that

$B_3 \neq 0$   $\forall s$

$u_\xi(t; v)$  — solution, equal  $v$  at  $t=0$ .

Now consider white-force:

$$n_0 = \sum b_s \dot{\beta}_s(z) e_s(x).$$

$u_0(t; v)$  — corresponding solutions.

## Donsker theorem:

$$\int_0^t n_\varepsilon(s) \, ds \xrightarrow[\varepsilon \rightarrow 0]{} \int_0^t n_0(s) \, ds,$$

"Because of that" we have the Splitting Up Method for SPDE:

$$\mathcal{D} u_\varepsilon(t; v) \xrightarrow[\varepsilon \rightarrow 0]{} \mathcal{D} u_0(t; v). \quad (6)$$

See work by Gyöngy - Krylov.

! The convergence (6) is not uniform in  $t$ .

$$\begin{array}{ccc} \mathcal{D} u_\varepsilon(t; v) & \xrightarrow[t \rightarrow \infty]{\text{Th 1}} & \mu_\varepsilon \\ (6) \downarrow \varepsilon \rightarrow 0 & & \downarrow \varepsilon \rightarrow 0 \quad \leftarrow \text{Theorem 3} \\ \mathcal{D} u_0(t; v) & \xrightarrow[t \rightarrow \infty]{\text{Th 1}} & \mu_0 \end{array}$$

For Thm 3 see

[SK, A. Shirikyan] Proc. A Royal Soc.  
Edinburgh 133 (2003),  
875 - 891

Conjecture. Thm 3 remains true if  
 $\eta(t, x) = \sqrt{\varepsilon} \eta^0(\frac{t}{\varepsilon}, x)$ , where  $\eta^0$  is

- 1) smooth in  $x$ ,
- 2) stationary in  $t$  with fast decaying correlation,
- + some technical assumptions