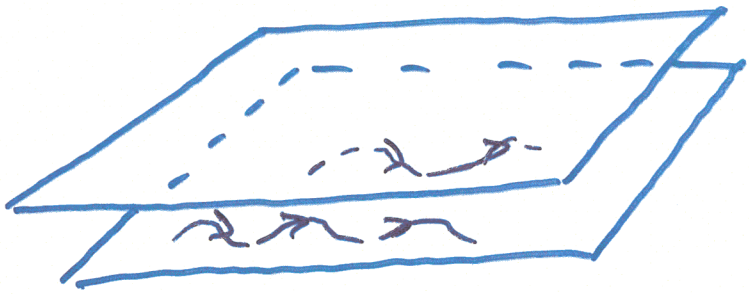
Mathematics of 2D Statistical Hydrodynamics

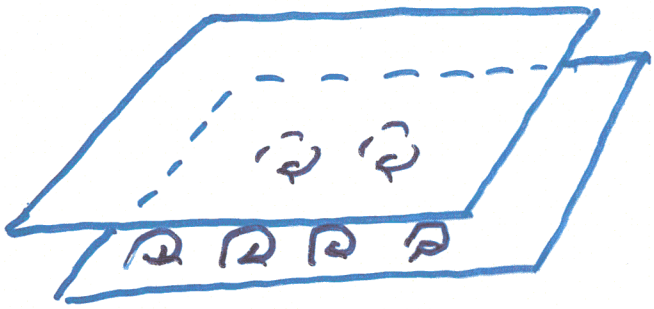
Introduction: Turbulence in 2D and in 3D

Does the 2D turbulence exist?

I) First try: flow between two close plate:

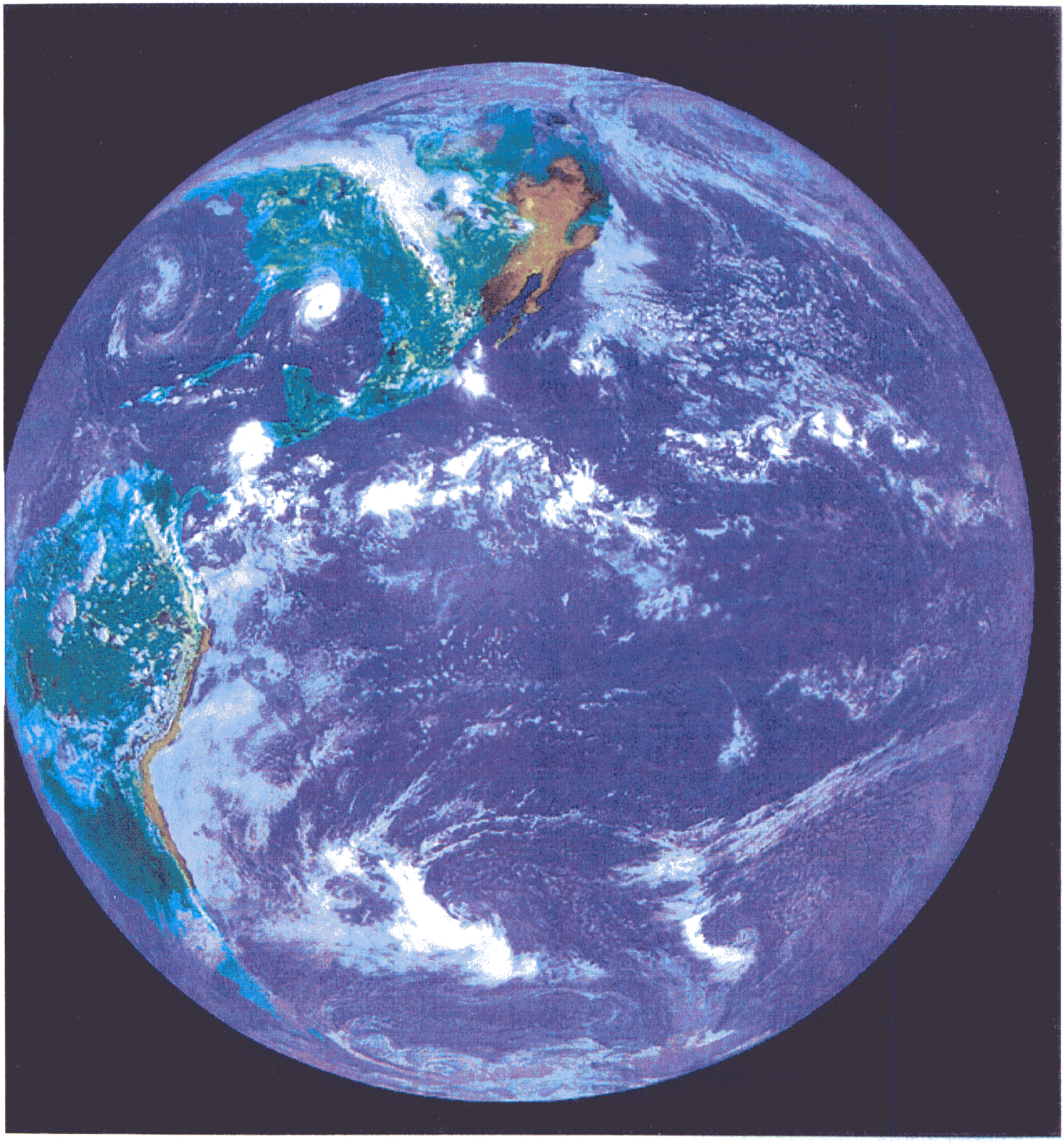


low velocity: 2D flow, parallel the plates

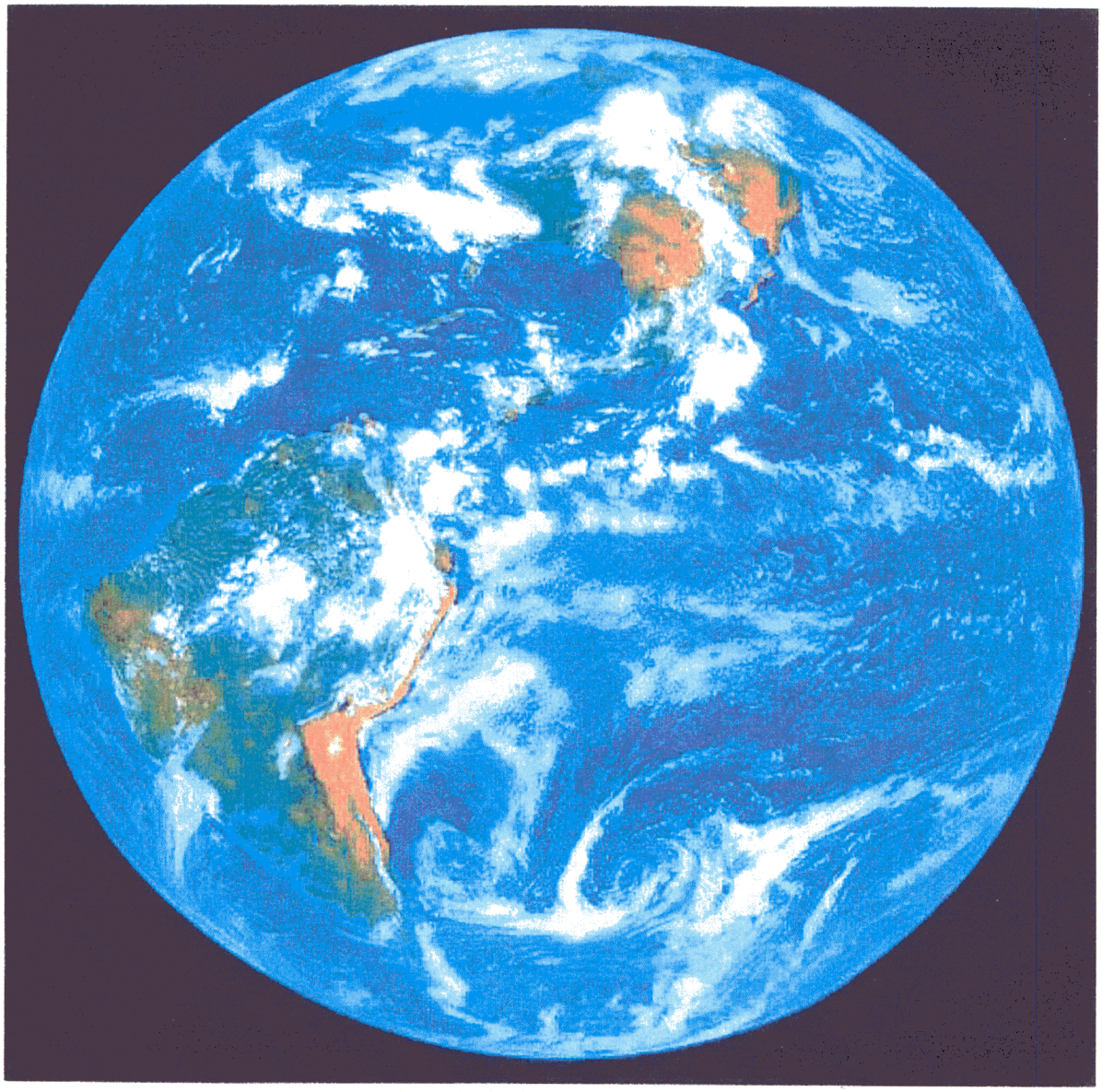


high velocity (= turbulent flow) ALWAYS is 3D!

II) Second try: the Earth atmosphere



the Earth Atmosphere, 1



the Eath atmosphere, 2

III There are other, more exotic examples

IV 2D turbulence is a good math. model for 3D

General References

[B] G.K. Batchelor, The Theory of Homogeneous Turbulence, CUP

[F] U. Frish, Turbulence, CUP

[SK] SK, in 'Rev. math. Phys' 14 (2002), 585-600

[VF] M.I. Vishik, A.V. Fursikov 'math. Problems in Statistical Hydromechanics', Kluwer, 1988

§1. 2D hydrodynamics in finite volume

$$\dot{u} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = \tilde{n}(t,x) \qquad (1)$$

$$u = u(t,x), \quad \operatorname{div} u = 0$$

either

$$x \in \Omega \Subset \mathbb{R}^2, \quad u|_{\partial\Omega} = 0 \qquad (D)$$

or

$$x \in \mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2, \quad \int \tilde{n}\,dx = \int u\,dx = 0. \qquad (P)$$

I shall talk about (P), although most of results hold for (D) as well.

$$H = \{u(x) \in L_2 \mid \operatorname{div} u = 0, \ \int u\,dx = 0\}$$

'div' applies in the sense of general functions

I often treat $u(t,x)$ as a curve $t \longmapsto u(t) \in H$

Elimination of the pressure p

$$v(x) \in L_2(\mathbb{T}^2;\mathbb{R}^2), \quad v = u(x) + \nabla p(x) + \text{const}$$
$$u(x) \in H$$

$$\Pi : L_2 \longrightarrow H, \quad v(x) \longmapsto u(x)$$

— the Leray projector

apply Π to eq. (1):

$$\dot{u} - \nu \Pi \Delta u + \Pi (u \cdot \nabla) u = \Pi \tilde{n} =: n$$

denote $A = -\Pi \Delta$, $B(u) = \Pi (u \cdot \nabla) u$

$$\dot{u} + \nu A u + B(u) = n(t), \qquad (1')$$

$$u(t), n(t) \in H$$

Basis in H

$$\{ e_s(x),\ s \in \mathbb{Z}^2 \setminus \{0\} \},$$

$$e_s = c_s s^{\perp} \cos s \cdot x, \quad e_{-s} = c_s s^{\perp} \sin s \cdot x$$

$\forall s \in \mathbb{Z}^2_+$, $\mathbb{Z}^2_+$ — "half of $\mathbb{Z}^2 \setminus 0$"

(say, $\mathbb{Z}^2_+ = \{ (s_1, s_2) \mid s_1 > 0$ and $s_2 > 0$ if $s_1 = 0 \}$)

For $s = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$, $s^{\perp} = \begin{pmatrix} -s_2 \\ s_1 \end{pmatrix}$

§2. 2D Statistical Hydrodyn.

I. Decaying Turbulence

force $\eta = 0$,

$\dot{u} + \nu A u + B(u) = 0$.

$u_{|t=0} = u_0(x)$ — a random field on $\mathbb{T}^2$

$u_0 = u_0^\omega(x)$, $\omega \in (\Omega, \mathcal{F}, P)$,

$|u_0|_{L^\infty} \sim 1$

$\Rightarrow$ $u(t) \sim e^{-\nu t}$. We study $u(t,\nu)$ while it is ~ 1

Kolmogorov-41, E.B. Hopf, Ch. Foias, Vishik-Fursikov

- decaying turb. cannot be stationary

II. Stationary Turbulence

Kolmogorov-42, etc

$\eta(t) = \eta^\omega(t) \in H$

$\eta(t,\nu)$

$\omega \in (\Omega, \mathcal{F}, P)$

Physical postulates on the force η

(after A.N. Kolmogorov)

1) $\eta(t,x)$ is smooth in x,

2) the random process $t \longmapsto \eta(t) \in H$ is stationary. The r.v.

$\eta(t)$ and $\eta(t+T)$

($T > 0$ is time-scale) are 'practically independent'

Fourier decomposition of η

$$H \ni \eta(t) = \sum_{s \in \mathbb{Z}^2 \setminus 0} b_s \, \eta_s^{\omega}(t) \, e_s(x)$$

$\{b_s\}$ – real constants, $b_s \to 0$

$\{\eta_s\}$ – random processes 'of order one'

Extra assumption:

3) processes $\{\eta_s\}$ are independent

b_s – intensity of Fourier mode s

$b_s \geq 0$. The most important case.

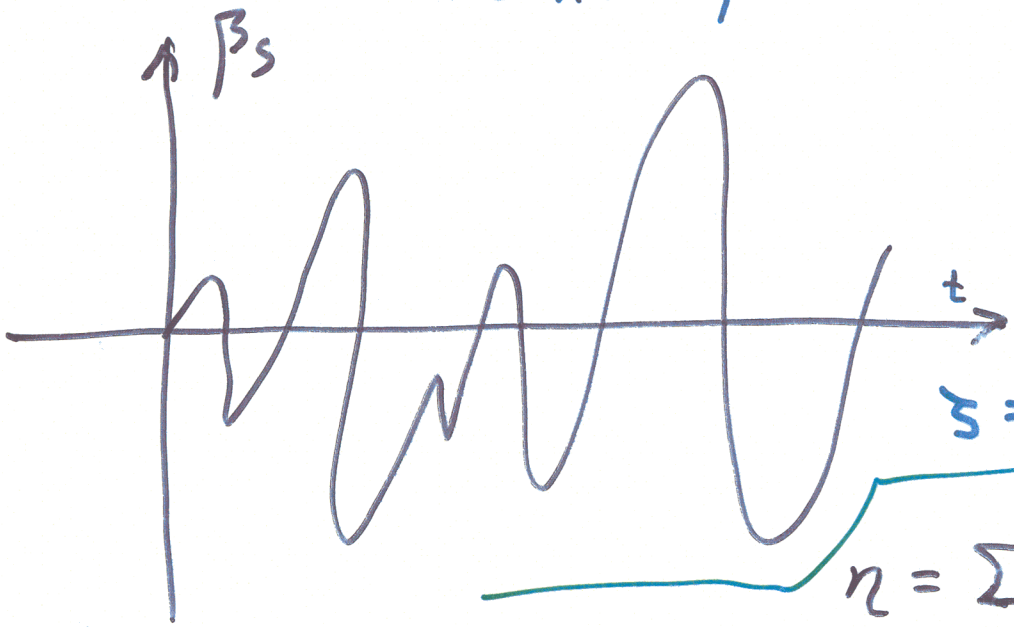
$$b_s \leq C_m |s|^{-m} \qquad \forall s, m$$

Examples of forces

A) white-noise forces

$$\eta = \sum b_s \eta_s(t) e_s(x), \quad \eta_s = \frac{\partial}{\partial t} \beta_s(t)$$

$\{\beta_s\}$ — independent standard Wiener processes; $\beta_s = \beta_s^{\omega}$.



$\eta = \frac{\partial}{\partial t} \xi$, $\xi(t)$ — W. process in H,

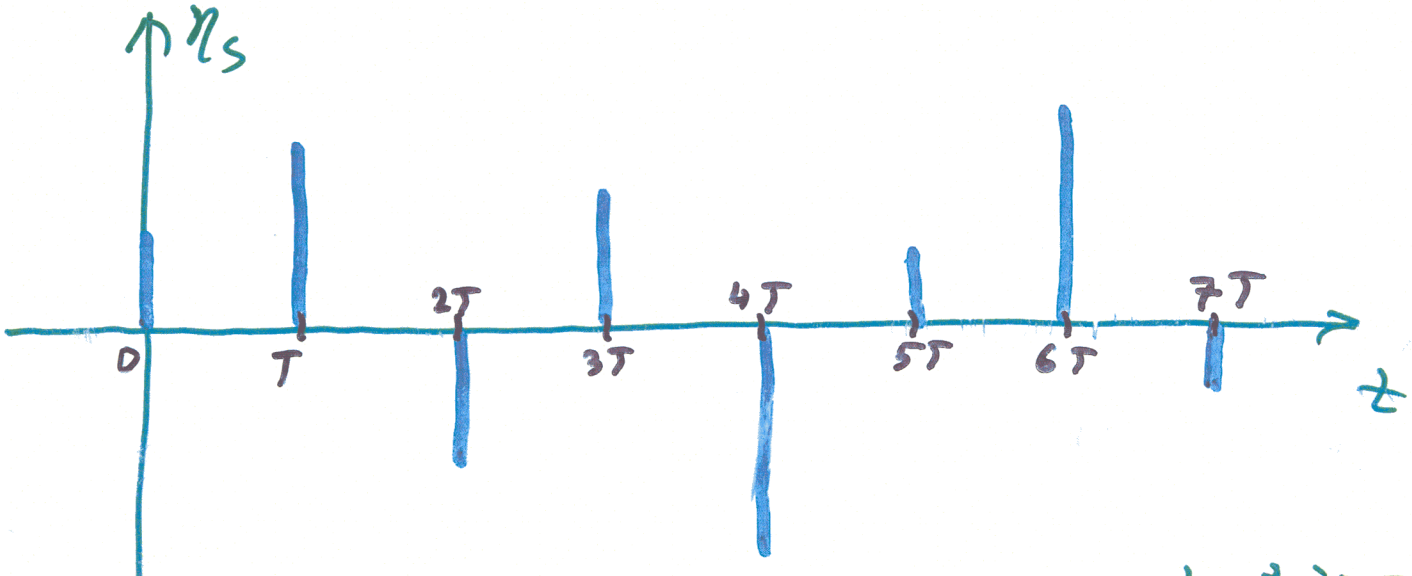
$$\xi = \sum_s b_s \beta_s(t) e_s(x)$$

$$\eta = \sum b_s \eta_s(t) e_s(x)$$

B) Kick-forces

$$\eta_s = \sum_j \xi_s^j \delta(t - jT)$$

$\{\xi_s^j\}$ — i.r.v., stationary in j.



These random processes are not stationary in t, but are T-periodic

The notion of a solution

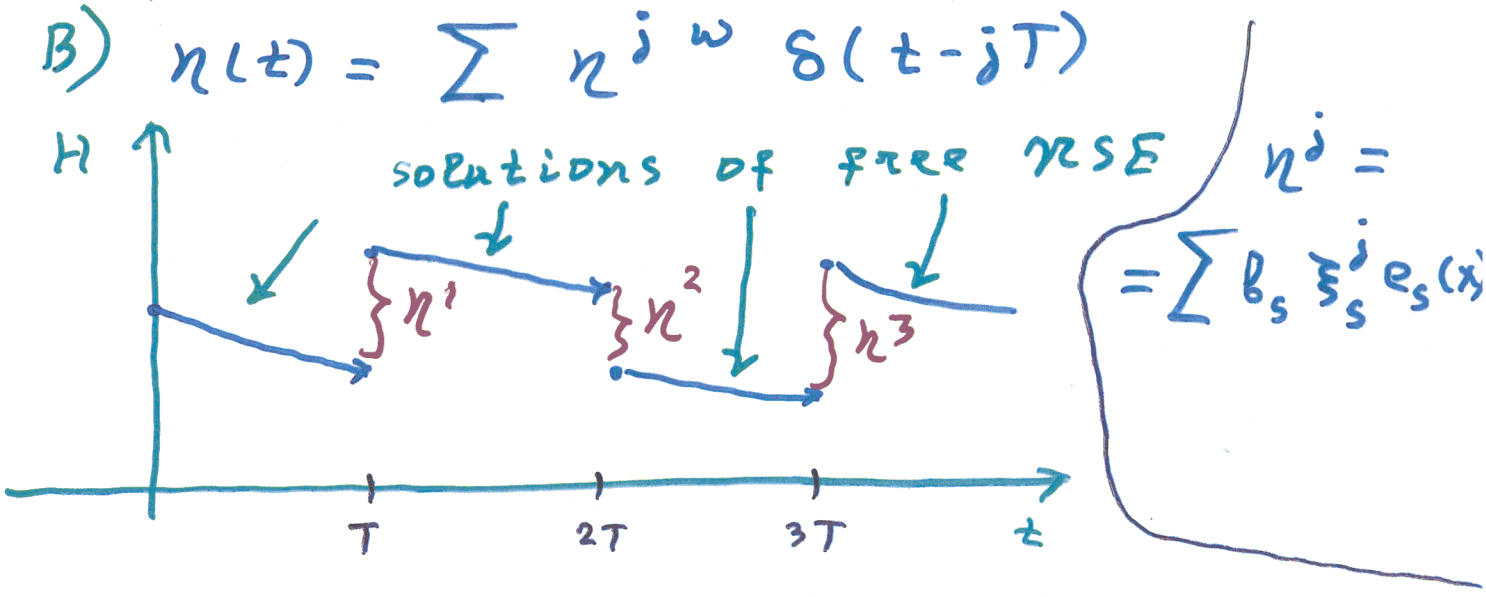
In A) and B), $\eta(t,x)$ is a generalised function of time t. Definition of a solution is straightforward:

A) $\eta(t) = \frac{\partial}{\partial t}\zeta(t)$, $\zeta(\cdot)$ - Wiener process in H

$u(t)$ is a $\mathcal{R}$ solution if see [VF]

$$u(t) - u(0) + \nu \int_0^t (A u(s) + B(u(s)))\, ds = \zeta(t) - \zeta(0), \quad \forall\, t \geq 0.$$

B) $\eta(t) = \sum \eta^j \omega\, \delta(t - jT)$



That is,

$$u(kT) = S^T(u((k-1)T)) + \eta^k,$$

where S^T is the time-T shift along trajectories of free nSE:

$$S^T(u(0)) = u(T),$$

$u(t)$ - solution of (1') with $\eta = 0$

• In case B) I consider $u(t)$ for $t \in \mathbb{N}T$ only

③ All results below are proven for A) and B). Certainly they generalise to forces as in 1) – 3) (+ some technical conditions)

§3 Stat. solutions and stationary measures

$$\dot u + \nu A u + B(u,u) = \eta(t) \qquad (1')$$

$$\mathcal{H} = H \cap C(\mathbb{T}^2;\mathbb{R}^2)$$

Def. $u(t) \in \mathcal{H}$ ($t \geq 0$ or $t \in \mathbb{R}$) is a stationary solution of (1') if $\mathcal{D}(u(t))$ is time-indep. measure in $\mathcal{H}$. $\mathcal{D}(u(t)) = \mu$ — stat. measure

Digression what kind of measure is $\mathcal{D}(u(t)) =: \mu_t$? Let $f : \mathcal{H} \to \mathbb{R}$ be bounded and continuous. Then

$$\int_{\mathcal{H}} f(v)\, \mu_t(dv) = \mathbb{E}^\omega f(u^\omega(t)).$$

If $U \subset \mathcal{H}$, then

$$\mu_t(U) = \mathbb{P}^\omega(u^\omega(t) \in U)$$

Examples. 1) Take $x_0 \in \mathbb{T}^2$. Set $f(u) = u(x_0)$. Then

$$\int f(v)\, \mu_t\, dv = \mathbb{E} f(u(t)) = \mathbb{E}\, u(t, x_0) \in \mathbb{R}^2$$

2) (Correlation Tensor) Fix $x_1, x_2 \in \mathbb{T}^2$. Set

$$f(u) = u(x_1)\, u(x_2)$$ (this is a 2×2 matrix)

$$\int f(v)\, \mu_t(dv) = \mathbb{E} f(u(t)) =$$

$$= \mathbb{E}\, u(t, x_1)\, u(t, x_2)$$

Existence of Stationary Measure

Meta Theorem (Bogoliubov - Krylov)

Stationary solutions and stat. measures exist, if we have "an apriori estimate".

§4. Uniqueness

$$\eta(t) = \sum b_s \eta_s(t) e_s(x), \qquad \eta_s(\cdot) \sim 1$$

Thm 1. Assume that

$$b_s \neq 0 \quad \forall\, |s| \leq N_2 \quad (s \in \mathbb{Z}^2 \setminus 0) \tag{2}$$

Then: a) (1') has a unique stat. measure μ,

b) $\forall$ solution $u(t)$ we have:

$$\mathcal{D}(u(t)) \longrightarrow \mu \quad \text{exp. fast} \tag{3}$$

c) $\exists$ stat. solution $U(t)$, $t \geq 0$, s.t. $\mathcal{D}(U(t)) \equiv \mu$

Note that b) + c) imply:

$$\operatorname{dist}(\mathcal{D}(u(t)), \mathcal{D}(U(t))) \leq C e^{-\varepsilon t}$$

$\forall$ solution $u(t)$

First proved in

[1] S.K., Armen Shirikyan

paper CMPh 213 (2000)

Proved for kick-forces; and for various randomly forced PDEs.

Main Idea of the Proof. Use the Foias-Prodi reduction to reduce (1') to a 1D Gibbs system in $\mathbb{R}^{N_v^2}$. Next prove that ∃! Gibbs measure. This reduction as a tool to prove the uniqueness is a discovery of [1].

Next [E-Mattingly-Sinai] CMPh 224 (2001), 83-106

[Bricmont-Kupiainen-Lefevere] CMPh 230, (2002), 87-132

prove of the uniqueness for white-forced NSE. (also under the assumption (2)). Their results do not imply the convergence (3); for Thm 1 for white-forced eq. see

[SK, A. Shirikyan], J. Math. Pures Appl. 81 (2002), 567-602

also see
[Mattingly] CMPh 230 (2002), 421-462.

See review [SK].

G. K. Batchelor, "The Theory of Homogeneous Turbulence", Introduction, pp. 6-7 :

Instead, we put our faith in the tendency for dynamical systems with a large number of degrees of freedom, and with coupling between these degrees of freedom, to approach a statistical state which is *independent* (partially, if not wholly) of the initial conditions. With this general property of dynamical systems in mind, rather than investigate the motion consequent upon a particular set of initial conditions, we explore the existence of solutions which are asymptotic in the sense that the further passage of time changes them in some simple way only. Since the energy of turbulent motion is being dissipated by viscosity continually, we cannot have the simple situation of the kinetic theory of gases, in which the asymptotic statistical state of the molecular motion is independent of time. The elucidation of the kind of asymptotic statistical state to be expected is the crux of the problem of homogeneous turbulence, and we shall have more to say about it in later chapters. Meanwhile it should be kept in mind that the general method of attacking the problem as formulated at the end of the preceding paragraph is indirect, inasmuch as we attempt to guess the ultimate statistical state of the turbulence and to show that this statistical state would follow from a whole class of different initial conditions.†

§5. Stationary in time and space solutions

Random solutions $u(t,x)$ of (1) that are stationary both in time and in space (i.e. in t and x) are the most important for the theory of turbulence, see [B, F]. Solutions $U(t,x)$ from Thm 1 are stationary in time t. Are they stat. in x?

Let

$$\eta(t,x) = \frac{\partial}{\partial t}\zeta(t,x), \quad \zeta = \sum_{s\in\mathbb{Z}^3\setminus 0} b_s \beta_s(t) e_s(x)$$

assume (2):

$$b_s \neq 0 \qquad \forall\, |s| \le N_2 \tag{2}$$

and assume that

$$b_s = b_{-s} \qquad \forall\, s \tag{2'}$$

I recall that

$$e_s = C_s s^{-1} \sin s\cdot x, \qquad e_{-s} = C_s s^{-1} \cos s\cdot x$$

Lemma. Process $\zeta(t,x)$ is stationary in x.

Since the force $\zeta(t,x)$ is station. in x then the corresponding stationary solution $U(t,x)$ also is stat. in x — since the stat. measure $\mu = \mathcal{D}U(t)$ is unique.

Theorem 2. If (2), (2') hold, then the corresponding stationary solution $U(t,x)$ is stationary in t and x, and the stationary measure μ is stationary w.r.t. translations of x.

In particular, we have:

$$0 = \mathbb{E}\int U(t,x)\,dx \overset{\text{Fubini}}{=} \int (\mathbb{E}\,U(t,x))\,d\mu$$

$$\overset{\text{the thm}}{=} (2\pi)^2\,\mathbb{E}\,U(t,x)$$

$$\Longrightarrow \quad \mathbb{E}\,U(t,x) = 0 \qquad \forall\, t, x$$

or

$$\int_{\mathcal{H}} u(x)\,\mu(du) = 0 \qquad \forall\, x.$$

This relation is used later.

§6. Strong Law of Large Numb.

Theorem 1 and some lemmas, used to prove it, imply the following result.

Let $f : \mathcal{H} \to \mathbb{R}$ be a Lipschitz functional of polynomial growth.

Thm (SLLN). For any $u_0 \in \mathcal{H}$,

$$\frac{1}{t}\int_0^t f(u(s;u_0))\,ds \xrightarrow[t\to\infty]{} \int_{\mathcal{H}} f\,d\mu,$$

almost surely.

See review [SH].

Amplification. $\forall\ \delta > 0$,

$$\left|\frac{1}{t}\int_0^t f(s;u_0)\,ds - \int_{\mathcal{H}} f\,d\mu\right| \le$$

$$\le C\,|f|_{L_\infty}\, t^{-\frac{1}{2}+\delta}$$

$\forall\ t \ge T_\delta(\omega)$.

See [A. Shirikyan] "Law of large numbers and CLT..." subm. to PTRF

Batchelor, The Theory...

which is approximately but not exactly identical with our idealized field. Since the experimental field of turbulence is such that the velocity at a point fixed relative to the grid is a stationary random function of time, we can anticipate, again assuming the applicability of ergodic theory under suitable conditions, that a time average is identical with a probability average for the experimental field.†

It seems, then, that the conventional measuring methods provide a time average, and thereby a probability average, for the experimental, statistically steady field of turbulence, while our theory is concerned with a probability average for the idealized spatially homogeneous field (at the same stage of decay). Hence to the extent that the two fields of turbulence are statistically identical at corresponding stages of the decay, experimental averages will (almost always) be identical with the theoretical averages to be discussed in this book.

2.2. The complete statistical specification of the field of turbulence

We return now to the idealized field which is spatially homogeneous, and consider how this field may be specified. Granted that we know how averages should be taken, what mean quantities are required for a specification that determines the field statistically?

It is a premise of probability theory‡ that a random function $f(\alpha)$, say, defined for all values of α, is determined statistically by the

† There is a simple physical picture which is consistent with this conclusion, just as there is in the case of turbulence which is spatially homogeneous. If we [illegible] of the variation with time of the velocity in the

§7. CLT

Let μ be a stat. measure, and $f: \mathcal{H} \to \mathbb{R}$ be a Lip. functional s.t.

$$\int f(u)\,\mu(du) = 0.$$

Let $u(t)$ be any solution. Then:

$$|\mathbb{E} f(u(t)) - 0| \overset{\text{Thm 1}}{\lesssim} e^{-\delta t}.$$

It means that the process $f(u(t))$ satisfies a 'mixing condition'. Therefore, a CLT a-la (M. Gordin, 1969) applies:

Thm.

$$\mathcal{D}\left(t^{-1/2}\int_0^t f(u(s))\,ds\right) \longrightarrow N(0,\sigma),$$

$$\sigma > 0.$$

See review [SK],
[A shir.] 'Law of large numbers and CLT...', subm. to PTRF

'On large time-scale solutions $u(t)$ of (1') have Gaussian statistics'

§8. Application of CLT.

Let $\eta(t,x) = \frac{\partial}{\partial t}\zeta(t,x)$,

$$\zeta = \sum b_s \beta_s(t) e_s(x).$$

Assume that

$$b_s \neq 0 \quad \forall\, |s| \leq N_\nu, \tag{2}$$

$$b_s = b_{-s} \quad \forall\, s \tag{2'}$$

Fix any $x_0 \in \mathbb{T}^2$.

Then the stat. measure μ is stationary in x (Theorem 2), and for

$$f(u) = u(x_0)$$

we have

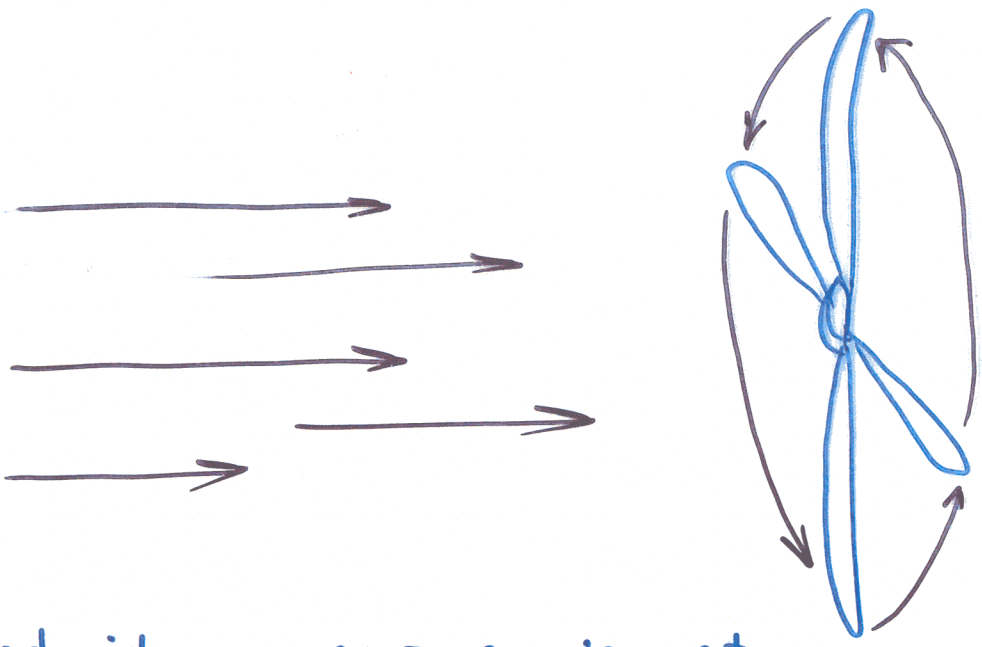
$$\int_{\mathcal{H}} f(u)\, \mu(du) = 0.$$

So CLT applies:

$$\mathcal{D}\left(T^{-1/2} \int_0^T u(s; u_0)\, ds\right) \longrightarrow \mathcal{N}(0, G).$$

Physical Application:

Animometer:



What it measures is not the instantenous velocity $u(0,x)$, but the averaged one, $\text{const} \cdot \int_0^T u(s,x)\,ds$, which is Gaussian!

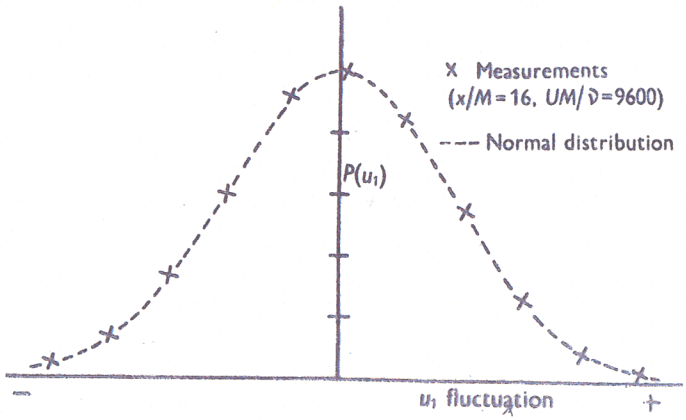


Fig. 8.1. Probability density function of u_1.

experiments By A. Townsend (1947)

from: Batchelor "The theory of homogeneous turbulence", p. 170