Matrix Analytic Methods, Explained Through State Reduction

by

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The QBD Process

The QBD process is a Markov Chain with the following transition matrix

$$Q = \begin{bmatrix} A_{b,b} & A_{b,0} & 0 & \dots & \\ A_{0,b} & A_{0,0} & A_0 & 0 & \ddots \\ 0 & A_2 & A_1 & A_0 & \ddots \\ 0 & \ddots & \ddots & \ddots & \end{bmatrix}$$

First, we discuss the special case

$$Q = \begin{bmatrix} A_{0,0} & A_0 & 0 & \dots \\ A_2 & A_1 & A_0 & \ddots \\ 0 & \ddots & \ddots & \ddots \end{bmatrix}.$$

All matrices square, dimension N
Problem: find the equilibrium vector π:

$$0 = \pi Q, \ \pi \neq 0, \ \pi = [\pi_0, \pi_1, \dots]$$

Equilibrium equations

$$\begin{aligned} 0 &= \pi_0 A_{00} + \pi_1 A_2 \\ 0 &= \pi_n A_0 + \pi_{n+1} A_1 + \pi_{n+2} A_2 \\ 1 &= \sum_{n=0}^{\infty} \pi_n e \end{aligned}$$

Why Important

Useful for processes with two or more state variables

X_1: Level, $X1 \geq 0$, e.g. line 1 in tandem queue
X_2: Phase, $0 \leq X_2 \leq N-1$, e.g. line 2 in tandem queue, type of customer, etc

Example: Customers arrive, join line 1, proceed to line 2 when served by server 1, leave when served by server 2. Blocking at $N-2$.

Event	level	phase	Rate	Condition	A_i
Arr.	$+1$		λ		A_0
1 to 2	-1	$+1$	μ_1	$X_1 > 0$	
				$X2 < N-1$	A_2
Dep.		-1	μ_2	Level > 0	A_1

Find: $p_{ij} = P\{X_1 = i, X_2 = j\}$
$\pi_i = [p_{ij}, j = 0, 1, \ldots, N-1]$

Block Elimination

Assume: $X_1 \le M$.

$$A^{(n)} = \begin{bmatrix} A_{0,0} & A_0 & 0 & \dots & \dots & \dots \\ A_2 & A_1 & A_0 & \ddots & \dots & \dots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & A_2 & A_1 & A_0 \\ 0 & \dots & \dots & \dots & A_2 & Y_M \end{bmatrix}$$

Y_M is given.

$$\begin{aligned} 0 &= \pi_0 A_{00} + \pi_1 A_2 \\ 0 &= \pi_n A_0 + \pi_{n+1} A_1 + \pi_{n+2} A_2, \ n = 0 : M-2 \\ 0 &= \pi_{M-1} A_0 + \pi_M Y_M \end{aligned}$$

Hence, $\pi_M = \pi_{M-1} A_0 (-Y_M)^{-1}$, and

$$\begin{aligned} 0 &= \pi_{M-2} A_0 + \pi_{M-1} A_1 + \pi_M A_2 \\ &= \pi_{M-2} A_0 + \pi_{M-1} A_1 + \pi_{M-1} A_0 (-Y_M)^{-1} \end{aligned}$$

If $Y_{M-1} = A_1 + A_0(-Y_M)^{-1} A_2$, this yields

$$0 = \pi_{M-2} A_0 + \pi_{M-1} Y_{M-1}$$

Censoring and State Reduction

Suppose the process is only observed while in levels 0 to $M-1$, that is, while in level M, the sample path is not observed, and the resulting gap is removed, as in a movie which has been censored.

To find the transition matrix of the censored process, we need to know the state which is reached when the censored part of the process ends. To this end, we need U, the matrix of the expected times in state (M,j) (level M, phase j), given the process started in (M,i) (Fundamental matrix)

$$U = \int_0^\infty \exp(Y_M t)dt = (Y_M)^{-1} \exp(Y_M t)|_0^\infty$$

Hence,

$$U = (-Y_M)^{-1}$$

. This suggests alternative interpretation of

$$Y_{M-1} = A_1 + A_0(-Y_M)^{-1}A_2$$

After eliminating level M, we can eliminate level $M-1$, level $M-2$, ..., level $n+1$ to obtain

$$A^{(n)} = \begin{bmatrix} A_{0,0} & A_0 & 0 & \dots & \dots & \dots \\ A_2 & A_1 & A_0 & \ddots & \dots & \dots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & A_2 & A_1 & A_0 \\ 0 & \dots & \dots & \dots & A_2 & Y_n \end{bmatrix}$$

With

$$\pi_n = \pi_{n-1} A_0 (-Y_n)^{-1}$$

and

$$Y_{n-1} = A_1 + A_0 (-Y_n)^{-1} A_2$$

We also define $R_n = A_0 (-Y_n)^{-1}$

Application of Censoring

$$Y_{n-1} = A_1 + A_0(-Y_n)^{-1}A_2$$

A_0: Rate of going from $n-1$ to n
A_0: Rate of going from n to $n-1$

$(-Y_n)^{-1}$: Expected stay in different states of level n

$R_n = A_0(-Y_n)^{-1}$ is the expected time spent in (n, j) for each time unit spent in $(n-1, i)$.

$G_n = (-Y_n)^{-1}A_2$ is the probability, given the sample function starts in (n, i), it will hit phase j when coming to level $n-1$ for the first time.

$A_0(-Y_n)^{-1}A_2$ is the rate of going from from phase i, level $n-1$, to phase j, level $n-1$, via some states of level n, but avoiding level $n-1$.

Increasing M

Above-n sojourn: sample function beginning when going from level n to $n+1$, and ending when returning to level n.

$A_0(-Y_{n+1})^{-1}A_2$: rate of above-n sojourns.
$n+s$: maximum of above-n sojourn. Exists if process recurrent. Hence:
$P\{s \geq d\} \to 0$ as $d \to \infty$.
$(A_0(-Y_{n+1})^{-1}A_2)_d$: rate of above-$n$ sojourns with maximum less than $n+d$
$(A_0(-Y_{n+1})^{-1}A_2)_d^C$: rate of other above-$n$ sojourns

$$A_0(-Y_{n+1})^{-1}A_2 = P\{s < d\}(A_0(-Y_{n+1})^{-1}A_2)_d \\ +P\{s \geq d\}(A_0(-Y_{n+1})^{-1}A_2)_d^C$$

Hence, if $d \to \infty$, the second term vanishes.

Set M to $n+d$, such that $P\{s \geq d\}$ is small
Means Y_n is close to Y.

The Rate Matrix

We have, according to Wallace (see also Neuts), adjusted to CTMCs

$$\pi_n = \pi_0 R^n$$

with R the minimal solution of

$$0 = A_0 + RA_1 + R^2 A_2$$

$R = A_0(-Y)^{-1}$. Proof: interpretation of R.
R is the expected time spend in phase j of level n on a above-$n-1$ sojourn for each time unit spent in phase i of level $n-1$. Also:
$0 = \pi_{n-1}A_0 + \pi_n Y$ yields $\pi_n = \pi_{n-1}A_0(-Y)^{-1}$.

Neuts also introduced a matrix G, with matches $(-Y)^{-1}A_2$
G is the probability that an above-$n-1$ sojourn starts in $(n-1, i)$ and ends $(n-1, j)$.

Summary for finding Equilibrium Probabilities

1. Select Y_M
2. Iterate $Y_{n-1} = A_1 + A_0(-Y_n)^{-1}A_2$. Stop when $|Y_{n-1} - Y_n| < \epsilon$
3. $R + A_0(-Y_n)^{-1}$
4. Find π_0, by solving
   $0 = \pi_0 A_{00} + \pi_1 A_2 = \pi_0(A_{00} + RA_2)$
5. Norm probabilities to make their sum 1

$\pi_n = \pi_0 R^n$
$1 = \sum_{n=0}^{\infty} \pi_n e = \pi_0 \sum_{n=0}^{\infty} R^n e = \pi_0(I - R)^{-1}e$

Irregular Initial Conditions

Method 1:

$0 = \pi_b A_{bb} + \pi_0 A_{0b}$

yields $\pi_b = \pi_0 A_{0b}(-A_{bb})^{-1}$

Substitute π_b, which yields

A_{00} by $\bar{A}_{00} = A_{00} + A_{0b}(-A_{bb})^{-1} A_{b0}$

Method 2:

$$\begin{bmatrix} A_{bb} & A_{b0} & 0 \\ A_{0b} & A_{00} & A_0 \\ 0 & A_2 & Y \end{bmatrix} \rightarrow \begin{bmatrix} A_{bb} & A_{b0} \\ A_{0b} & \widehat{A}_{00} \end{bmatrix}$$

with $\widehat{A}_{00} = A_{00} + RA_2$

Make Probabilities Sum to 1

$$
\begin{aligned}
1 &= \pi_b e_b + \sum_{n=0}^{\infty} \pi_0 R^n e \\
1 &= \pi_b e_b + \pi_0 (I - R)^{-1} e
\end{aligned}
$$

Suggestion: Choose some state of level 0 or possibly b, and initially set its probability equal to 1. Then calculate sum above, and norm

Improvement over truncation 1. Truncation would yield $Y_n \neq Y$

1. We go truly to infinity for summation

Cyclic Reduction

We have

$$
\begin{aligned}
0 &= \pi_{n-1}A_0 + \pi_n A_1 + \pi_{n+1}A_2 \qquad (1)\\
0 &= \pi_n A_0 + \pi_{n+1}A_1 + \pi_{n+2}A_2 \qquad (2)\\
0 &= \pi_{n+1}A_0 + \pi_{n+2}A_1 + \pi_{n+3}A_2 \qquad (3)
\end{aligned}
$$

$(1)*(-A_1)^{-1}A_0 + (2) + (3)*(-A_1)^{-1}A_2$ yields:

$$
\begin{aligned}
0 &= \pi_{n-1}A_0(-A_1)^{-1}A_0\\
&+ \pi_{n+1}(A_1 + A_2(-A_1)^{-1}A_0 + A_0(-A_1)^{-1}A_2)\\
&+ \pi_{n+3}A_2(-A_1)^{-1}A_2
\end{aligned}
$$

Reduces number of equations by factor of two
Can be applied repeatedly
Leads to the logarithmic algorithm of Latouche and Ramaswami