

# **Twelve Limit Cycles in a Cubic Case of the 16th Hilbert Problem**

PEI YU

Department of Applied Mathematics  
University of Western Ontario  
London, Ontario, Canada N6A 5B7

[pyu@pyu1.apmaths.uwo.ca](mailto:pyu@pyu1.apmaths.uwo.ca)

Collaborator: Maoan Han  
Shanghai Jiaotong University

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# **Outline**

- Introduction
- Computation of Focus Value
- Existence of Small Limit Cycles
- 10 and 12 Limit Cycles
- Possible 14 Limit Cycles
- Conclusion

# Introduction

- The 2nd Part of the Hilbert's 16th Problem:  
For planar system

$$\begin{aligned}\dot{x} &= P(x, y) \\ \dot{y} &= Q(x, y),\end{aligned}$$

What is maximal number of limit cycles?

- A very difficult problem in general,  
in particular, for global analysis.
- Local Problem: Restricted to the neighborhood of isolated fixed points.
  - Degenerate Hopf bifurcation problem
  - Calculation of focus value
  - Criterion for determining center
- Quadratic System [Bautin, 1952]:

$$\begin{aligned}\dot{x} &= \lambda_1 x - y - \lambda_3 x + (2 \lambda_2 + \lambda_5) x y + \lambda_6 y^2 \\ \dot{y} &= x + \lambda_1 y + \lambda_2 x^2 + (2 \lambda_3 + \lambda_4) x y - \lambda_2 y^2\end{aligned}$$

where  $\lambda_i$ 's are constant coefficients.

- At most 3 small limit cycles near the origin.
- A mistake found in the 3rd focus value (Shi, 1980;  
Qin & Liu, 1981; Farr *et al.*, 1989; Yu, 1998).
- One global limit cycle exists (Shi, 1980).

## Introduction (Cond.)

- Cubic System (the origin is a center):

- A necessary and sufficient condition obtained for the system:

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -x + \sum_{2 \leq i+j \leq 3} b_{ij} x^i y^j\end{aligned}$$

to have a center at the origin (Kukles [1944], Cherkas [1978], Christopher and Lloyd [1990], Jin and Wang [1990], Lloyd and Pearson [1992], Wang [1999], W. Ye and Y. Ye [2001], etc.)

- Lloyd and Pearson [1992]: 6 small amplitude limit cycles exist in the vicinity of the origin.

- A necessary and sufficient condition obtained for the central symmetric cubic system:

$$\begin{aligned}\dot{x} &= -y + \sum_{i+j=3} a_{ij} x^i y^j \\ \dot{y} &= x + \sum_{i+j=3} b_{ij} x^i y^j\end{aligned}$$

to have a center at the origin (Malkin [1964]).

- Lunkevich & Sibirskii [1965]: Integrability
  - Liu [1987]: At most 5 small amplitude limit cycles

## Introduction (Cond.)

- **Cubic System (the origin is not a center):**

- Liu and Li [1989] found 5 conditions which are necessary and sufficient for the system:

$$\dot{x} = x + \sum_{i+j=3} a_{ij} x^i y^j$$

$$\dot{y} = -y - \sum_{i+j=3} b_{ij} x^i y^j$$

to be integrable.

- Li and Liu [1991] constructed a central symmetric cubic system:

$$\begin{aligned}\dot{x} &= y(1 + x^2 - a y^2) + \epsilon x(m x^2 + n y^2 - \mu) \\ \dot{y} &= -x(1 - c x^2 + y^2) + \epsilon y(m x^2 + n y^2 - \mu)\end{aligned}$$

where  $a$  and  $c$  are constants satisfying  $a > c > 0$ ,  $ac > 1$ , and  $\epsilon, m, n$  and  $\mu$  are parameters with  $0 < \epsilon \ll 1$ .

They proved that the system can have 11 limit cycles: 8 local (small) and 3 global (large).

- **A More General Cubic System:**

[M. Han, Y. Lin and P. Yu, 2004]

$$\begin{aligned}\dot{x}_1 &= ax_1 + bx_2 + a_{30} x_1^3 + a_{21} x_1^2 x_2 + a_{12} x_1 x_2^2 + a_{03} x_2^3 \\ \dot{x}_2 &= bx_1 + ax_2 + b_{30} x_1^3 + b_{21} x_1^2 x_2 + b_{12} x_1 x_2^2 + b_{03} x_2^3\end{aligned}$$

**Eigenvalues:**  $a \pm |b|$  at  $(x_1, x_2) = (0, 0)$ , where  $a \neq 0$ ,  $|b| > |a| \implies (0, 0)$  is a saddle point.  
 $\implies$  10 small limit cycles.

# Computation of Focus Value

- Hopf Bifurcation and Normal Form Theory

- The key step in finding small amplitude (local) limit cycles is to calculate the focus values.
- Small limit cycle solutions are related to degenerate Hopf bifurcations.
- Analysis of Hopf bifurcation depends on normal form theory.
- Focus values are the normal form coefficients.

- Consider the General System:

$$\dot{\mathbf{x}} = J\mathbf{x} + \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in R^n, \quad \mathbf{f}: R^n \rightarrow R^n$$

$\mathbf{x} = \mathbf{0}$ : an equilibrium.

$J\mathbf{x}$ : linear term,

$\mathbf{f}$ : analytic nonlinear function.

Jacobian: 
$$J_{\mathbf{x}=\mathbf{0}} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & A \end{bmatrix},$$

where  $A \in R^{(n-2) \times (n-2)}$  is stable.

# Computation of Focus Value (Cond.)

- Computation of Normal Forms

- Poincaré Normal Form Theory
- Birkhoff Normal Form Theory
- Taken’s Normal Form Theory

- Other Approaches

- Lyapunov constants
- Succession function
- Time Averaging
- Lindstedt-Poincaré method
- Lyapunov-Schmidt reduction
- KB and KBM methods
- Multiple time scales (MTS)
- Intrinsic harmonic balancing
- etc.

## Computation of Focus Value (Cond.)

- MTS:

$$\dot{r} = v_1 r^3 + v_2 r^5 + v_3 r^7 + \dots$$

$$\dot{\theta} = 1 + \beta_1 r^2 + \beta_2 r^4 + \beta_3 r^6 + \dots$$

where  $r$  and  $\theta$  denote the amplitude and phase of motion, respectively.

- $v_1, v_2, v_3, \dots$  are called the 1st, 2nd, 3rd, ... order focus values.

- An Example:

$$\dot{x}_1 = x_2 + x_1^2 - x_1 x_3$$

$$\dot{x}_2 = -x_1 + x_2^2 + x_1 x_4 + x_2^3$$

$$\dot{x}_3 = -x_3 + x_1^2$$

$$\dot{x}_4 = -x_4 + x_5 + x_1^2$$

$$\dot{x}_5 = -x_4 - x_5 + x_2^2$$

- Normal Form Using MTS (Yu [1998]):

$$\dot{r} = \frac{3}{40} r^3 - \frac{14867}{68000} r^5 - \frac{26912070343}{103873536000} r^7 + \dots$$

$$\dot{\theta} = 1 - \frac{7}{12} r^2 + \frac{8093503}{14688000} r^4 - \frac{1887495055097}{3895257600000} r^6 + \dots$$

**Maple programs:** For numbers, symbolic notations and their combinations.

## Existence of Small Limit Cycles

- The System:

$$(*) \begin{cases} \dot{x}_1 = ax_1 + bx_2 + a_{30}x_1^3 + a_{21}x_1^2x_2 + a_{12}x_1x_2^2 + a_{03}x_2^3 \\ \dot{x}_2 = bx_1 + ax_2 + b_{30}x_1^3 + b_{21}x_1^2x_2 + b_{12}x_1x_2^2 + b_{03}x_2^3 \end{cases}$$

$a_{03} = -b, \quad b_{03} = -a \implies \text{two foci } (0, 1), (0, -1)$

$a_{12} = a \implies \text{Hopf bifurcation}$

$b_{12} = \frac{1+4a^2}{2b} - b \implies \omega = 1$

- For Simplicity in Computation: Choose

$$a = \frac{1}{2}, \quad b = 1 \implies b_{12} = 0$$

$$(**) \begin{cases} \dot{x}_1 = \frac{1}{2}x_1 + x_2 + a_{30}x_1^3 + a_{21}x_1^2x_2 + \frac{1}{2}x_1x_2^2 - x_2^3 \\ \dot{x}_2 = x_1 + \frac{1}{2}x_2 + b_{30}x_1^3 + b_{21}x_1^2x_2 - \frac{1}{2}x_2^3 \end{cases}$$

- Transformed to the Point  $(0, 1)$ :

$$\begin{aligned} \dot{u}_1 &= u_2 + (a_{21} - b_{21})u_1^2 + (2b_{21} - 2a_{21} - 1)u_1u_2 + (a_{21} - b_{21} - \frac{1}{2})u_2^2 \\ &\quad + (a_{30} - b_{30})u_1^3 + (3b_{30} - 3a_{30} - b_{21} - a_{21})u_1^2u_2 \\ &\quad + (3a_{30} - 3b_{30} + 2a_{21} - 2b_{21} + \frac{1}{2})u_1u_2^2 + (b_{30} - a_{30} + b_{21} - a_{21})u_2^3 \\ \dot{u}_2 &= -u_1 - b_{21}u_1^2 + 2b_{21}u_1u_2 - (b_{21} - \frac{3}{2})u_2^2 \\ &\quad - b_{30}u_1^3 + (3b_{30} + b_{21})u_1^2u_2 - (3b_{30} + 2b_{21})u_1u_2^2 + (b_{30} + b_{21} - \frac{1}{2})u_2^3 \end{aligned}$$

## Existence of Small Limit Cycles (Cond.)

- Focus Values:

$$v_1 = \frac{3}{4}a_{30} + \frac{1}{2}a_{21}^2 - a_{21}b_{21} + \frac{3}{4}b_{21} = 0$$

$$\implies a_{30} = -b_{21} + \frac{4}{3}b_{21}a_{21} - \frac{2}{3}a_{21}^2$$

$$v_2 = \frac{1}{3}b_{30} \left[ 5 \underbrace{b_{21}(2a_{21} - 1)}_{- \frac{1}{18}(a_{21} - 2b_{21})} - (5a_{21} - 3)(a_{21} + 1) \right] \\ - \frac{1}{18}(a_{21} - 2b_{21}) \left[ (10a_{21}^3 - 22a_{21}^2 - 9a_{21} + 18) - 10b_{21}(a_{21} - 1)(2a_{21} - 3) \right]$$

There are two cases depending upon if

$$\underline{b_{21}(2a_{21} - 1) - (5a_{21} - 3)(a_{21} + 1) = 0}.$$

Each case has several sub-cases. It has a total of 16 sub-cases, among which 7 are limit cycles, and the remains are centers.

These cases are listed in Table 1.

Computer output of  $v_i$ 's:

$$v_3 = \dots \text{ (24 lines)}$$

$$v_4 = \dots \text{ (78 lines)}$$

$$v_5 = \dots \text{ (145 lines)}$$

# Existence of Limit Cycles (Cond.)

**Table 1. Classification of Point  $(x_1, x_2) = (0, 1)$ .**

$a$	Other Conditions	Center/Focus
$a = 0$	$a_{21} = 0, a_{30} = -b_{21}/3$	Center (?) <b>Rigorous proof needed !</b>
	$b_{21} = a_{30} = 0$	
	$a_{21} = 2b, a_{30} = b_{21}, b_{30} = b$	
$a = \frac{1}{2}$	$a_{21} = 0, a_{30} = -b_{21}, b_{30} = -2b_{21}$	Center (?) <b>Rigorous proof needed !</b>
	$a_{21} = 2b_{21}, a_{30} = -b_{21}, b_{30} = 0$	
	$a_{21} = \frac{2}{3}, a_{30} = -\frac{1}{27}(3b_{21}+8), b_{30} = \frac{2}{27}(15b_{21}-14)$	
	$a_{21} = \frac{1}{4}(1 \pm \sqrt{5}), b_{21} = \frac{1}{20}(10 \pm \sqrt{5}),$ $a_{30} = \frac{1}{20}(-10 \pm \sqrt{5}), b_{30} = \frac{1}{4}(-3 \pm \sqrt{5})$	
	$b_{21} = \frac{2a_{21}^2-3}{2(2a_{21}-3)}, a_{30} = \frac{2a_{21}^2-4a_{21}+3}{2(2a_{21}-3)}, b_{30} = a_{21}-1$	
	$b_{21} = \pm \sqrt{\frac{3}{35}}b, a_{30} = \pm \frac{(2a_{21}-b)}{\sqrt{105}b}, b_{30} = \frac{3a_{21}-b}{5}$	
$a = 0$	$b_{21} = \pm \sqrt{\frac{3}{35}}b, a_{30} = \pm \frac{(2a_{21}-b)}{\sqrt{105}b}, b_{30} = \frac{3a_{21}-b}{5}$	<b>Focus :</b> $v_1 = v_2 = v_3 = 0,$ $v_4 = \mp \frac{32\sqrt{210}a_{21}(a_{21}-2b)}{91875b^2}$ <b>(4 Limit Cycles)</b>
$a = \frac{1}{2}$	$a_{21} = 0, b_{21} = \frac{3}{5}, a_{30} = -\frac{3}{5}, b_{30} = -\frac{28}{25}$	<b>Focus :</b> $v_1 = v_2 = v_3 = 0,$ $v_4 = -\frac{3024}{78125} \approx -0.0387$ <b>(4 Limit Cycles)</b>
	$a_{21} = \frac{1}{4}(1 \pm \sqrt{5}), b_{21} = \frac{1}{20}(10 \pm \sqrt{5}),$ $a_{30} = \frac{1}{20}(-10 \pm \sqrt{5}), b_{30} = \frac{1}{40}(-35 \pm 11\sqrt{5})$	<b>Focus :</b> $v_1 = v_2 = v_3 = 0,$ $v_4 = \frac{3003 \mp 1281\sqrt{5}}{64000} \approx \begin{cases} 0.0021 \\ 0.0916 \end{cases}$ <b>(4 Limit Cycles)</b>
	$a_{21} = \frac{417 \mp 15\sqrt{385}}{3232}, b_{21} = \frac{49027 \pm 643\sqrt{385}}{71104},$ $a_{30} = \frac{-43307 \mp 779\sqrt{385}}{71104}, b_{30} = \frac{-22115 \mp 463\sqrt{385}}{17776}$	<b>Focus :</b> $v_1 = v_2 = v_3 = v_4 = 0,$ $v_5 = -\frac{875(6172327 \pm 306137\sqrt{385})}{27806783488}$ $\approx \begin{cases} -0.3832 \\ -0.0052 \end{cases}$ <b>(5 Limit Cycles)</b>

# Existence of Limit Cycles (Cond.)

- **Sufficient Conditions:**

○ **Theorem 1.** Consider the equation:

$$\dot{r} = r(v_0 + v_1 r^2 + \cdots + v_k r^{2k}) \quad (k \geq 1). \quad (1)$$

Let  $x = r^2$ , and  $a_{i-1} = \frac{v_i}{v_k}$ , for  $i = 1, 2, \dots, k$ , we have

$$x^k + a_{k-1} x^{k-1} + \cdots + a_1 x + a_0 = 0, \quad (2)$$

Let  $N_0, N_1, \dots, N_{k-1}$  be constants such that the following equation

$$u^k + N_{k-1} u^{k-1} + \cdots + N_1 u + N_0 = 0$$

has  $k$  simple positive roots  $u_j$ ,  $j = 1, 2, \dots, k$ . Then for any continuous functions  $a_j$  satisfying

$$a_j(\epsilon) = N_j \epsilon^{k-j} + o(\epsilon^{k-j}), \quad j = 0, 1, \dots, k-1,$$

equation (2) has exactly  $k$  simple positive roots in the form of  $x_j = \epsilon u_j + o(\epsilon)$  for sufficiently small  $\epsilon > 0$ . Therefore, if

$$v_j = N_j \epsilon^{k-j} + o(\epsilon^{k-j}) \quad \text{for } j = 0, 1, \dots, k,$$

with  $N_k \neq 0$ , then equation (1) has exactly  $k$  real positive roots, i.e., the original system has exactly  $k$  limit cycles in a neighborhood of the origin for sufficiently small  $\epsilon > 0$ .

○ **Main idea:**  $v_i v_{i+1} < 0$  ( $i = 0, 1, 2, \dots, k-1$ ), and

$$0 < |v_0| \ll |v_1| \ll \cdots \ll |v_{k-1}| \ll |v_k|$$

**For separated parameters: one for each order.**

## Existence of Limit Cycles (Cond.)

- ⊕ **Theorem 2.** Suppose the following condition

$$v_j = a_j(\epsilon_1, \epsilon_2, \dots, \epsilon_k), \quad j = 0, 1, \dots, k.$$

holds, and further assume that

$$a_k(0) \neq 0, \quad a_j(0) = 0, \quad \text{for } j = 0, 1, \dots, k-1,$$

$$\text{and} \quad \det \left[ \frac{\partial(a_0, a_1, \dots, a_{k-1})}{\partial(\epsilon_1, \epsilon_2, \dots, \epsilon_k)}(0) \right] \neq 0.$$

Then for any given  $\epsilon_0 > 0$ , there exist  $\epsilon_1, \epsilon_2, \dots, \epsilon_k$  and  $\delta > 0$  with  $|\epsilon_j| < \epsilon_0$ ,  $j = 1, 2, \dots, k$  such that equation (1) has exactly  $k$  real positive roots, i.e., the original system has exactly  $k$  limit cycles in a  $\delta$ -ball with the center at the origin.

- ⊕ **For non-separated parameters:** solving  $v_i = 0$  simultaneously to determine several parameters.

## 8 Limit Cycles

- Phase Portrait for 8 Limit Cycles

**Take**  $\epsilon_1 = 0.05$ ,  $\epsilon_2 = 0.0008$ ,  $\epsilon_3 = 0.000002$ ,  $\epsilon_4 = 0.000000006$

$$\begin{aligned}\dot{x}_1 &= 0.49999994 x_1 + x_2 - 0.600798 x_1^3 + 0.5 x_1 x_2^2 - x_2^3, \\ \dot{x}_2 &= x_1 + 0.49999994 x_2 - 1.07 x_1^3 + 0.6008 x_1^2 x_2 - 0.5 x_2^3,\end{aligned}$$

$(0, \pm 1)$ : **4th-order fine foci**,  $(0, 0)$ ,  $(\pm 1.273830, \pm 1.007518)$ ,  $(\pm 0.650790, \mp 0.737127)$ : **saddle points**,  $(\pm 1.045669, \pm 0.155235)$ : **unstable foci**.

**Limit cycles:**  $r_1 = 0.00288$ ,  $r_2 = 0.00309$ ,  $r_3 = 0.00904$ ,  $r_4 = 0.0967$ .

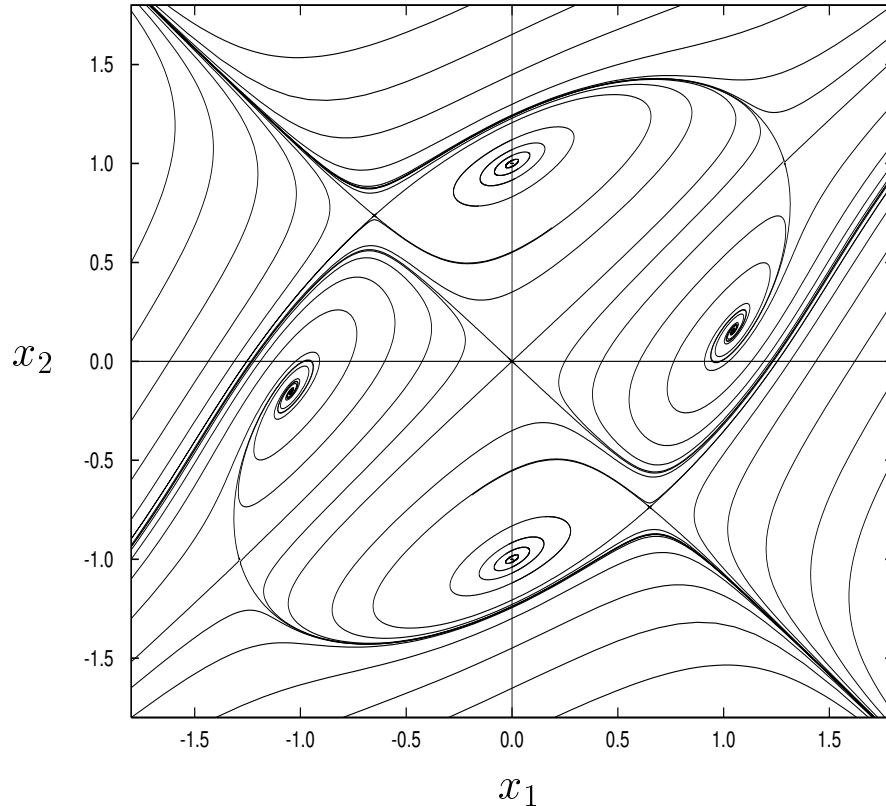


Fig. 1. The phase portrait for 8 limit cycles.

## 8 Limit Cycles (Cond.)

- No Global Limit Cycles from Saddle Points

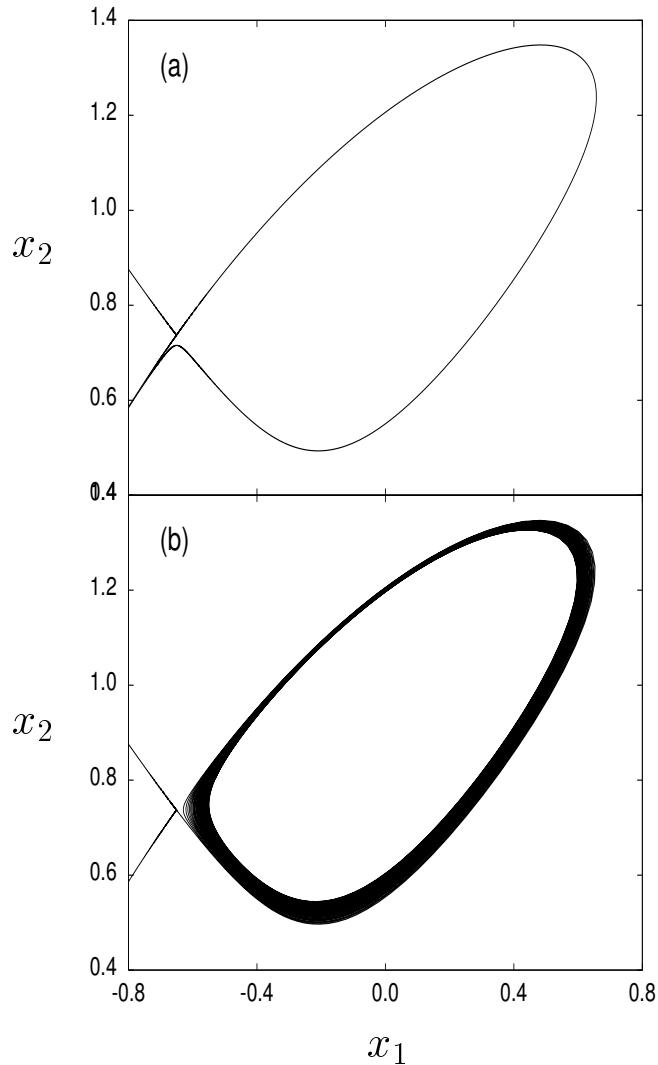


Fig. 2. The trajectories connecting the saddle point  $(-0.650780, 0.737127)$ .

# 10 Limit Cycles

- Under the Conditions:

$$a = a_{12} = -b_{03} = \frac{1}{2}, \quad b = -a_{03} = 1, \quad b_{12} = 0,$$

and  $a_{30} = -b_{21} + \frac{4}{3}b_{21}a_{21} - \frac{2}{3}a_{21}^2 \implies v_1 = 0$

$$b_{30} = \frac{(a_{21}-2b_{21})[(10a_{21}^3-22a_{21}^2-9a_{21}+18)-10b_{21}(a_{21}-1)(2a_{21}-3)]}{6K_1}$$

$$\implies v_2 = 0$$

then  $v_3$  and  $v_4$  are given by

$$v_3 = \frac{35K_2}{288K_1^2} f_1 \quad \text{and} \quad v_4 = \frac{7K_2}{10368K_1^3} f_2,$$

where the factors  $K_2$ ,  $f_1$  and  $f_2$  are

$$K_2 = a_{21}(3a_{21}-2)(a_{21}-2b_{21})(6b_{21}-4b_{21}a_{21}+2a_{21}^2-3)$$

$$f_1 = 20a_{21}^4 - 120a_{21}^3b_{21} + 240a_{21}^2b_{21}^2 - 160a_{21}b_{21}^3 + 38a_{21}^3 - 132a_{21}^2b_{21} \\ + 72a_{21}b_{21}^2 + 80b_{21}^3 + 8a_{21}^2 + 58a_{21}b_{21} - 148b_{21}^2 - 27a_{21} + 90b_{21} - 18$$

$$f_2 = 99400a_{21}^8 - 1098400a_{21}^7b_{21} + 5066400a_{21}^6b_{21}^2 - 12499200a_{21}^5b_{21}^3 \\ + 17417600a_{21}^4b_{21}^4 - 13017600a_{21}^3b_{21}^5 + 4083200a_{21}^2b_{21}^6 + 276180a_{21}^7 \\ - 2386360a_{21}^6b_{21} + 7941600a_{21}^5b_{21}^2 + 5792960b_{21}^4a_{21}^3 - 11960960b_{21}^3a_{21}^4 \\ + 4110720a_{21}^2b_{21}^5 - 4083200a_{21}b_{21}^6 + 226728a_{21}^6 - 1066352a_{21}^5b_{21} \\ + 194744a_{21}^4b_{21}^2 + 6567392a_{21}^3b_{21}^3 - 12366272a_{21}^2b_{21}^4 + 620096a_{21}b_{21}^5 \\ + 1020800b_{21}^6 - 38648a_{21}^5 + 1084512a_{21}^4b_{21} - 5022028a_{21}^3b_{21}^2 \\ + 7956904a_{21}^2b_{21}^3 - 2632944a_{21}b_{21}^4 - 2500960b_{21}^5 - 175114a_{21}^4 \\ + 1260968a_{21}^3b_{21} - 2394864a_{21}^2b_{21}^2 - 2008a_{21}b_{21}^3 + 2447888b_{21}^4 \\ - 138919a_{21}^3 + 481530a_{21}^2b_{21} + 217080a_{21}b_{21}^2 - 1344400b_{21}^3 \\ - 55020a_{21}^2 - 84984a_{21}b_{21} + 536712b_{21}^2 + 22824a_{21} - 162720b_{21} + 24408$$

## 10 Limit Cycles (Cond.)

**Eliminating  $b_{21}$  from  $f_1$  (3rd-deg polynomial of  $b_{21}$ ) and  $f_2$  (6th-deg polynomial of  $b_{21}$ ):**

$\implies b_{21} = b_{21}(a_{21})$  and

$$\frac{a_{21}(a_{21}-1)(4a_{21}^2-2a_{21}-1)}{\text{centers}} \frac{(3232a_{21}^2-834a_{21}+27)}{} = 0$$

$\implies$

$$a_{21} = \frac{417 \pm 15\sqrt{385}}{3232} \equiv a_{21}^*, \quad b_{21} = \frac{49027 \mp 643\sqrt{385}}{71104} \equiv b_{21}^*$$

$$a_{30} = \frac{-43307 \pm 779\sqrt{385}}{71104}, \quad b_{30} = \frac{-22115 \pm 463\sqrt{385}}{17776}$$

**such that**  $v_1 = v_2 = v_3 = v_4 = 0$ ,

**and**  $v_5 = -\frac{875(6172327 \mp 306137\sqrt{385})}{27806783488} \approx \begin{cases} -0.0052 \\ -0.3832 \end{cases}$

**Give the following perturbations:**

$$a_{21} = a_{21}^* - \epsilon_1$$

$$b_{21} = b_{21}^* - \epsilon_2$$

$$b_{30} = b_{30}^*(a_{21}^* - \epsilon_1, b_{21}^* - \epsilon_2) + \epsilon_3$$

$$a_{30} = a_{30}^*(b_{30}^*(a_{21}^* - \epsilon_1, b_{21}^* - \epsilon_2) + \epsilon_3), a_{21}^* - \epsilon_1, b_{21}^* - \epsilon_2) - \epsilon_4$$

$$a = \frac{1}{2} + \epsilon_5$$

**where**  $0 < \epsilon_5 \ll \epsilon_4 \ll \epsilon_3 \ll \epsilon_2$ , **and**

$$0.1437 \epsilon_1 < \epsilon_2 < 0.2126 \epsilon_1, \quad \epsilon_1 \ll 1$$

**resulting in**

$$0 < v_1 \ll -v_3 \ll v_5 \ll -v_7 \ll v_9 \ll -v_{11}$$

## 10 Limit Cycles (Cond.)

- Phase Portrait for  $v_5 \approx -0.0052$ .

Take  $\epsilon_1 = 0.1 \times 10^{-1}$ ,  $\epsilon_2 = 0.144 \times 10^{-2}$ ,  
 $\epsilon_3 = 0.2 \times 10^{-7}$ ,  $\epsilon_4 = 0.6 \times 10^{-11}$ ,  $\epsilon_5 = 0.2 \times 10^{-14}$

$$\begin{aligned}\dot{x}_1 &= 0.49999999999998 x_1 + x_2 - 0.397020785466680 x_1^3 + 0.210087021366098 x_1^2 x_2 + 0.5 x_1 x_2^2 - x_2^3 \\ \dot{x}_2 &= x_1 + 0.49999999999998 x_2 - 0.742338874076202 x_1^3 + 0.510632864429088 x_1^2 x_2 - 0.5 x_2^3\end{aligned}$$

$(0, \pm 1)$ : **5th-order fine foci**,  $(0, 0)$ ,  $(\pm 1.322688, \pm 1.475976)$ ,  $(\pm 0.810283, \mp 0.758273)$ : **saddle points**,  $(\pm 1.196814, \pm 0.061615)$ : **unstable foci**.

Limit cycles:  $r_1 = 0.00041$ ,  $r_2 = 0.00291$ ,  $r_3 = 0.00632$ ,  $r_4 = 0.00831$ ,  $r_5 = 0.03799$ .

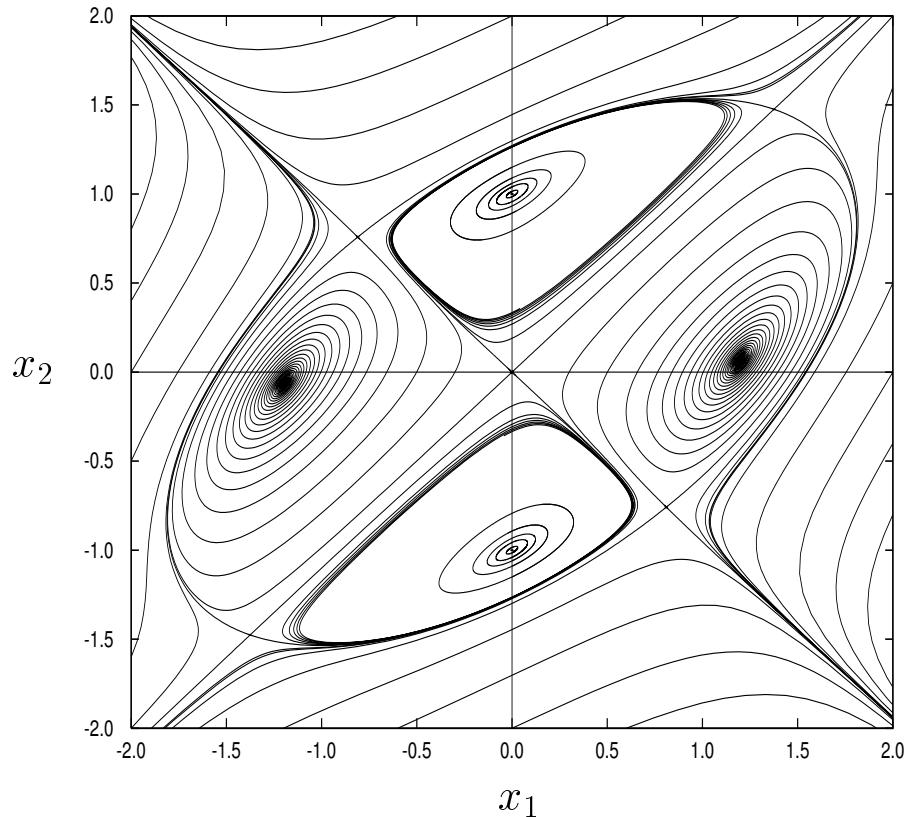


Fig. 3. The phase portrait for the first 10 limit cycles.

## 10 Limit Cycles (Cond.)

- Phase Portrait for  $v_5 \approx -0.3832$ .

**Take**  $\epsilon_1 = 0.2 \times 10^{-1}$ ,  $\epsilon_2 = 0.7 \times 10^{-2}$ ,  
 $\epsilon_3 = 0.2 \times 10^{-7}$ ,  $\epsilon_4 = 0.2 \times 10^{-10}$ ,  $\epsilon_5 = 0.1 \times 10^{-14}$

$$\begin{aligned}\dot{x}_1 &= 0.4999999999999999 x_1 + x_2 - 0.812629039426331 x_1^3 + 0.030957533089348 x_1^2 x_2 + 0.5 x_1 x_2^2 - x_2^3 \\ \dot{x}_2 &= x_1 + 0.4999999999999999 x_2 - 1.71030313062685 x_1^3 + 0.846949412798635 x_1^2 x_2 - 0.5 x_2^3\end{aligned}$$

(0, ±1): **5th-order fine foci**, (0, 0), ( $\pm 1.038313, \pm 1.128288$ ), ( $\pm 0.466513, \mp 0.794779$ ): **saddle points**, ( $\pm 0.747376, \pm 0.034333$ ): **unstable foci**.

**Limit cycles:**  $r_1 = 0.00004$ ,  $r_2 = 0.00023$ ,  $r_3 = 0.00128$ ,  $r_4 = 0.00463$ ,  $r_5 = 0.07468$ .

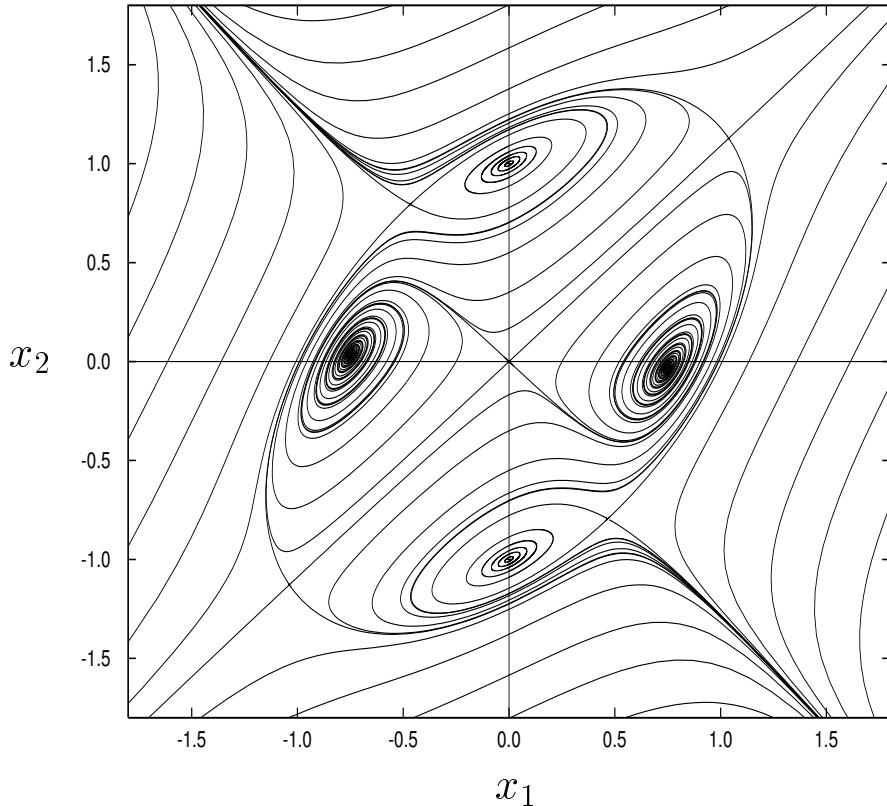


Fig. 4. The phase portrait for the second 10 limit cycles.

## 12 Limit Cycles

- The Cubic System:

$$\begin{aligned}\dot{x}_1 &= a x_1 + b x_2 + a_{30} x_1^3 + a_{21} x_1^2 x_2 + a_{12} x_1 x_2^2 + a_{03} x_2^3 \\ \dot{x}_2 &= b x_1 + a x_2 + b_{30} x_1^3 + b_{21} x_1^2 x_2 + b_{12} x_1 x_2^2 + b_{03} x_2^3\end{aligned}$$

- Under the Conditions:

$$(***) \quad a_{03} = -b, \quad a_{12} = -b_{03} = a, \quad b_{12} = \frac{1+4a^2}{2b} - b \quad (\neq 0),$$

the origin  $(0, 0)$  is a saddle point, and  $(0, 1)$  and  $(0, -1)$  are linear centers.

- There are 6 free parameters  $a, b, a_{21}, b_{21}, a_{20}$  and  $b_{30}$ . Therefore, it might be possible to choose them such that

$$v_1 = v_2 = v_3 = v_4 = v_5 = 0, \text{ but } v_6 \neq 0,$$

and leave 1 parameter for linear perturbation.

- Note that under the condition  $(***)$ , the system has a normalized frequency  $\omega = 1$ .

## 12 Limit Cycles (Cond.)

- Transformation:

$$x = 2 b u_1, \quad y = 1 + 2 a u_1 - u_2$$

- Transformed System Near the Point  $(0, 1)$ :

$$\begin{aligned}\dot{u}_1 &= u_2 + 2(a_{21}b - a^2)u_1^2 + 4a u_1 u_2 - \frac{3}{2}u_2^2 \\ &\quad + 4b(a_{21}b + a_{30}b)u_1^3 - 2(a_{21}b - a^2)u_1^2 u_2 - 2a u_1 u_2^2 + \frac{1}{2}u_2^3\end{aligned}$$

$$\begin{aligned}\dot{u}_2 &= -u_1 + 4(2ab^2 - 2a^3 - b_{21}b^2 + a_{21}ab - a)u_1^2 \\ &\quad + 2(2a^2 - 2b^2 + 1)u_1 u_2 + 4(2a_{21}a^2b - 2b_{21}ab^2 \\ &\quad + 2a_{30}ab^2 - 2b_{30}b^3 + 2a^2b^2 - 2a^4 - a^2)u_1^3 \\ &\quad + 4(2a^3 - 2ab^2 + a - a_{21}ab + b_{21}b^2)u_1^2 u_2 \\ &\quad - (2a^2 - b^2 + 1)u_1 u_2^2\end{aligned}$$

- Based on the above system, 6 focus values have been obtained using MTS method:

$$v_i = v_i(a, b, a_{21}, b_{21}, a_{30}, b_{30}), \quad i = 1, 2, \dots, 6.$$

## 12 Limit Cycles (Cond.)

$$v_1 = \frac{1}{2} b \left[ 2ab(4a^2 - 4b^2 + 3) + 3ba_{30} + b_{21}b(4b^2 - 4a_{21}b - 1) + 2a_{21}a(2b^2 + 2ba_{21} - 4a^2 - 1) \right].$$

$$v_1 = 0 \implies$$

$$\begin{aligned} a_{30} &= a_{30}(b, a_{21}, a, b_{21}) \\ &= -\frac{1}{3} \left[ b_{21}(4b^2 - 4ba_{21} - 1) + 2\left(\frac{a}{b}\right)a_{21}(2b^2 + 2ba_{21} - 4a^2 - 1) \right. \\ &\quad \left. + 2ab(4a^2 - 4b^2 + 3) \right] \quad (b \neq 0) \end{aligned}$$

**Computer output:**

$v_2 = \dots$  (**26** lines)

$v_3 = \dots$  (**180** lines)

$v_4 = \dots$  (**774** lines)

$v_5 = \dots$  (**2740** lines)

$v_6 = \dots$  (**7855** lines)

## 12 Limit Cycles (Cond.)

$$v_2 = \frac{1}{9} b^2 (b_N + 6 b b_D b_{30})$$

$$\begin{aligned} b_N = & 640 a^7 (a_{21} - b)^2 \\ & - 16 a^5 [40 b (a_{21} - b)^2 (a_{21} + 2 b) - 4 (a_{21} - b) (7 a_{21} - 12 b) + 15] \\ & + 160 a^4 b b_{21} (a_{21} - b) [4 b (a_{21} - b) - 1] \\ & + a^3 [160 b^2 (a_{21} - b)^2 (a_{21} + 2 b)^2 \\ & \quad + 16 b (a_{21} - b) (54 b^2 + 5 b a_{21} - 14 a_{21}^2) \\ & \quad + 8 (69 b^2 - 18 b a_{21} + 9 a_{21}^2) - 168] \\ & - a^2 b b_{21} [320 b^2 (a_{21} - b)^2 (a_{21} + 2 b) \\ & \quad - 16 b (a_{21} - b) (19 a_{21} - 14 b) + 24 (3 a_{21} + 2 b)] \\ & + a [160 b^4 b_{21}^2 (a_{21} - b)^2 \\ & \quad - 8 b^3 (a_{21} - b) (12 b^2 + 24 b a_{21} + 9 a_{21}^2 + 10 b_{21}^2) \\ & \quad - 12 b^2 (14 b^2 + 7 b a_{21} - 6 a_{21}^2) + 12 b (8 b + 3 a_{21}) - 27] \\ & + b_{21} [24 b^4 (a_{21} - b) (3 a_{21} + 2 b) + 6 b^2 (10 b^2 - 3)] \\ b_D = & a [(2 a_{21} b (10 b^2 + 10 b a_{21} - 1) - (40 b^4 - 32 b^2 + 9)] \\ & + 10 (b_{21} b^2 + 2 a^3) (2 b^2 - 2 b a_{21} - 1). \end{aligned}$$

•  $v_2 = 0 \implies \text{two cases:}$

**Generic case:**  $b_D \neq 0$

**Special case:**  $b_D = 0$

## 12 Limit Cycles (Cond.)

- Generic case ( $b_D \neq 0$ ):  $b_{30}$  is uniquely determined,

$$b_{30} = -\frac{b_N}{6 b b_D} \quad (b \neq 0, b_D \neq 0)$$

- Special case ( $b_D = 0$ ):  $b_{30}$  is not used at this order.

$b_D = 0 \implies (\text{needs 2 parameters for } v_2 = 0)$

$$b_{21} = -a \left\{ \frac{2a^2}{b^2} + \frac{\left[ (2a_{21}b(10b^2 + 10ba_{21} - 1) - (40b^4 - 32b^2 + 9)) \right]}{10b^2(2b^2 - 2ba_{21} - 1)} \right\}$$
$$(b \neq 0, a \neq 0)$$

$\implies$

$$v_2 = \frac{2ab^2}{15(2b^2 - 2ba_{21} + 1)^2} (2ab - 2aa_{21} + 2ba_{21} + 1 - 2b^2)$$
$$\times (2ab - 2aa_{21} - 2ba_{21} - 1 + 2b^2)$$
$$\times [20a^2(2b^2 - 2ba_{21} - 1)$$
$$- (16b^4 + 8b^3a_{21} - 24b^2a_{21}^2 - 14b^2 - 6a_{21} + 9)]$$

- Four variables:  $b_{30}, a_{21}, a, b$

$\implies$  four equations:  $v_i = 0, i = 2, 3, 4, 5.$

## 12 Limit Cycles (Cond.)

- $a$  or  $a_{21} \implies v_2 = 0$
- The first two factors of  $v_2 \implies$  centers
- The third factor of  $v_2 \implies$

$$a = \pm \left[ \frac{16 b^4 + 8 b^3 a_{21} - 24 b^2 a_{21}^2 - 14 b^2 - 6 a_{21} + 9}{20 (2 b^2 - 2 b a_{21} - 1)} \right]^{1/2}$$

$\implies$

$$v_3 = \frac{2 a b^2}{625 (2 b^2 - 2 b a_{21} - 1)^4} F F_1$$

$$v_4 = \frac{-2 a b^2}{3515625 (2 b^2 - 2 b a_{21} - 1)^7} F F_2$$

$$v_5 = \frac{a b^2}{31640625000 (2 b^2 - 2 b a_{21} - 1)^{10}} F F_3$$

$$F = 5b(2b^2 - 2ba_{21} - 1)b_{30} + 8b^4 - 6b^3a_{21} - 2b^2(a_{21}^2 + 1) - 3ba_{21} + 2$$

$F_i = F_i(b_{30}, a_{21}, b)$  (ith-degree polynomial of  $b_{30}$ ,  
 $i = 1, 2, 3$ )

- $F = 0 \implies$  center

## 12 Limit Cycles (Cond.)

- $F_1 = 0 \implies$

$$b_{30} = \frac{-1}{875 b^3 (2 b^2 - 2 b a_{21} - 1)} \\ \times [5104 b^{10} - 10360 a_{21} b^9 - 100 (26 a_{21}^2 + 43) b^8 \\ + 20 a_{21} (946 a_{21}^2 + 53) b^7 - 10 (1412 a_{21}^4 - 1212 a_{21}^2 - 199) b^6 \\ + 4 a_{21} (764 a_{21}^4 - 2555 a_{21}^2 - 5) b^5 + 20 (67 a_{21}^4 - 34 a_{21}^2 - 33) b^4 \\ - 15 a_{21} (86 a_{21}^2 - 7) b^3 - 10 (32 a_{21}^2 - 9) b^2 + 100 a_{21} b + 12]$$

- Then  $F_2 = 0$  and  $F_3 = 0$  become

$F_2^*(a_{21}, b) = 0$  (9th-degree polynomial of  $a_{21}$ )

$F_3^*(a_{21}, b) = 0$  (14th-degree polynomial of  $a_{21}$ )

**Eliminating**  $a_{21} \implies a_{21} = a_{21}(b^2)$  **and**

$$F_4(b^2) = b (264 b^2 + 7) (56 b^2 + 3) (64 b^2 + 5) (75 b^2 + 11) F_{41} F_{42} \\ = 0$$

$\implies$

$$F_{41} = (3 b^2 - 5) (26 b^2 - 45) = 0 \implies \text{centers}$$

$$F_{42} = 10195528 b^6 - 21299025 b^4 + 3454965 b^2 + 259405 \\ = 0 \quad (\text{3rd-degree polynomial of } b^2)$$

## 12 Limit Cycles (Cond.)

- $F_{42} = 0 \implies$

$$b_{\pm} = \pm 0.4904504000 \implies a^2 < 0 \text{ (NOT solutions)}$$

$$b_{\pm} = \pm 1.3798788398 \implies \text{4 solutions } (|b| > |a|):$$

$$b = 1.3798788398, a_{21} = 1.0897036998, b_{30} = 0.7093364483 \quad \left. \right\}$$

$$b = -1.3798788398, a_{21} = -1.0897036998, b_{30} = -0.7093364483 \quad \left. \right\}$$

$$a = \pm 0.0877426100, b_{21} = \pm 0.1470131077, a_{30} = \mp 0.0284721124$$

Consider one of the four solutions:

$$b^* = 1.3798788398$$

$$a_{21}^* = 1.0897036998$$

$$b_{30}^* = 0.7093364483$$

$$a^* = 0.0877426100$$

$$b_{21}^* = 0.1470131077$$

$$a_{30}^* = -0.0284721124$$

$$b_{12}^* = -1.0063695748$$

## 12 Limit Cycles (Cond.)

- **Main Theorem:** *Given the cubic system (\*) which is assumed to have a saddle point at the origin and a pair of symmetric fine focus points at  $(x, y) = (0, 1)$  and  $(0, -1)$ . Further suppose*

$$\begin{aligned}a_{12} &= -b_{03} = 0.0877426100, \\a_{03} &= -1.3798788398, \\b_{12} &= -1.0063695748.\end{aligned}$$

*Then if  $b, a_{21}, b_{30}(b, a_{21}), a(b, a_{21}), b_{21}(b, a_{21}, a)$  and  $a_{30}(b, a_{21}, a, b_{21})$  are perturbed as*

$$\begin{aligned}b &= b^* + \epsilon_1 \\a_{21} &= a_{21}^* + \epsilon_2 \\b_{30} &= b_{30}(b^* + \epsilon_1, a_{21}^* + \epsilon_2) + \epsilon_3 \\a &= a(b^* + \epsilon_1, a_{21}^* + \epsilon_2) - \epsilon_6 \\b_{21} &= b_{21}(b^* + \epsilon_1, a_{21}^* + \epsilon_2, a(b^* + \epsilon_1, a_{21}^* + \epsilon_2) - \epsilon_6) - \epsilon_4 \\a_{30} &= a_{30}(b^* + \epsilon_1, a_{21}^* + \epsilon_2, a(b^* + \epsilon_1, a_{21}^* + \epsilon_2) - \epsilon_6, \\&\quad b_{21}(b^* + \epsilon_1, a_{21}^* + \epsilon_2, a(b^* + \epsilon_1, a_{21}^* + \epsilon_2) - \epsilon_6) - \epsilon_4) - \epsilon_5\end{aligned}$$

*where  $0 < \epsilon_6 \ll \epsilon_5 \ll \epsilon_4 \ll \epsilon_3 \ll (\epsilon_2, \epsilon_1) \ll 1$ , system (\*) has exact twelve small limit cycles. The notation  $(\epsilon_2, \epsilon_1)$  means that  $\epsilon_2$  and  $\epsilon_1$  are in the same order, with  $\epsilon_2 = (\delta + \bar{\epsilon})\epsilon_1$  for some  $\delta > 0$  and some small  $\bar{\epsilon} > 0$ .*

## 12 Limit Cycles (Cond.)

- **Proof (Sketch):**

- Consider perturbations to  $v_5 = 0$  and  $v_4 = 0$  simultaneously, s.t.  $0 < v_4 \ll -v_5 \ll v_6$  ( $v_6 > 0$ ).

$$J(b^*, a_{21}^*) = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \equiv \begin{bmatrix} \frac{\partial v_5}{\partial b} & \frac{\partial v_5}{\partial a_{21}} \\ \frac{\partial v_4}{\partial b} & \frac{\partial v_4}{\partial a_{21}} \end{bmatrix}_{(b^*, a_{21}^*)} = \begin{bmatrix} -6.1361037671 & 0.0000000000 \\ -2.3694030045 & 2.1866028190 \end{bmatrix}$$

Let  $b = b^* + \epsilon_1$ ,  $a_{21} = a_{21}^* + \epsilon_2 \implies 0 < -\frac{J_{21}}{J_{22}} \epsilon_1 < \epsilon_2$

Let  $\epsilon_2 = (\delta + \bar{\epsilon}) \epsilon_1$ , where  $\delta = -\frac{J_{21}}{J_{22}} = 1.0836000868$

$\implies v_4 \approx J_{22} \epsilon_1 \bar{\epsilon}$ ,  $v_5 \approx J_{11} \epsilon_1$  ( $\epsilon_1, \bar{\epsilon} \ll 1$ )

- Perturbation to  $v_3$ : let

$$\begin{aligned} b_{30} &= b_{30}(b^* + \epsilon_1, a_{21}^* + \epsilon_2) + \epsilon_3 \implies \\ v_3 &\approx (-0.0581541802 - 533.7000900191 \epsilon_1 + 471.0210964726 \epsilon_2) \epsilon_3 \end{aligned}$$

- Perturbation to  $v_2$ : let

$$\begin{aligned} b_{21} &= b_{21}(b^* + \epsilon_1, a_{21}^* + \epsilon_2, a) - \epsilon_4 \implies \\ v_2 &\approx (1.6661691420 - 20.9332869624 \epsilon_1 + 29.2490912085 \epsilon_2 - 6.6431544667 \epsilon_3) \epsilon_4 \end{aligned}$$

- Perturbation to  $v_1$ : let

$$\begin{aligned} a_{30} &= a_{30}(b^* + \epsilon_1, a_{21}^* + \epsilon_2, a, b_{21}(b^* + \epsilon_1, a_{21}^* + \epsilon_2, a) - \epsilon_4) - \epsilon_5 \\ \implies v_1 &\approx -(2.856098420 + 4.139636520 \epsilon_1) \epsilon_5 \end{aligned}$$

- Perturbation to linear term: let

$$a = a(b^* + \epsilon_1, a_{21}^* + \epsilon_2) - \epsilon_6 \implies v_0 = \frac{1}{2} \epsilon_6 > 0$$

- Normal form:

$$\begin{aligned} \dot{r} &= r(v_0 + v_1 r^2 + v_2 r^4 + v_3 r^6 + v_4 r^8 + v_5 r^{10} + v_6 r^{12}) \\ (0 < v_0 &\ll -v_1 \ll v_2 \ll -v_3 \ll v_4 \ll -v_5 \ll v_6) \end{aligned}$$

## 12 Limit Cycles (Cond.)

- An Numerical Example:

$$\epsilon_1 = 0.01$$

$$\epsilon_2 = 0.010846$$

$$\epsilon_3 = 0.00000001$$

$$\epsilon_4 = 0.00000000000006$$

$$\epsilon_5 = 0.0000000000000001$$

$$\epsilon_6 = 0.000000000000000002$$

$$b = 1.389878839792798029819223084829,$$

$$a_{21} = 1.100549699750322460204379462971,$$

$$b_{30} = 0.723083443246592425047289189844,$$

$$a = 0.088317687330421587849539710365,$$

$$b_{21} = 0.148351368647488295561789258216,$$

$$a_{30} = -0.028284006013282710541331939148,$$

$$v_0 = 0.1 \times 10^{-22},$$

$$v_1 = -0.289764478395566149818268055720 \times 10^{-17},$$

$$v_2 = 0.100880924993181495233228330467 \times 10^{-12},$$

$$v_3 = -0.578965080933760443062740846761 \times 10^{-9},$$

$$v_4 = 0.736608095896203656125558382977 \times 10^{-6},$$

$$v_5 = -0.176685951291921753599727302890 \times 10^{-3},$$

$$v_6 = 0.688118224089944005454377118998 \times 10^{-2}.$$

$$r_1 = 0.001998424223,$$

$$r_2 = 0.005539772256,$$

Amplitudes of

$$r_3 = 0.013978579010,$$

the 6 LC:

$$r_4 = 0.027014927622,$$

$$r_5 = 0.063377651980,$$

$$r_6 = 0.143875184886.$$

## Possibility of 14 Limit Cycles

Under the condition (\*\*), there are  
four parameters  $a, b, a_{21}$  and  $b_{21}$  to give

$$\begin{aligned}v_3 &= v_3(a, b, a_{21}, b_{21}) \\v_4 &= v_4(a, b, a_{21}, b_{21}) \\v_5 &= v_5(a, b, a_{21}, b_{21}) \\v_6 &= v_6(a, b, a_{21}, b_{21})\end{aligned}$$

Thus, it may be possible to choose the four  
parameters such that, in addition to  $v_1 = v_2 = 0$ ,

$$v_3 = v_4 = v_5 = v_6 = 0, \text{ but } v_7 \neq 0.$$

Further, we may use  $a_{12}$  to perturb the  
linear term.

This gives a possibility to find 14 limit cycles!

A Very Hard Computational Problem!

## Conclusion

- A planar system with 3rd-degree polynomial functions can have 12 small limit cycles.
- A sufficient condition for the existence of multiple small amplitude limit cycles is established.
- Such a system may have 14 small amplitude limit cycles.
- Conjecture of a sufficient condition for a center:

*If there exist two consecutive focus values which can be set zero by choosing a same parameter, then all the followed focus values equal zero under this choice, leading to a center.*

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- What is the maximal number of the limit cycles (local and global) of cubic systems?