

Sy met a Home on a  
n Reversible systems

JEROEN LAMB (Turing College London)  
(www.maths.ac.uk/~jwlan)

Ale Jan Harboe (University of Amsterdam)

## Motivation

For example

Reaction Diffusion Equation

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + f$$

$x \in \mathbb{R}$   
 $u \in \mathbb{R}$

Stationary state  $u(x) = v(x)$

$$\Rightarrow v = D f(v)$$

$$\begin{cases} v = w & \text{or } v = w \\ w = D f(v) & \end{cases}$$

(v, w)  $\in \mathbb{R}^n$

$\mathcal{L}_2$  plays reversible

$$R: \mathbb{R} \hookrightarrow Rvw \quad (v, w)$$

$$RF = F R$$

$(v, w)$  soln of  $\mathcal{L}_2$  then also

$$Rv(x) = w(x) \quad (v(x), w(x))$$

In this talk:

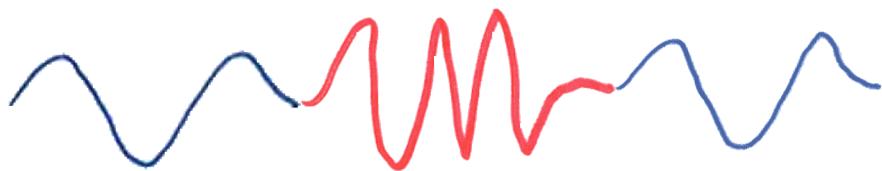
R-Reversible vector field in  $\mathbb{R}^{2n}$

$$R^2 = I, \dim \text{Fix } R = n$$

Periodic solutions of  $F$       spatially periodic  
stationary states

Homoclinic solution to periodic solution

spatially periodic stationary state with  
defect



Question:

What is the structure (and nature) of  
the nonwandering set near such a  
homoclinic?

Answer may depend on structure of vector field

general dissipative vector field

$W^s(p) \cap W^u \Rightarrow$  nontrivial hyperbolic  
basic set  
(horse shoe)

[also applies to nonsymmetric periodic so  
of reversible vector field]

### Questions:

- What about nonwandering set near symmetric homoclinic to symmetric periodic solution?
- How does this compare to situation for (reversible) Hamiltonian vector fields?

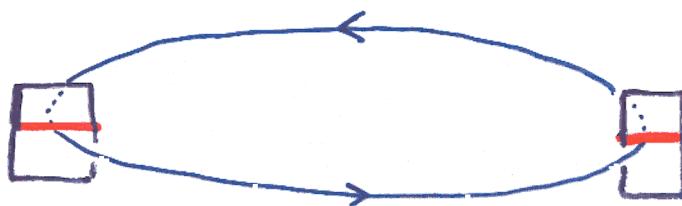
# Periodic solutions of reversible vector fields

$$x \in \mathbb{R}^2 \quad \frac{dx}{dt} = F(x) \quad R \subset \mathbb{R}^2 \quad \text{dim } F = n$$

Consider  $R$  as an symm to per. sol. suit

$$x + T \quad \text{and} \quad R(x+t) \quad (t)$$

$$[ \Leftrightarrow x \text{ reverses } F \text{ exactly twice} ]$$



$$F: S \rightarrow S$$

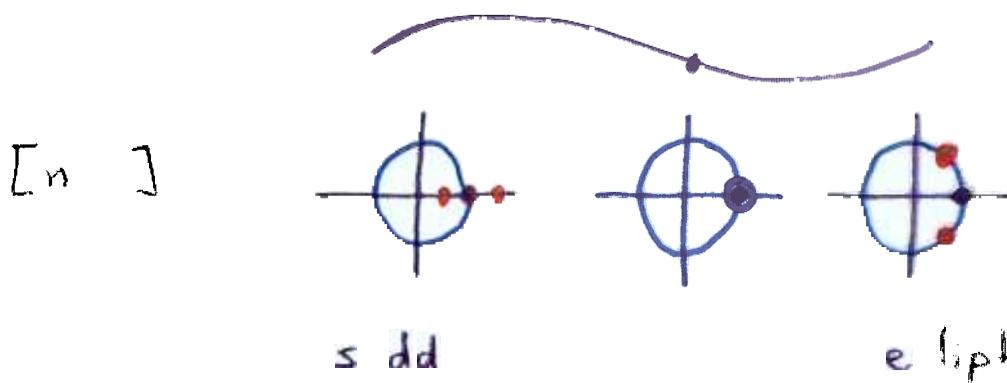
$$\dim S = n$$

$$\dim F|_S = n$$

$$F: S \rightarrow \phi(F|_S) \Rightarrow \dim = 1$$

Locally symmetric per. sol. suit

form 2D manifold (1 parameter family)



Comparison to reversible Hamiltonian case

a 1-parameter family of periodic solution  
where parameter  $H$  foliation by level  
sets of Hamiltonian).

N.B. for reversible vector field such a  
foliation **does not** exist

### Homoclinic tangle

1 par family of symmetric periodic solutions of saddle  
type  
 $\exists a \quad a \in (\delta \delta) \text{ near } f(a_0) \quad a_0 = 0$

traversely

$$W_{\pm}(a_0) \cap F_{\rho(a_0)}$$

$\rho(a_0)$  symmetric homoclinic solution to  $f(a_0)$

$\Rightarrow$  a 1 par family of symmetric homoclinic  
 $\rho(a)$  to  $(a)$

[idem Hamiltonian setting]

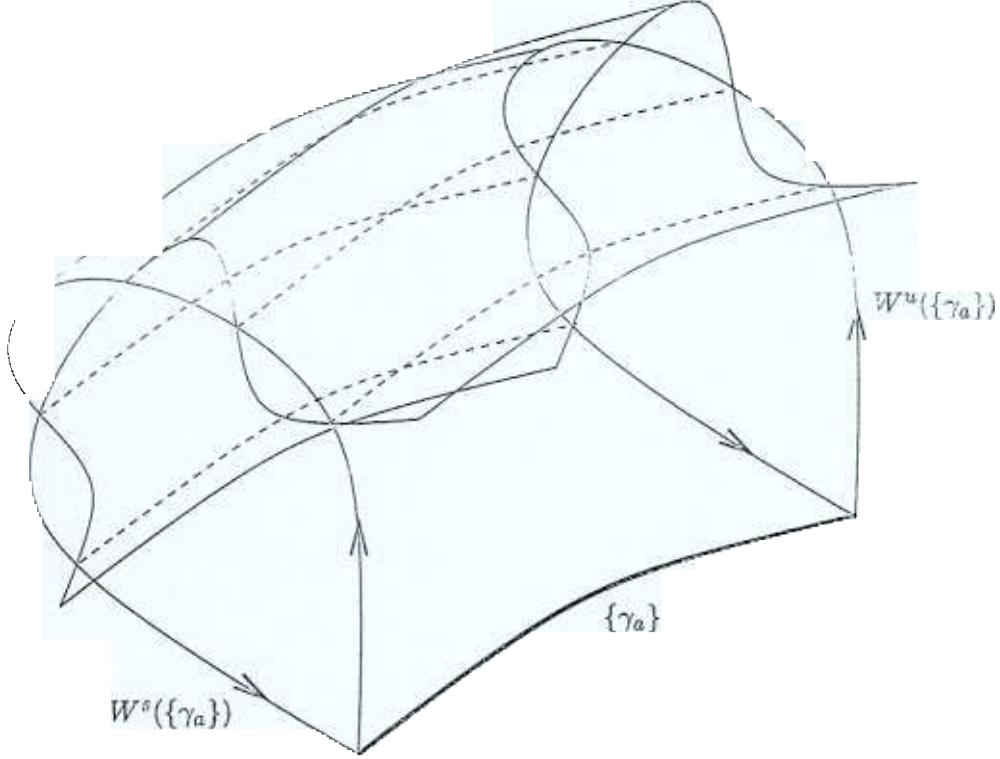


Figure 1: Homoclinic tangles to a family  $\{\gamma_a\}$  of symmetric periodic orbits. The picture indicates the manifolds for a return map on a global cross-section; in general only local cross-sections near  $\{\gamma_a\}$  and  $\{\rho_a\}$  would be considered.

**Theorem 2.1** For each  $\eta \in \mathcal{B}$ , there is a one-dimensional center manifold  $W_\eta^c$  for  $\Psi$ , so that any orbit  $x$  with itinerary  $\Upsilon(x) = \eta$ , satisfies  $x \in W_\eta^c$ . The curve  $W_\eta^c$  is smooth and depends continuously on  $\eta$ . Moreover,  $W_{\sigma(\eta)}^c = \Psi(W_\eta^c)$ .

PROOF. The invariant curves are obtained as intersections of center stable with center unstable manifolds. We will construct invariant center stable manifolds. Center unstable manifolds are constructed analogously. For the proof we combine constructions for center stable manifolds from [Irw80b, GilVan87] with constructions for horseshoes from [Irw80a, HomVilSan03].

The family  $\{\gamma_a\}$  gives a curve  $\{p_a\}$  of fixed points  $p_a = \gamma_a \cap \Sigma_0$  for  $\Psi$  in  $\Sigma_0$ . Likewise,  $\{\rho_a\}$  gives a curve  $\{r_a\}$  of homoclinic points  $r_a = \rho_a \cap \Sigma_1$  for  $\Psi$  in  $\Sigma_1$ . Take coordinates  $x = (x_s, x_c, x_u)$  in  $\mathbb{R}^{2n-1}$  on  $\Sigma_1$  corresponding to the projection of the  $DX_4$ -invariant splitting  $E^s \oplus T\{\gamma_a\} \oplus E^u$  at  $\gamma_{a_0}$  onto  $\Sigma_0$ . The  $x_c$ -axis is tangent to the curve of fixed points  $\{p_a\}$  at  $p_{a_0}$ . Take similar coordinates on  $\Sigma_1$ , which we also denote by  $x = (x_s, x_c, x_u)$ . The  $x_c$ -axis is here tangent to the curve  $\{r_a\}$  at  $r_{a_0}$ . We may assume that the involution  $R$  acts linearly in the  $x$ -coordinates on  $\Sigma_0$  and  $\Sigma_1$  by

$$R(x_s, x_c, x_u) = (x_u, x_c, x_s).$$

## Homoclinic tangle in Hamiltonian case:

~ [n=2] return  
is !  
with

on level set  $H=\text{constant}$   
and unstable

$\Rightarrow$  horse-shoe  
(nontrivial hyperbolic basic set)

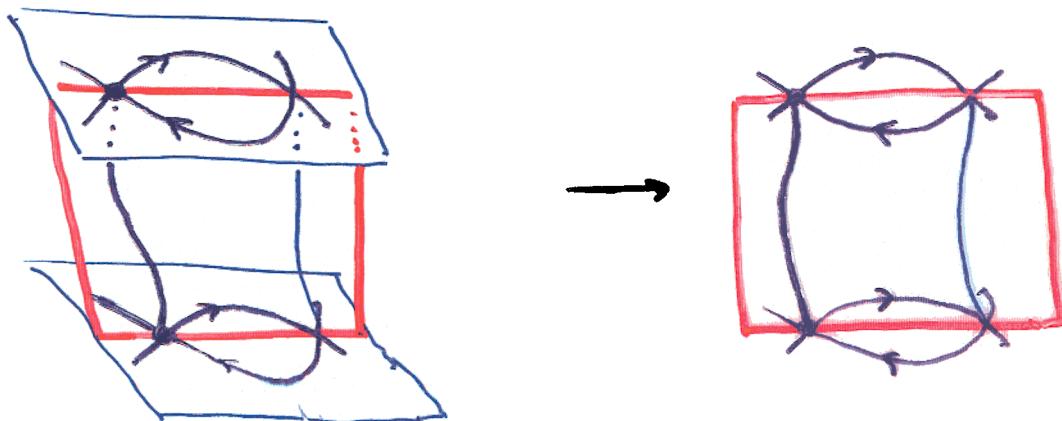
considering interval of level sets of  $H$   
 $\Rightarrow$  1-par family of horseshoes

[same result in reversible Hamiltonian case]

## Thought experiment:

Consider reversible Hamiltonian vector field  
with symmetric homoclinic tangle.

Now **perturb** to destroy Hamiltonian structure  
but retain the reversibility



These observations:

after non-Hamiltonian perturbation, symmetric periodic solutions will continue to arise in 1-par families

families of nonsymmetric periodic solutions will **not** persist (typically), by Kupka-Smale [Devaney '76]

(3) In Hamiltonian case,

$\mathcal{H} = \text{horseshoe} \times H\text{-interval}$

is normally hyperbolic invariant set

This will survive the nonHamiltonian perturbation

and thus nonwandering set will be inside

Questions:

- Does the nonwandering set contain nontrivial basic sets?
- Is the nonwandering set structurally stable?

Study nonwandering set using return map  
on sections

$\Sigma_0$  through  $p(a_0) \in \text{Fix } R \cap f(a_0)$

and

$\Sigma_1$  through  $r(a_0) \in \text{Fix } R \cap p(a_0)$

$\Rightarrow \exists$  lamination of normally hyperbolic  
center manifolds near  $p(a_0)$  and  $r(a_0)$

- The nonwandering set is contained in the lamination.
- Reduction of dynamics to skew product of interval maps over a subshift of finite type

let  $\beta$  be subset of sequences  $\mathbb{Z} \rightarrow \{0, 1\}$   
with transition matrix  $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

and  $\sigma: \beta \rightarrow \beta$  the shift operator

then  $\psi$  reduces to  $\phi: \beta \times I \hookrightarrow$

$$\phi(\eta, y) = (\sigma(\eta), g(\eta, y))$$

Define:  $R: \beta \rightarrow \beta$        $R\gamma(h) = \gamma(-h)$

$\eta \in \beta$  is symmetric if  $R\eta = \sigma^s \eta$  for some  $s$ .

First (expected) result.

If  $\eta$  symmetric, then  $g(\eta, y) = y$

$\Rightarrow W^c(\eta)$  contains a 1-param family of periodic solutions

Lemma:  $\exists$  neighborhood  $U$  of  $\{\psi^n(r(a_0))\}_{n \in \mathbb{Z}}$  and a  $C^1$ -small perturbation  $\psi \rightarrow \tilde{\psi}$  (preserving orbit) such that  $\exists H: U \rightarrow \mathbb{R}$  and  $H \circ \tilde{\psi} = H$  with  $H$  smooth (quadratic).

i.e. near the homoclinic tangle,

$\psi$  is  $C^1$ -close to a conservative diffeomorphism.

## Theorem

dynamics near symmetric  
homoclinic tangle in reversible system  
**not** C<sup>1</sup> structurally stable

stark dr. with

general dissipative  
Hamiltonian

case where structural stability near  
homoclinic tangle follows from  
uniform hyperbolicity level set of H

The following useful proposition illustrates this.

## Proposition

by C small perturbation we can  
create a hyperbolic saddle  
periodic orbit arbitrarily close  
to the symmetric homoclinic tangle

[note this hyperbolic saddle must  
be nonsymmetric]

## Proof

use lemma to obtain occ. conservative  
diffeo which highly degenerate by reversible  
Kupka Smale  $\Rightarrow$  by small perturbation  
one creates hyperbolic periodic solution.

Now suppose we have a hyperbolic saddle period orbit near the homoclinic tangle.

Then, if close enough to homoclinic orbit, its

④ stable and unstable manifolds will intersect  $\text{Fix } R$

Note that each such saddle has index  $n$ , or  $n-1$

Let  $\tau$  be hyperbolic saddle of index  $n$ , then

$R(\tau)$  has index  $n-1$ .

By symmetry, if ④ is satisfied, one obtains a **heterodimensional cycle**

(heteroclinic cycle between saddle points of different index  $[n \text{ and } n-1]$ )

N.B.: these heterodimensional cycles arise in a persistent way, due to **reversibility**

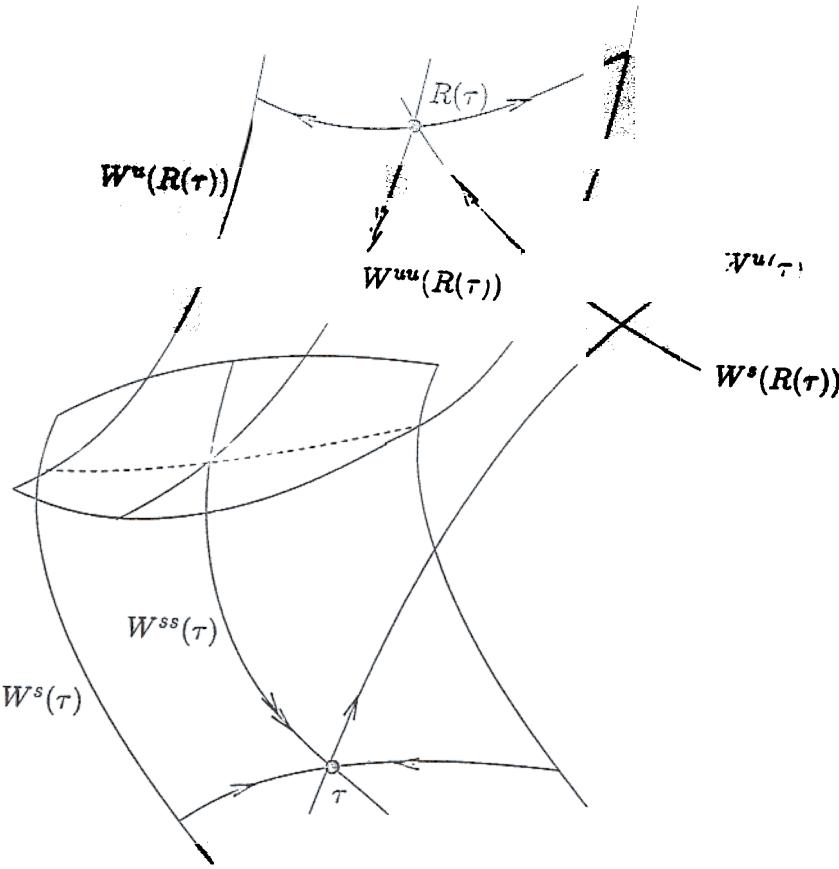


Figure 2: A heterodimensional cycle involving a hyperbolic periodic orbit  $\tau$  and its symmetric image  $R(\tau)$ .

$$(I1) \quad W^{s,u}(\tau) \pitchfork_{\rho^1} W^s(R(\tau)), \quad W^{s,u}(R(\tau)) \pitchfork_{\rho^1} W^u(\tau)$$

$$(I2) \quad W^{ss}(\tau) \pitchfork_{\rho^2} W^u(R(\tau)), \quad W^{uu}(R(\tau)) \pitchfork_{\rho^2} W^s(\tau),$$

For both (I1) and (I2), the two conditions imply each other by reversibility.

Consider a small neighborhood of the periodic orbits  $\tau, R(\tau)$  and the heteroclinic orbits  $\rho^1, \rho^2$ . Take small cross-sections  $\Sigma_0$  and  $\Sigma_2 = R(\Sigma_0)$  near  $\tau$  and  $R(\tau)$  respectively. Take small symmetric cross-sections  $\Sigma_1, \Sigma_3$  near  $\rho^1 \cap \text{Fix}(R)$  and  $\rho^2 \cap \text{Fix}(R)$  respectively. Consider the first return map  $\Psi$  on the union of these four cross-sections, following orbits only as long as they are near the heterodimensional cycle. Abstracted from the previous sections, we can reduce the dynamics near the heterodimensional cycle to a skew product of interval maps.

**Theorem 4.1** *There is a subshift of finite type  $\mathcal{B}$  so that for each  $\eta \in \mathcal{B}$ , there is a one dimensional center manifold  $W_\eta^c$  for  $\Psi$ , so that any orbit  $x$  with itinerary  $\Upsilon(x) = \eta$ , satisfies  $x \in W_\eta^c$ . the curve*

With some mild condition involving transversality  
of the strong stable and unstable manifold we  
obtain by application of the strong  $\Lambda$ -lemma:

type

size

### Theorem:

Consider a reversible vector field with a symmetric homoclinic tangle

Then there exists a  $C^1$ -neighborhood of this system, such that for an  $C^1$  open and dense subset, the vector fields have a nontrivial hyperbolic basic set.

+

Theorem:

## Conclusion:

Do symmetric homoclinic tangles in reversible systems give rise to nontrivial hyperbolic basic sets?

Yes that is most of the time in  $C^1$  sense

is the dynamics near such a tangle structurally stable?

No!

## Applications

dynamics near homoclinic orbit to symmetric saddle-focus. [eg Härterich q $\varphi$ ]

dynamics near homoclinic "bellows" [Homburg & Knobloch]

Open problems: many eg bifurcations!