Canards and mixed mode oscillations

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Canard Solutions

• Slow/fast systems often have strongly attracting, locally invariant slow manifolds

• Slow manifolds sometimes end abruptly and trajectories must follow fast directions.

• Canards are exceptional trajectories which cross over to the repelling slow manifold and continue moving on the slow scale.

• Some very interesting dynamical phenomena, like mixed mode oscillatins or localization may arise due to the presence of canards.

Mixed Mode Oscillations

In singular perturbations one encounters two basic types of periodic solutions: *small oscillations* and *relaxation oscillations*. Mixed mode oscillations are a oscillatory solutions combining both types.

Example equation

$$\begin{aligned}\varepsilon\dot{x} &= -y + \frac{1}{2}x^2 - \frac{1}{3}x^3\\ \dot{y} &= x - z\\ \dot{z} &= \varepsilon(\mu - z + \varphi(x)).\end{aligned}$$

- The presence and type of mixed mode oscillations depends on the function φ.

Selected references

Canards::

Benoit

Wechselberger

Guckenheimer and Haiduc

Mixed mode oscillations:

Millik, Szmolyan, Löffelman and Gröller

Moehlis

Drover, Rubin, Su ...

Singular Perturbation Theory

$$\varepsilon \dot{x} = f(x,y) \qquad\qquad x' = f(x,y)$$

$$\dot{y} = g(x,y) \qquad x \in \mathbb{R}^n,\ y \in \mathbb{R}^m \qquad y' = \varepsilon g(x,y),$$

slow equation — fast equation

0th order approximations are given by:

$$f(x,y) = 0 \qquad\qquad x' = f(x,y)$$

$$\dot{y} = g(x,y) \qquad\qquad y' = 0,$$

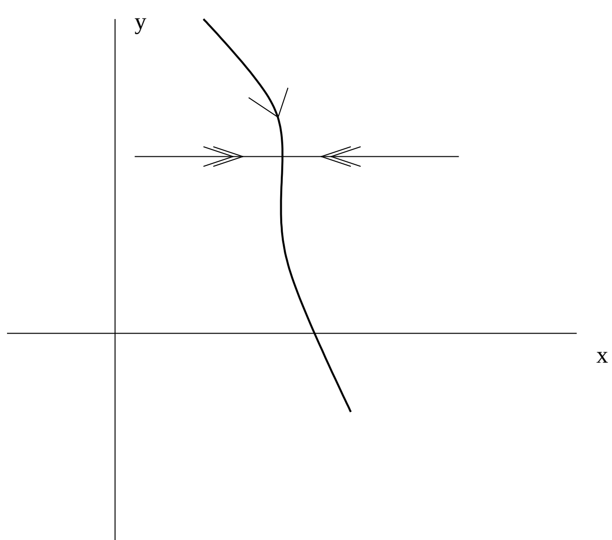
reduced equation — layer equation

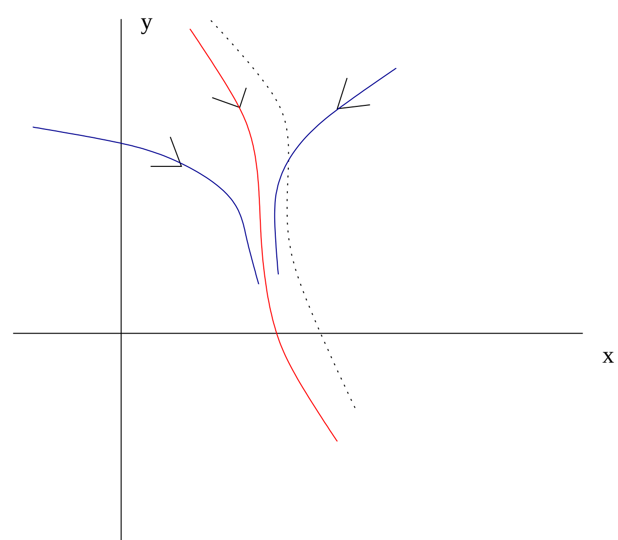
- The set $S_0 = \{(x,y) \ : \ f(x,y) = 0\}$ is called the reduced manifold.
- S_0 is the phase space for the reduced problem and the set of equlibria for the layer problem.

Fenichel Theorem

$$\begin{aligned} x' &= f(x,y) \\ y' &= \varepsilon g(x,y). \end{aligned} \tag{1}$$

Theorem (Fenichel) Suppose $\tilde{S}_0$ is an open subset of S_0 such that for every $(x,y) \in \tilde{S}_0$ the matrix $D_x f$ has no eigenvalues on the imaginary axis. Then, for $\varepsilon > 0$, there exists a locally invariant manifold for (1) S_ε close to $\tilde{S}_0$ and the flow on S_ε is close to the flow of the reduced equation on S_0.



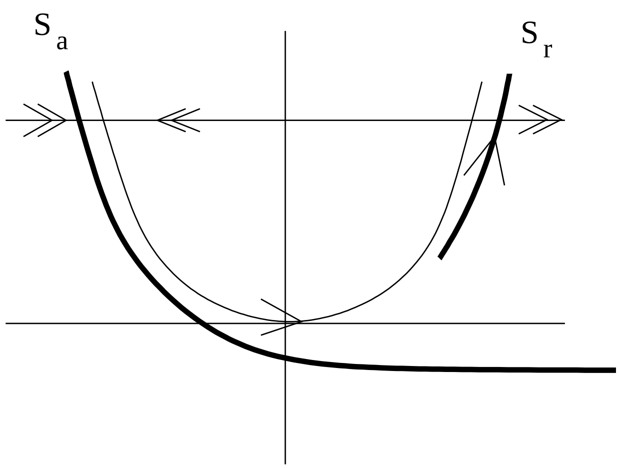


Non-hyperbolic points

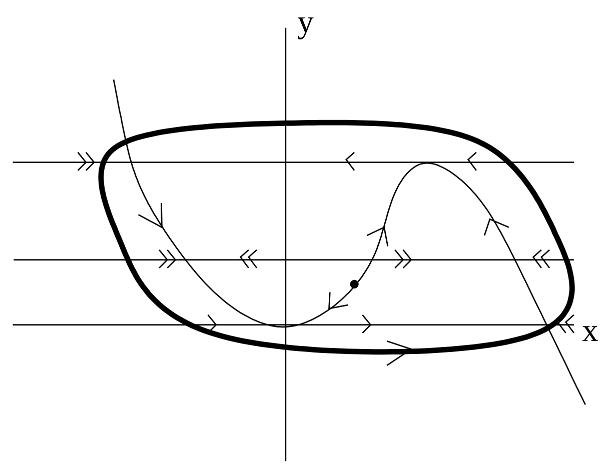
Interesting dynamics involving jumps between different locally invariant slow manifolds is related to the loss of normal hyperbolicity of S_0. Simplest example: *fold*. The following equations give an example:

$$\varepsilon \dot{x} = -y + x^2$$

$$\varepsilon y = g(x, y), \qquad g(0,0) < 0.$$



(a) simple fold



(a) relaxation oscillation

Fold in Systems with Two Slow Variables

$$x' = -y + x^2$$
$$y' = \varepsilon g_1(x, y, z)$$
$$z' = \varepsilon g_2(x, y, z)$$

• $S_0 = \{(x, y, z) \ : \ y = x^2\}$ is a parabolic cylinder

• Fold line $\mathcal{F} = \{0, 0, z) \ : \ z$ is arbitrary $\}$

• reduced equation can be obtained in variables (x, z) by setting $y = x^2$

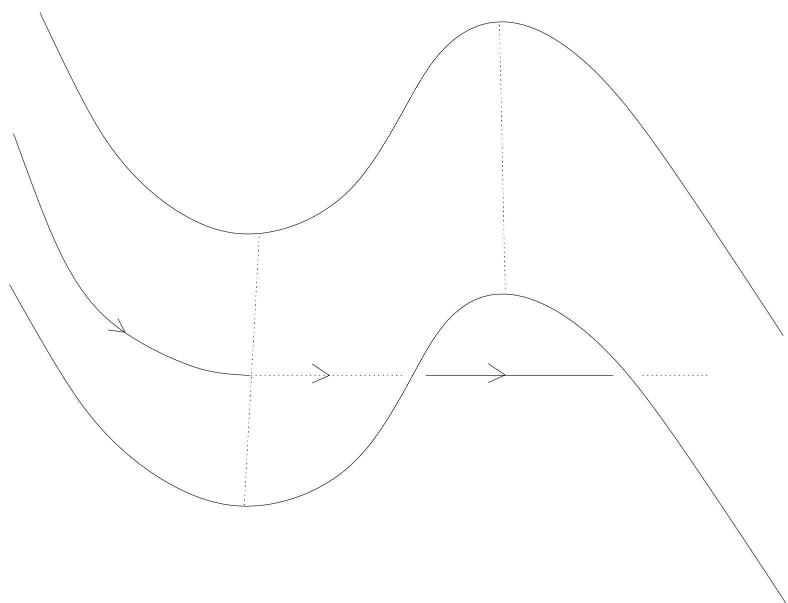
$$2x\dot{x} = g_1(x, y, z)$$
$$\dot{z} = g_2(x, y, z)$$

• Fold line $\mathcal{F} = \{x = 0\}$ corresponds to the set of singularities of the reduced equation.

Simple fold

$$2x\dot{x} = g_1(x, y, z)$$
$$\dot{z} = g_2(x, y, z)$$

Consider $p_0 = (0, 0, z_0) \in \mathcal{F}$. If $g_1(p_0) \neq 0$ then p_0 is like a simple fold point in a system with one slow variable.



Folded singularities correspond to the case of $g_1(p_0) = 0$.

Folded node

$$\begin{aligned} \varepsilon\dot{x} &= -y + x^2 \\ \dot{y} &= x - z \\ \dot{z} &= (\mu - x - z), \qquad \mu > 0. \end{aligned}$$

Reduced equation:

$$\begin{aligned} 2x\dot{x} &= x - z \\ \dot{z} &= (\mu - x - z). \end{aligned}$$

We desingularize (rescale time by $-2x$):

$$\begin{aligned} 2x\dot{x} &= -2x(x - z) \\ \dot{z} &= -2x(\mu - x - z), \end{aligned}$$

or, after cancelation:

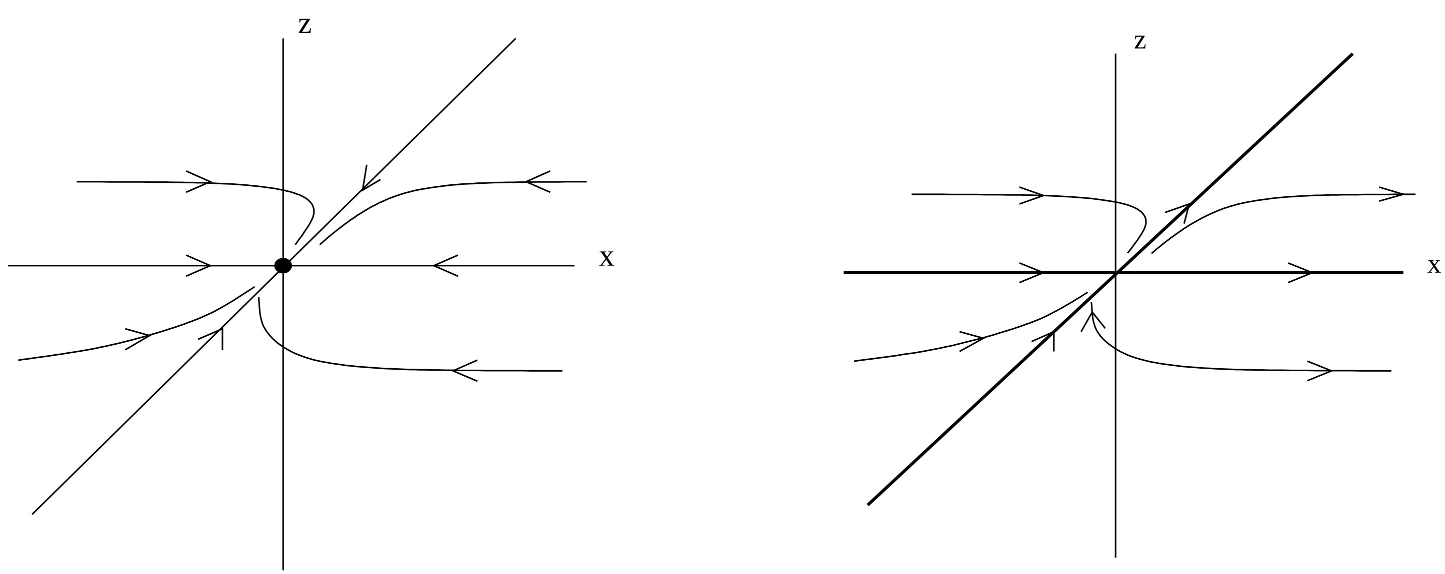
$$\begin{aligned} \dot{x} &= z - x \\ \dot{z} &= -2x(\mu - x - z), \end{aligned}$$

Folded node cont.

$$\dot{x} = z - x$$

$$\dot{z} = -2x(\mu - x - z),$$

- Away from the fold line $\{x = 0\}$ the trajectories of the reduced and desingularized system are equal, up to time parametrization. Time direction has to be reversed for $x > 0$.
- $(0, 0)$ is an equilibrium of node type.



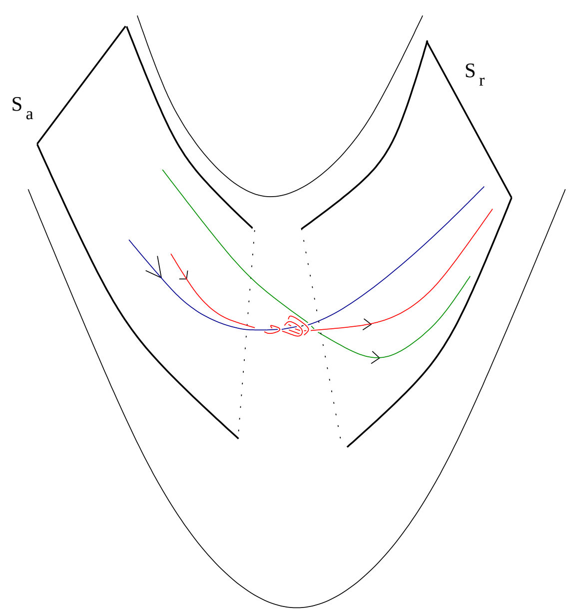
(a) (b)

Canard solutions

• By Fenichel theory away from $\mathcal{F}$ there exist two dimensional slow manifolds S_a and S_r. Canard solutions are solutions that pass from S_a to S_r.

• Near folded node there are two primary canards corresponding to principal directions. We call them weak canard and strong canard.

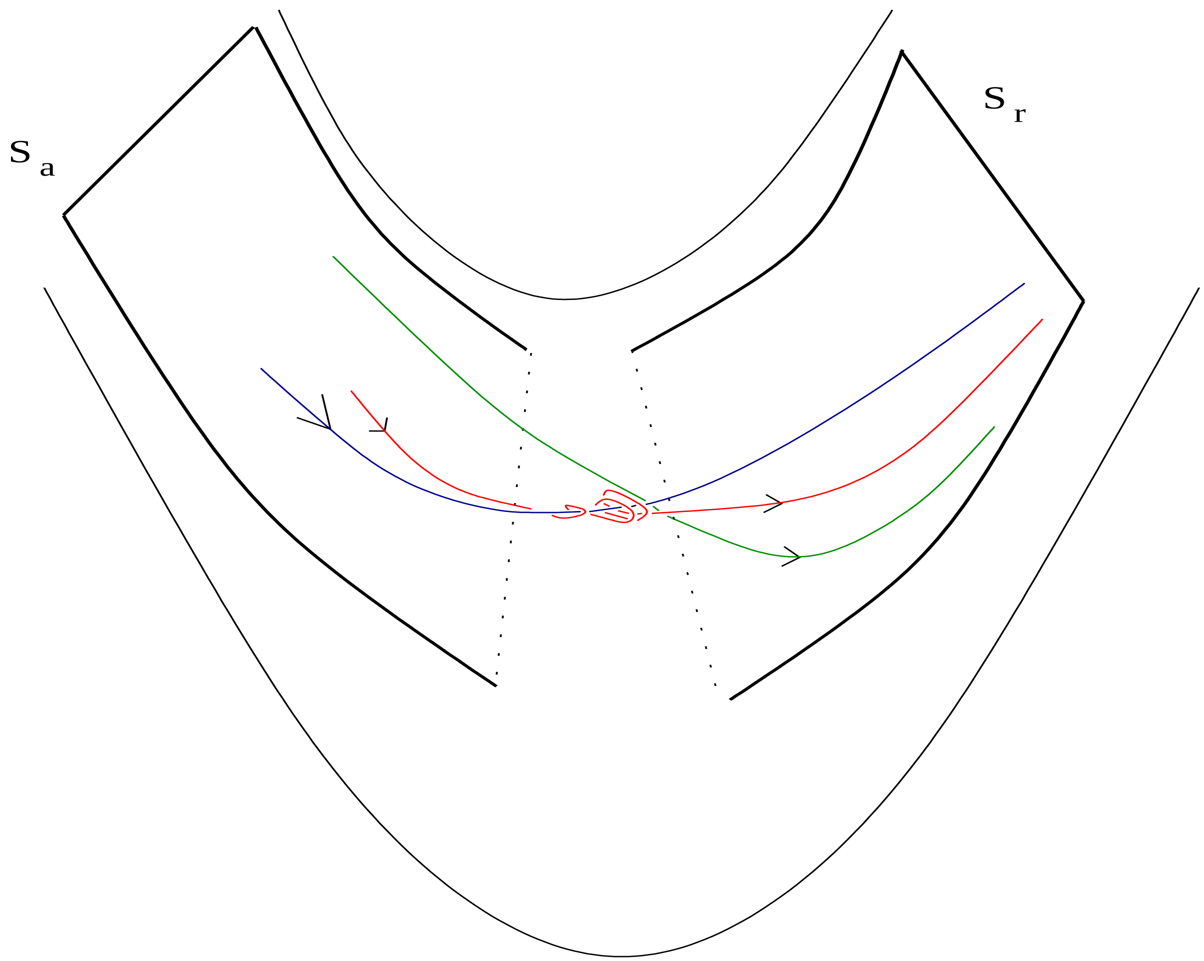
• Let μ be the ratio of the weak to the strong eigenvalue of the desingularized system and let m be such that $m-1 \leq \mu \leq m$. For each $1 \leq k \leq m-1$ there exists a secondary canard which winds k times around the weak canard.



Dynamics near Folded Node

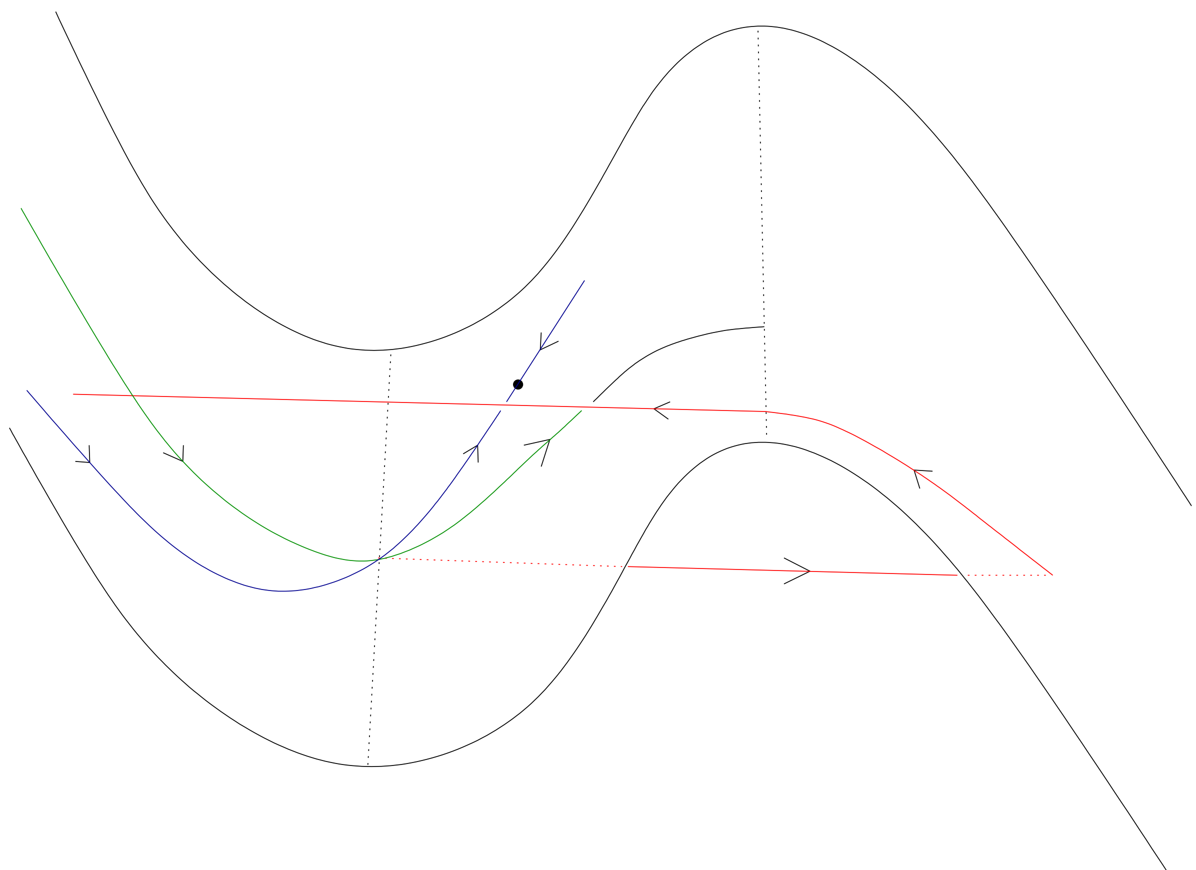
- Trajectories on S_a to the left of the strong canard (in the sense of the figure) are attracted to the weak canard. They rotate about the weak canard as they pass near the fold. They may leave the neighborhood either through a a relaxation mechanism or following one of the canards.

- Trajectories right of the strong canard follow the relaxation mechanism.



Global problem

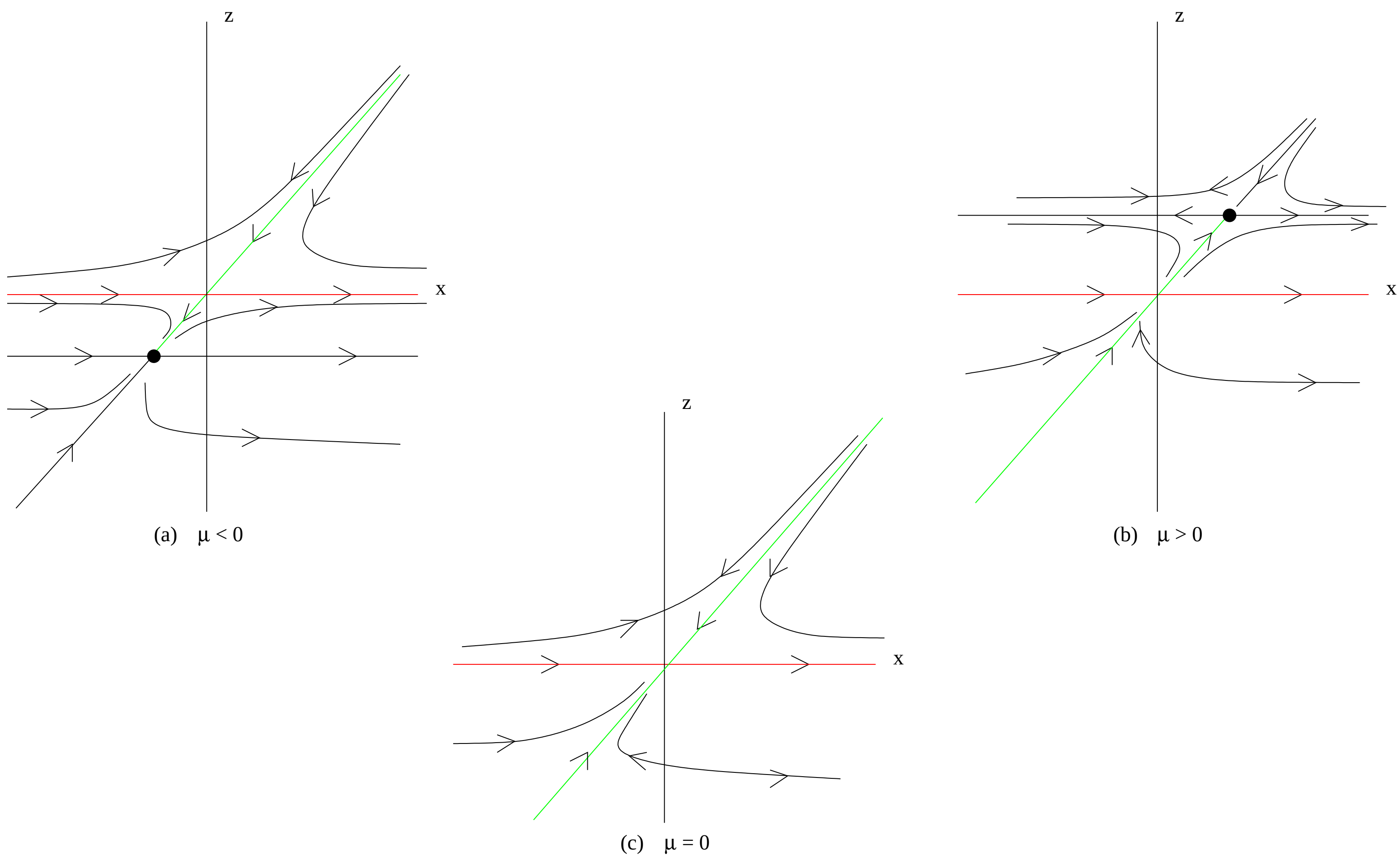
Mechanism of complicated dynamics: The trajectries following the relaxation route come back to S_{a-} to the left of the strong canard and subsequently converge to the weak canard.



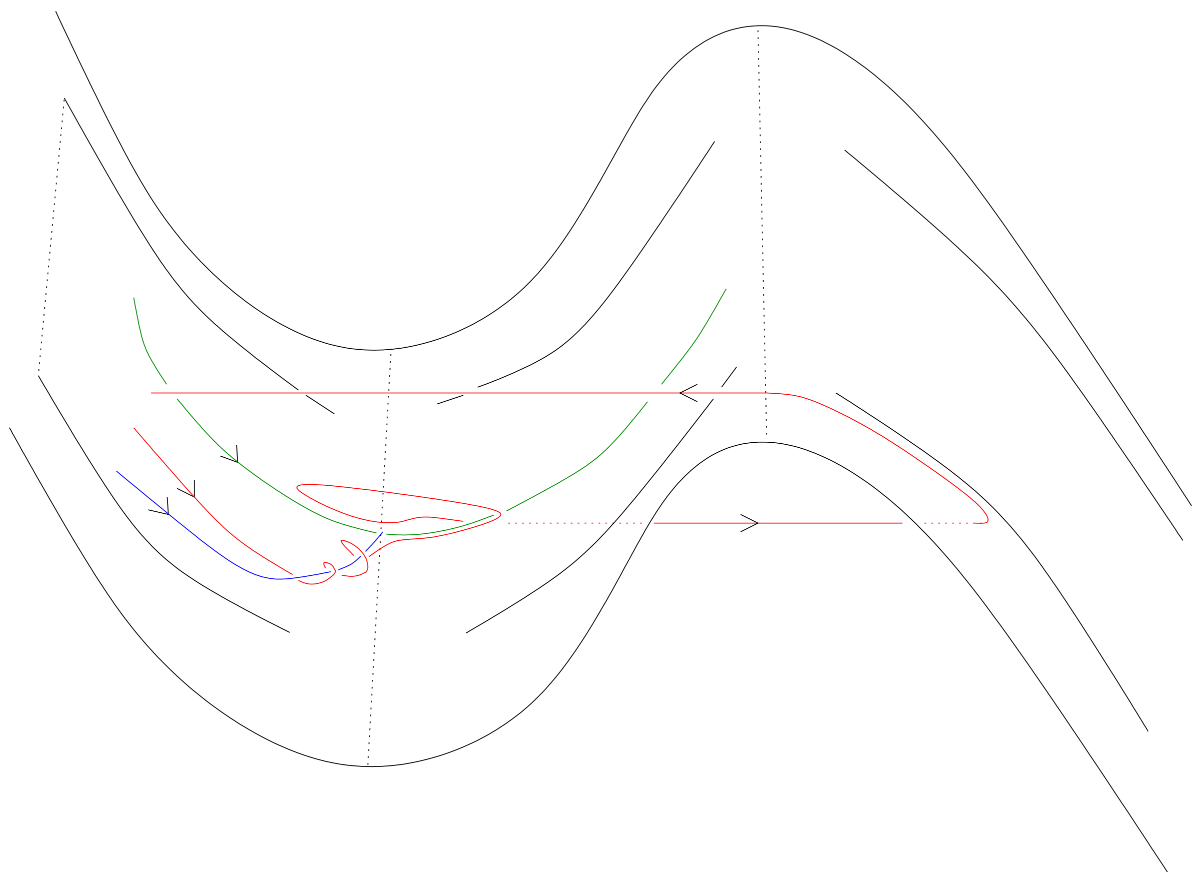
Folded saddle-node (easier?!)

$$\varepsilon \dot{x} = -y + x^2$$
$$\dot{y} = x - z$$
$$\dot{z} = (\mu - x - z), \qquad \mu \approx 0.$$

Phase portraits for reduced system:



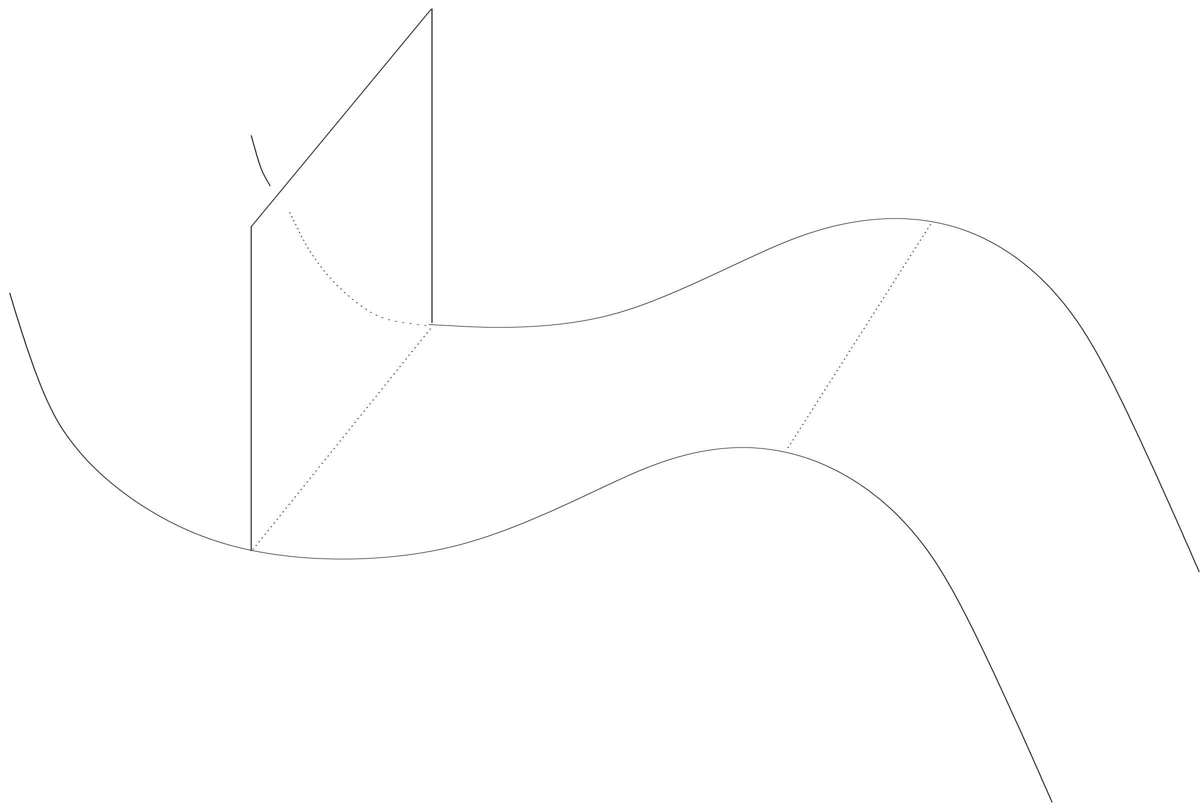
Dynamics near folded saddle-node There is no weak canard. There are infinitely many secondary canards, all very close to the strong canard. Most of the trajectories spiral as they pass near the fold and exit either along the strong canard or follow the relaxation mechanism.



Section of the flow

We define our return map π on the hyperplane

$$\Delta = \{(0, y, z),\ (y, z) \in \mathbb{R}^2\}.$$

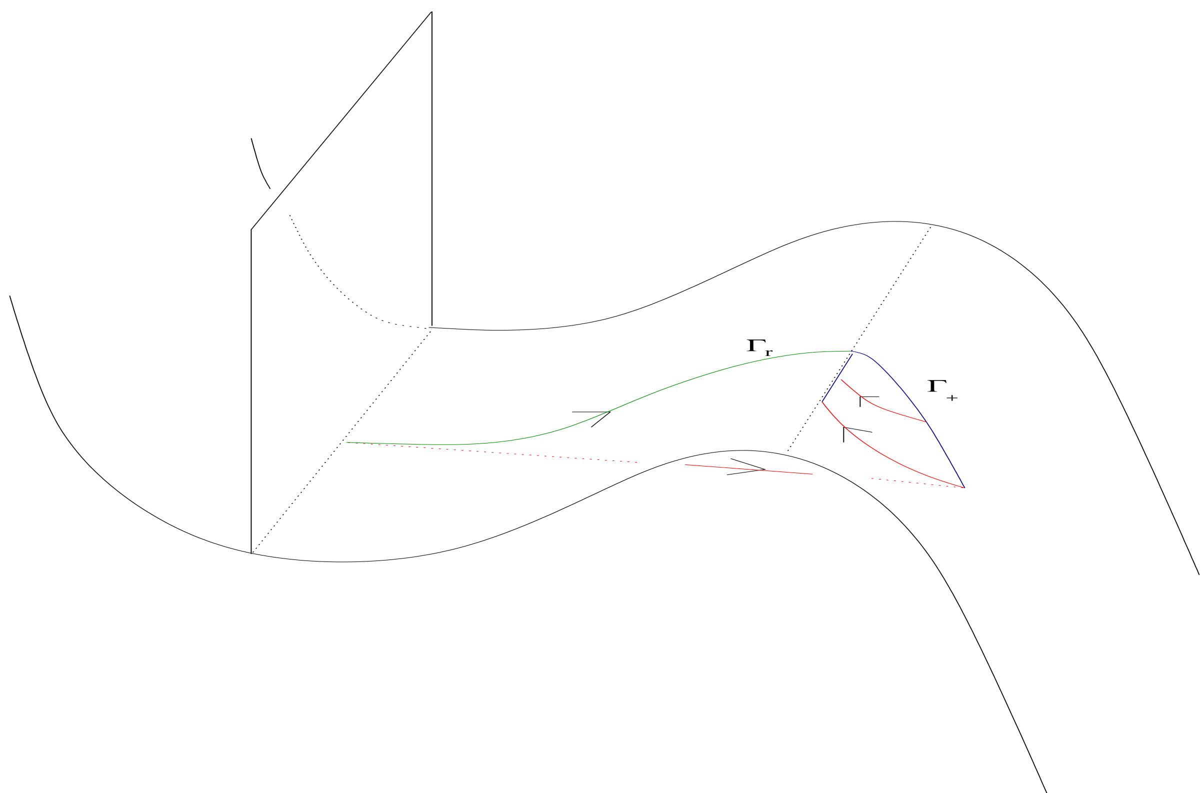


- Although Δ may fail to be a section for some trajectories it is still the best candidate.

Return map

Notation:

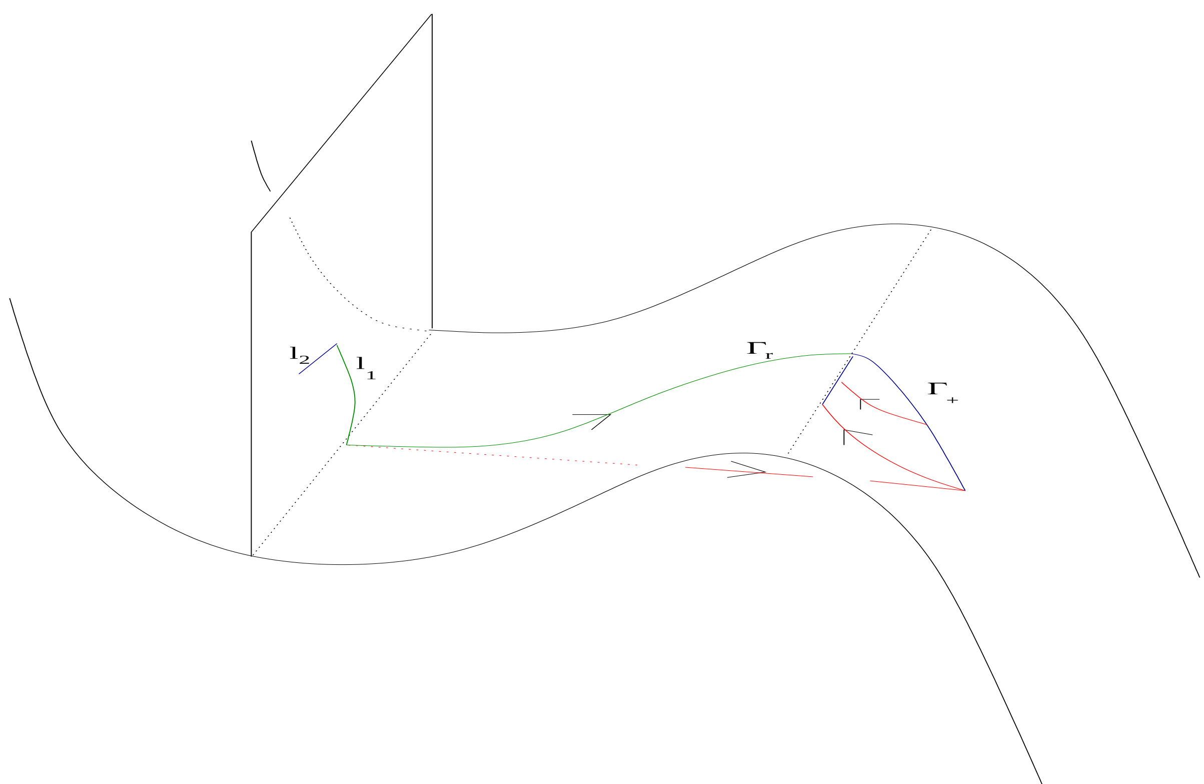
- Γ is the strong canard
- $\Gamma_r = \Gamma \cap S_r$
- Γ_+ is the projection of Γ_r onto S_{a+}.



Return map cont.

Proposition The part of the attractor outside of a small neighborhood of the folded saddle node is close the union of two curves $l_1 \cup l_2$, where:

- l_1 is the orthogonal projection along x of Γ_r onto Δ,
- l_2 is the projection by the flow of Γ_+ onto Δ.

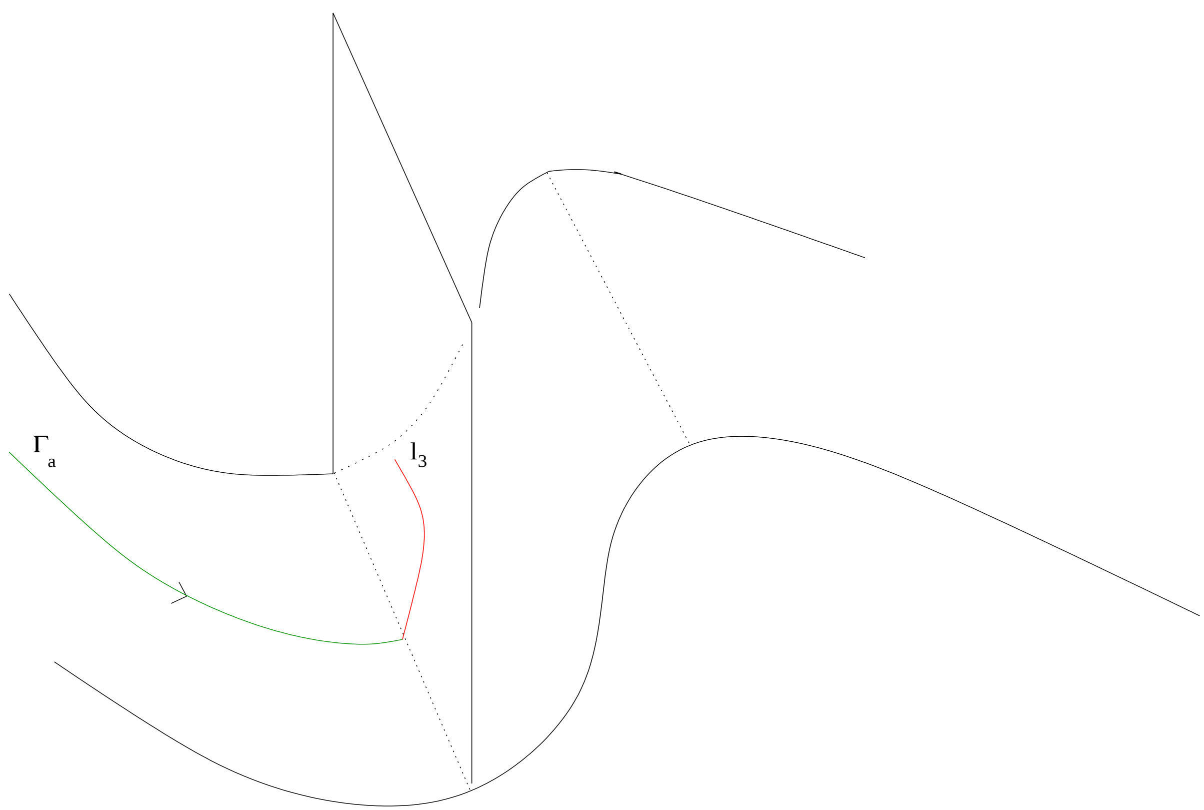


The curve $\tilde{W}$

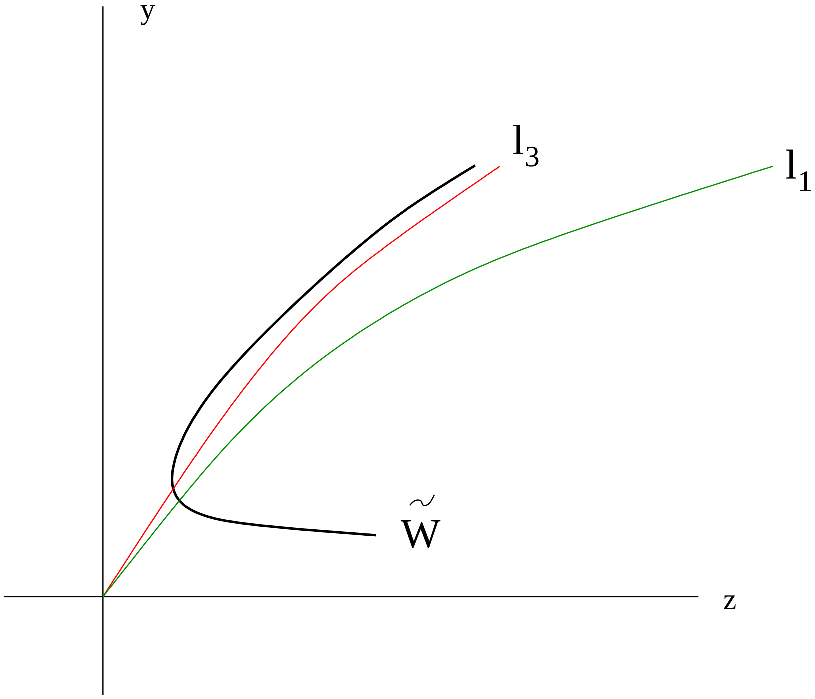
Let $\tilde{W}$ be the set of points in Δ whose forward trajectories end up in S_r after one passage near the fold.

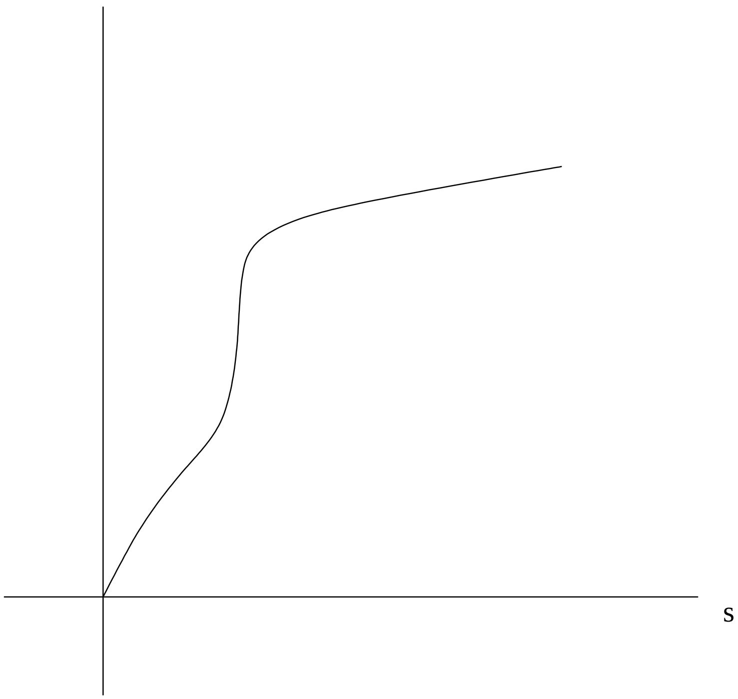
- $\tilde{W}$ divides Δ into points whose trajectories jump to S_{a+} before returning to Δ and trajectories that remain close to $S_{a-} \cup S_r$ and then jump to Δ.
- Let $\Gamma_a = \Gamma \cap S_{a-}$ and let l_3 be the orthogonal projection of Γ_a onto Δ.

Outside a small neighborhood of the fold $\tilde{W}$ is close to l_3



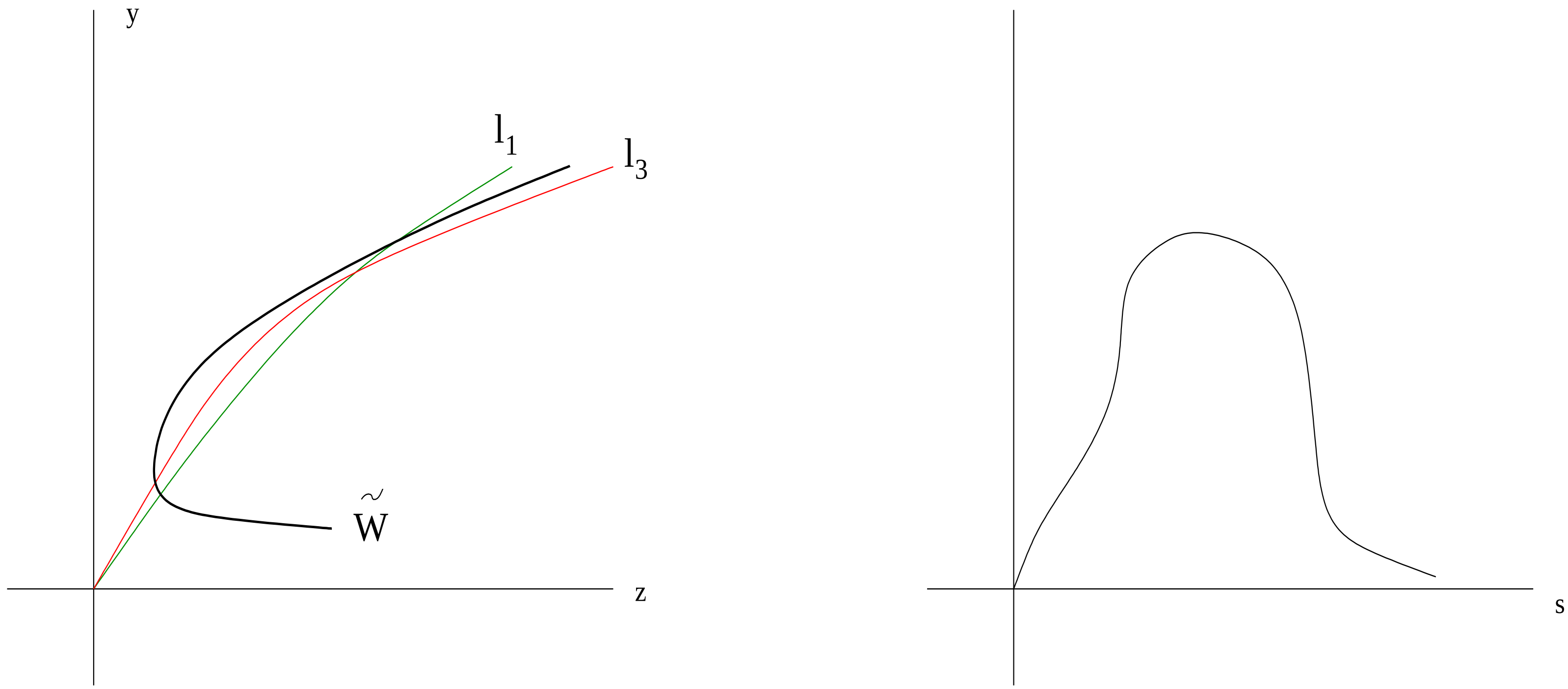
Return map - scenario 1





s = arclength of the attractor

Return map - scenario 2



Conclusions

- Canard induced mixed mode oscillations can happen in a large parameter regime ($\Delta\mu = O(1)$)
- In the interval of mixed mode oscillations there are infinitely many canard explosions
- Geometric approach is well suited for the analysis of this problem.