# Coupled Cell Systems

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# Overview

Coupled cell system: discrete space, continuous time system
Has information that cannot be understood by phase space theory alone

1) symmetry

   synchrony, phase shifts, multirhythms

2) groupoids

   input sets, balanced relations, quotient networks

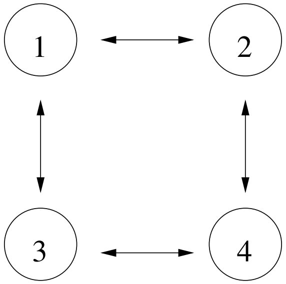
3) new states

   different dynamics on different cells

Primary Question: What aspects of the dynamics of coupled cell systems are due to network architecture?

# Part I: Symmetry and Synchrony

- Coupled cell systems described by graph

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$$f(x, y, z) = f(x, z, y)$$

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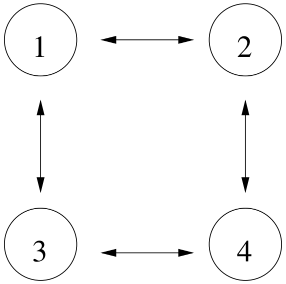


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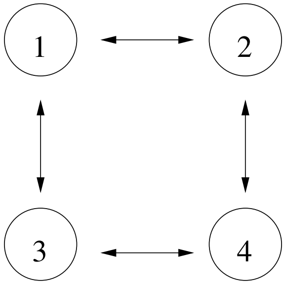


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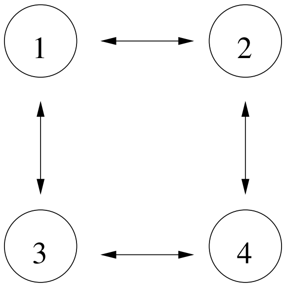


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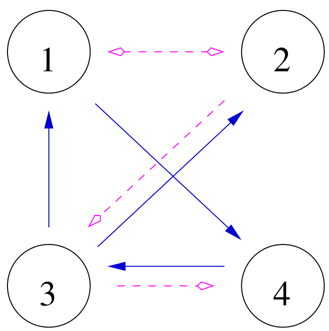
- Question: Are all synchrony spaces fixed-point spaces?

Answer: No




$$\begin{aligned}
\dot{x}_1 &= g(x_1, x_3, x_2) \\
\dot{x}_2 &= g(x_1, x_3, x_1) \\
\dot{x}_3 &= g(x_1, x_4, x_2) \\
\dot{x}_4 &= g(x_1, x_1, x_3)
\end{aligned}$$

# Spatio-Temporal Symmetries

Let $x(t)$ be a time-periodic solution

- $K = \{\gamma \in \Gamma : \gamma x(t) = x(t)\}$ **space symmetries**
- $H = \{\gamma \in \Gamma : \gamma\{x(t)\} = \{x(t)\}\}$ **spatiotemporal symmetries**

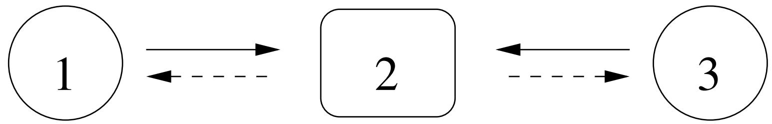
**Facts**:

- $\gamma \in H \Longrightarrow \theta \in \mathbf{S}^1$ such that $\gamma x(t) = x(t + \theta)$
- $H/K$ is cyclic

**Question**:

How do spatiotemporal symmetries manifest themselves in coupled cell systems?

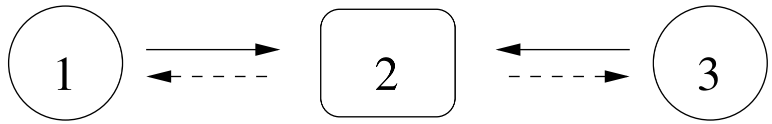
# A Three-Cell System




$$\begin{aligned}\dot{x}_1 &= f(x_1, x_2)\\ \dot{x}_2 &= g(x_2, x_1, x_3)\\ \dot{x}_3 &= f(x_3, x_2)\end{aligned}$$

$$g(x_2, x_1, x_3) = g(x_2, x_3, x_1)$$

# A Three-Cell System




- $$\begin{aligned}\dot{x}_1 &= f(x_1, x_2)\\ \dot{x}_2 &= g(x_2, x_1, x_3) \qquad\qquad g(x_2, x_1, x_3) = g(x_2, x_3, x_1)\\ \dot{x}_3 &= f(x_3, x_2)\end{aligned}$$

- Symmetry: $\sigma(x_1, x_2, x_3) = (x_3, x_2, x_1)$

  $\mathrm{Fix}(\sigma) = \{x_1 = x_3\}$ is flow-invariant. Robust synchrony

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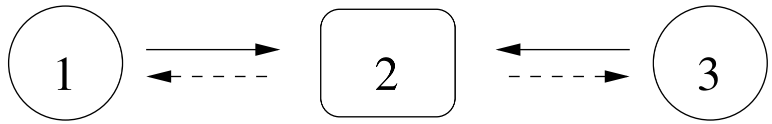


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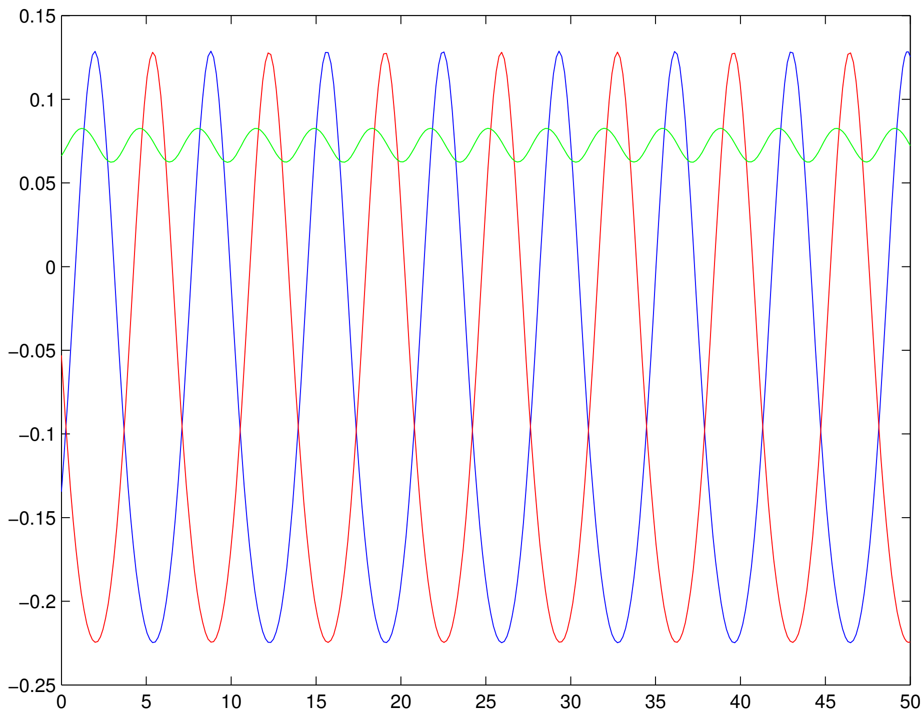
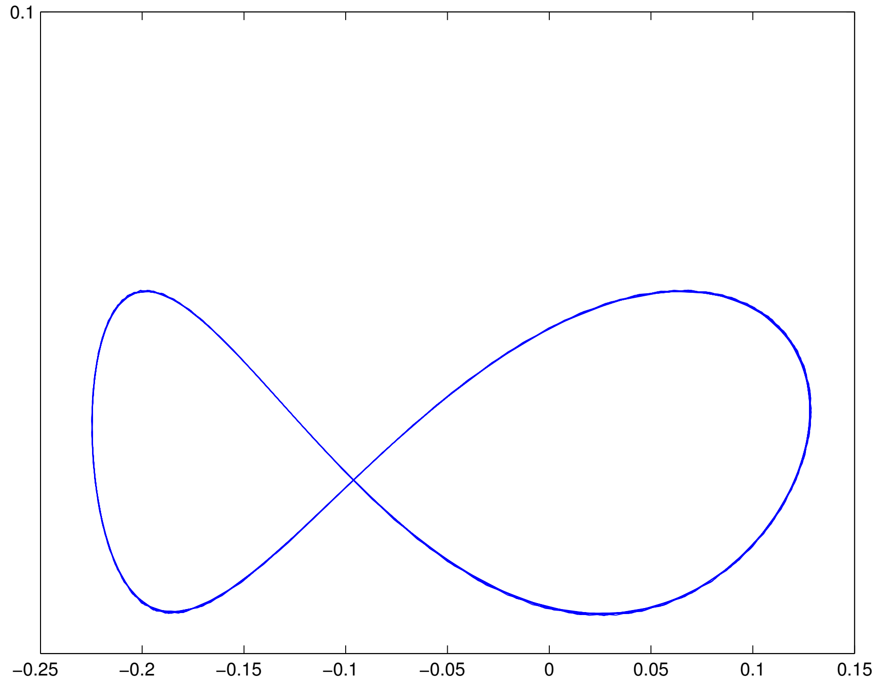
- Out-of-phase periodic solutions ($H = \mathbf{Z}_2(\sigma)$, $K = \mathbf{1}$):

$$\sigma X(t) = X\left(t + \tfrac{1}{2}\right)$$

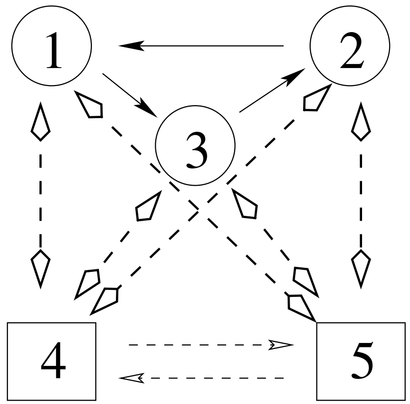
$$x_3(t) = x_1\left(t + \tfrac{1}{2}\right) \quad \text{and} \quad x_2(t) = x_2\left(t + \tfrac{1}{2}\right)$$

# A Three-Cell System (2)



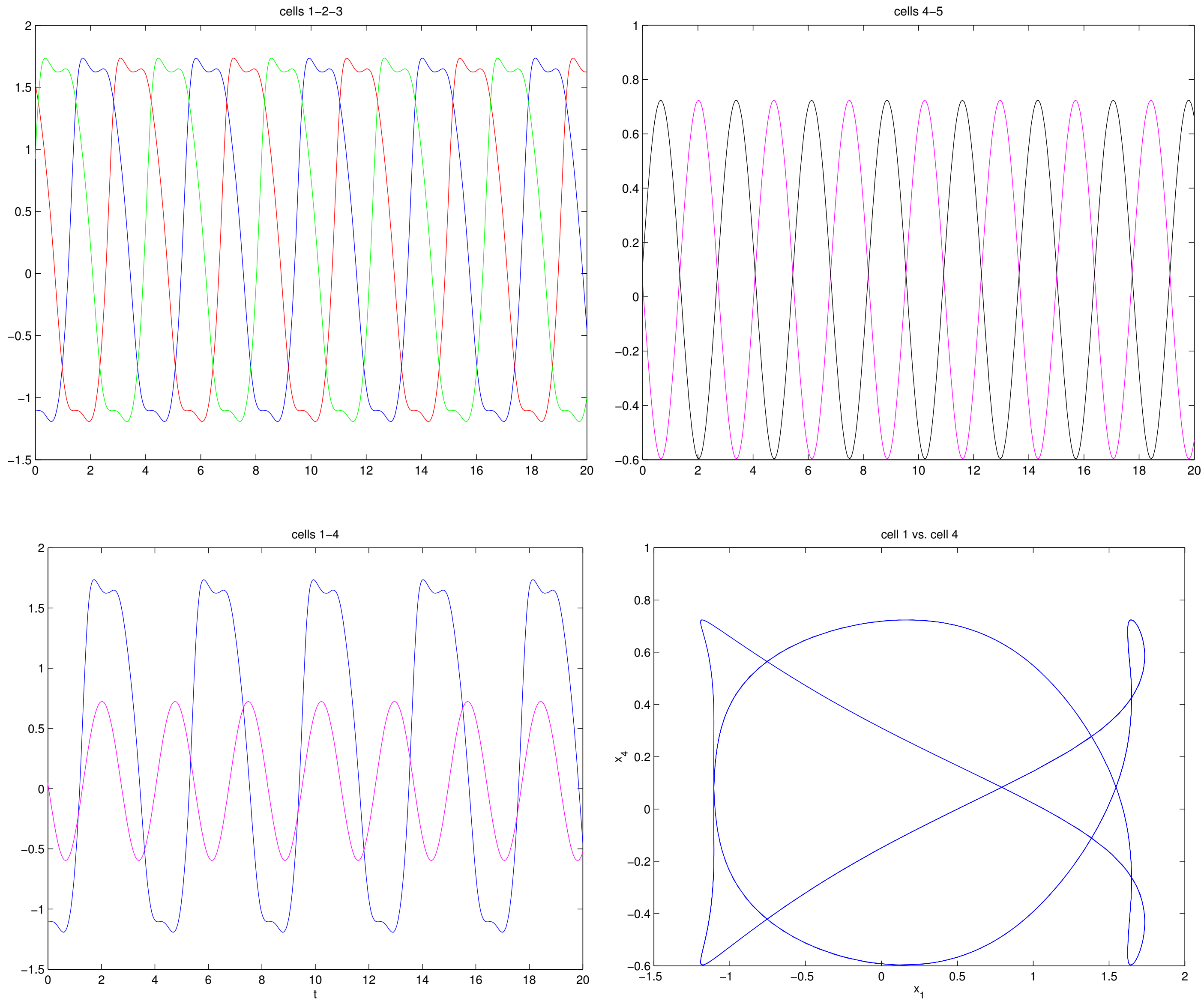
# Polyrhythms




- Symmetry group of five-cell system is $\mathbf{Z}_3 \times \mathbf{Z}_2 \cong \mathbf{Z}_6$
- Periodic solutions with $(H, K) = (\mathbf{Z}_6, \mathbf{1})$ can exist
- Let $\sigma = (\rho, \tau)$ be generator of $\mathbf{Z}_3 \times \mathbf{Z}_2$.

$(\sigma^2, 1/3) \Longrightarrow$ 3-cell ring exhibits rotating wave
$(\sigma^3, 1/2) \Longrightarrow$ 2-cell ring is out-of-phase
$(\sigma \;\;, 1/6) \Longrightarrow$ triple 2-cell freq = double 3-cell freq

# Polyrhythms (2)



# Summary on Symmetry

Permutation symmetries of coupled cell systems lead to

- synchrony
- discrete rotating waves
- multifrequency motions

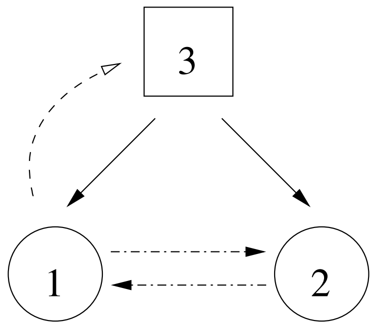
# Part II: Coupled Cell Theory

- input sets and input isomorphisms
- network architecture and symmetry groupoids
- balanced colorings and synchrony subspaces
- quotient networks (discussed with examples)

## Main Results

1) synchrony subspace iff balanced coloring
2) restriction to synchrony subspace is a coupled cell system — the quotient network
3) every quotient cell system lifts

# Asymmetric Three-Cell Network

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- Robust synchrony exists in networks without symmetry
- Polydiagonal $Y = \{x : x_1 = x_2\}$ is flow-invariant

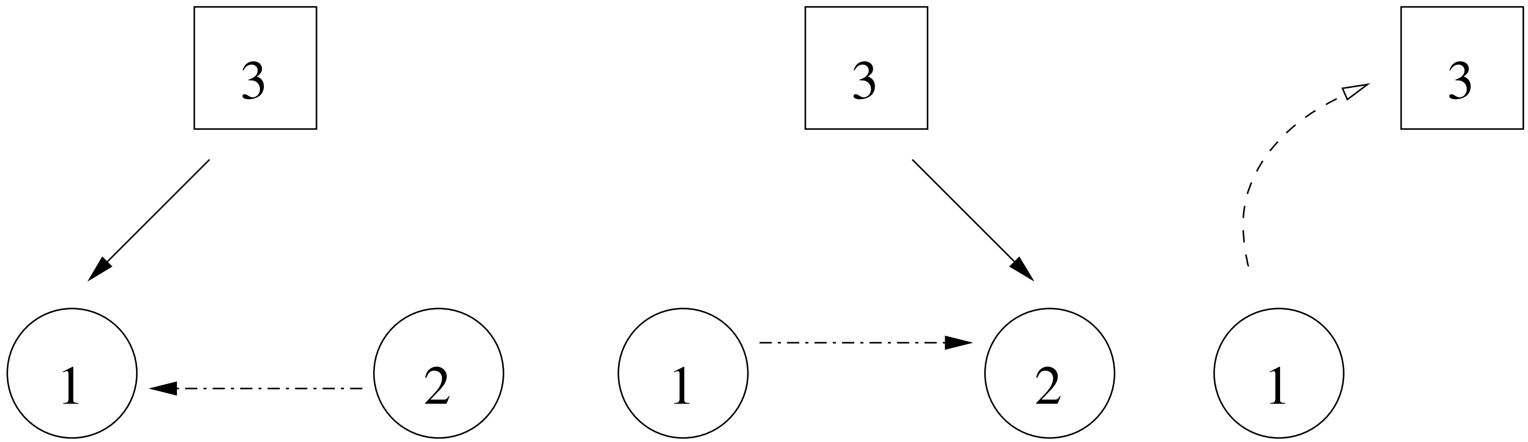
Restrict equations $\dot{x}_1, \dot{x}_2$ to $Y$

$$\begin{aligned}\dot{x}_1 &= f(x_1, x_1, x_3)\\ \dot{x}_2 &= f(x_1, x_1, x_3)\end{aligned}$$

- Cells 1 and 2 are **identical within the network**

# Input Sets

- Input set of cell $j$: Cell $j$ & cells $i$ that connect to $j$
- Key idea: cells 1, 2 have isomorphic input sets



# Coupled Cell Network Definition

(a) A set $\mathcal{C} = \{1, \ldots, N\}$ of *cells*

(b) An equivalence relation $\sim_C$ on cells in $\mathcal{C}$

(c) Each node $c$ has a finite set of *input terminals* $I(c)$. Each $i \in I(c)$ corresponds to an *arrow* $(\tau(i), i)$ beginning at $\tau(i)$ and ending at $i$. $\mathcal{E}$ = set of arrows

(d) An equivalence relation $\sim_E$ on arrows in $\mathcal{E}$

(e) Equivalent arrows have equivalent tails and heads

A coupled cell network is represented by a graph

- For each class of cells choose node symbol $\bigcirc, \Box, \triangle$
- For each class of arrows choose arrow symbol $\longrightarrow, \Rightarrow, \rightsquigarrow$

# Symmetry Groupoid

- Cells $c, d$ are input equivalent $\sim_I$ if there is a bijection

$$\beta : I(c) \to I(d)$$

such that $(i, c) \sim_E (\beta(i), d)$ for all $i \in I(c)$

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- Groupoid is like group; but products not always defined
- Coupled cell systems: ODEs that commute with $\mathcal{B}_G$

# Patterns of Synchrony

- Color cells in $\mathcal{C}$

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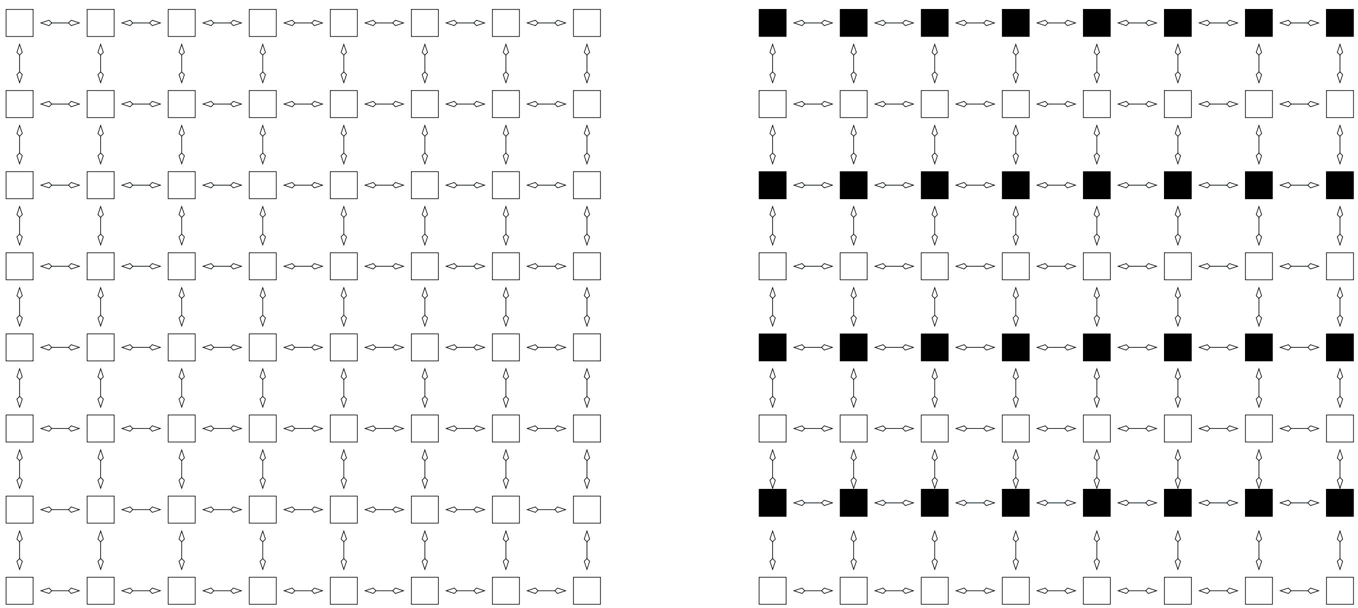
- Coloring is balanced if every pair of cells with same color has a color preserving input isomorphism
- **Thm**: Coloring is pattern of synchrony iff coloring is balanced

# Part III: Examples

- Lattice dynamical systems
  - Classify balanced two colorings up to symmetry
  - Balanced two colorings occur in codimension one bifurcations (use quotient networks)
- Feed-forward network
  - Amplitude enhancement in Hopf bifurcation
  - Different dynamics in different cells

# Lattice Dynamical Systems

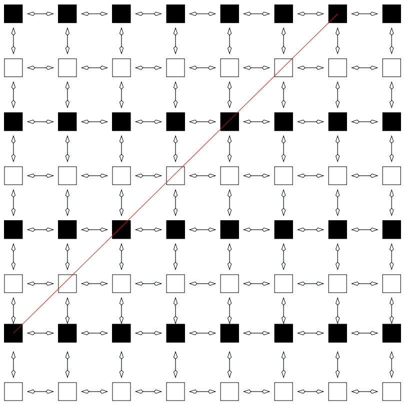
- Consider square lattice with nearest neighbor coupling
- Form a two-color balanced relation

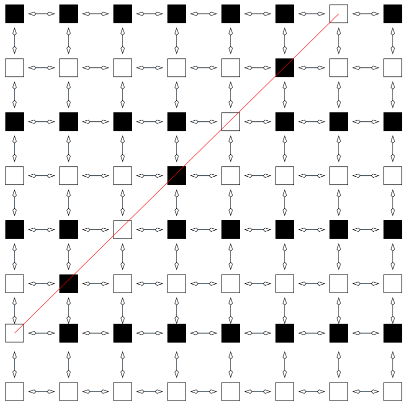


- Each black cell connected to two black and two white
  Each white cell connected to two black and two white

# Lattice Dynamical Systems (2)

- On Black/White diagonal interchange black and white





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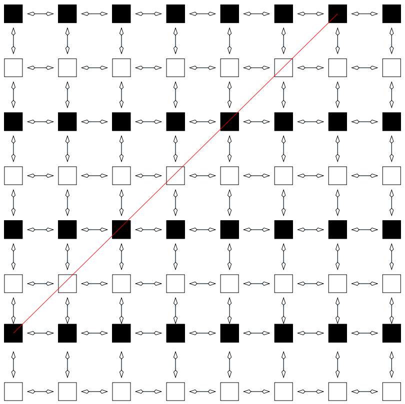
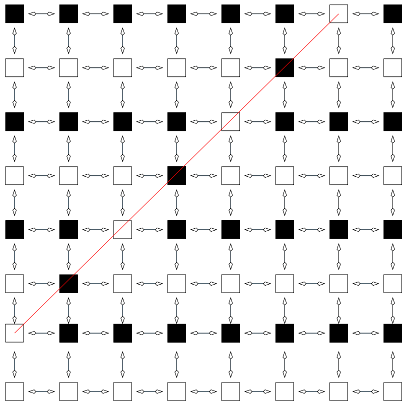
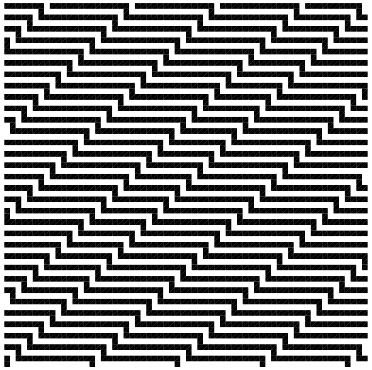
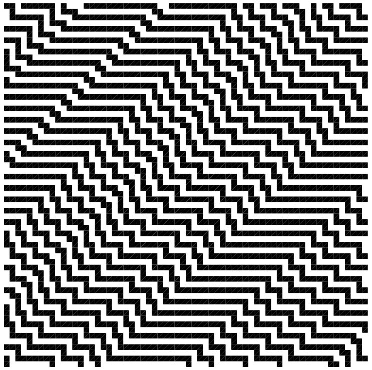




Result is balanced

- A continuum of different patterns of synchrony exist

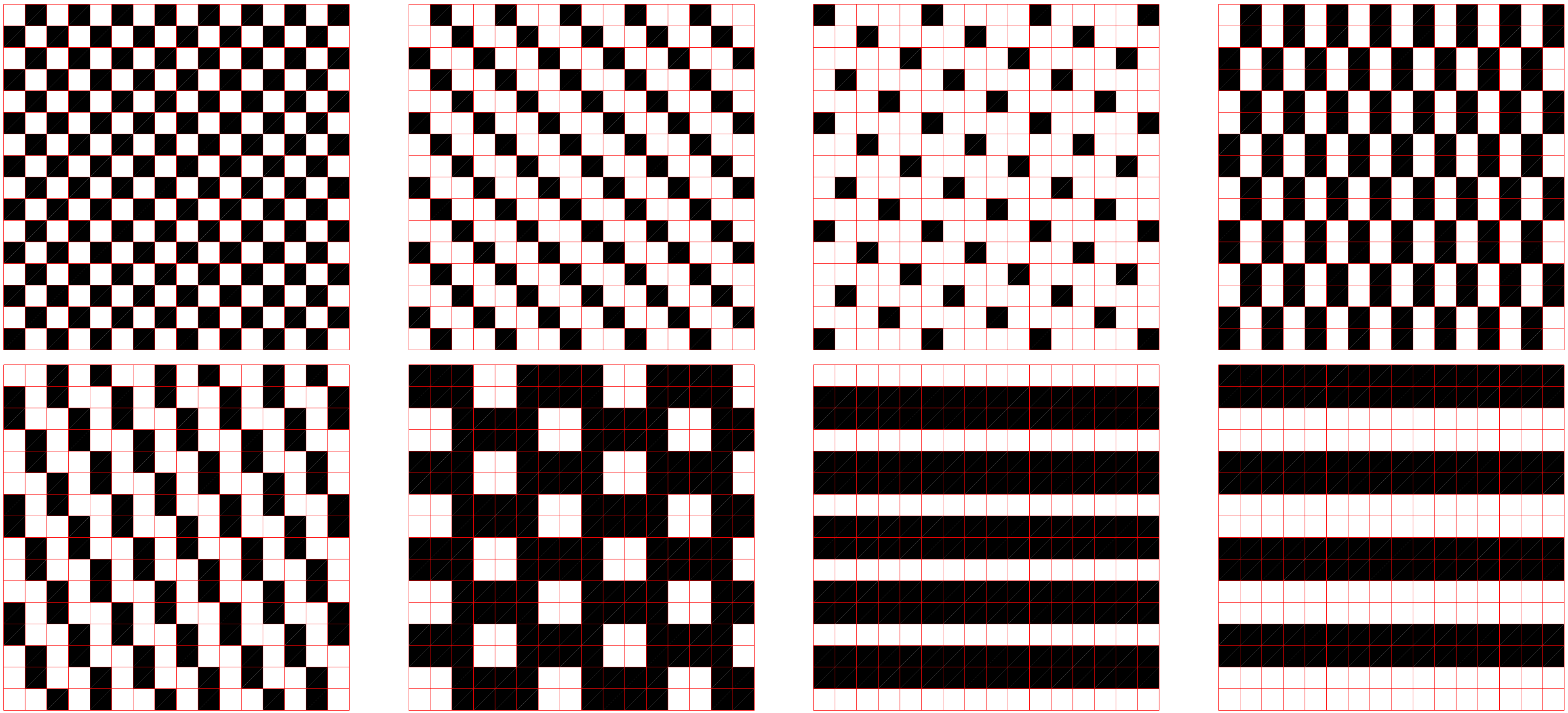




# Lattice Dynamical Systems (3)

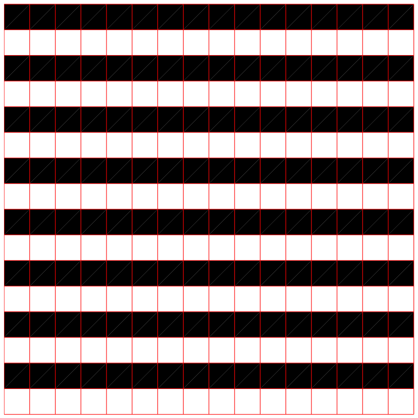
Yunjiao Wang

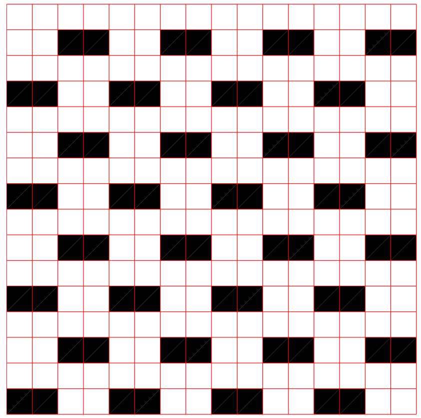
There are eight isolated balanced two-colorings on square lattice with nearest neighbor coupling



# Lattice Dynamical Systems (4)

There are two infinite families of balanced two-colorings generated by interchanging black and white along diagonals on which black and white cells alternate





Up to symmetry, these are the two-color patterns of synchrony

# Quotient Cell Systems

Given $\mathcal{G} = (\mathcal{C}, \sim_C, \mathcal{E}, \sim_E)$ and balanced coloring $\bowtie$

Define: *quotient network* $\mathcal{G}_{\bowtie} = (\mathcal{C}_{\bowtie}, \sim_{\mathcal{C}_{\bowtie}}, \mathcal{E}_{\bowtie}, \sim_{\mathcal{E}_{\bowtie}})$ by

(a) $\mathcal{C}_{\bowtie} = \{\overline{c} : c \in \mathcal{C}\} = \mathcal{C}/\bowtie$

(b) Define $\overline{c} \sim_{C_{\bowtie}} \overline{d} \iff c \sim_C d$

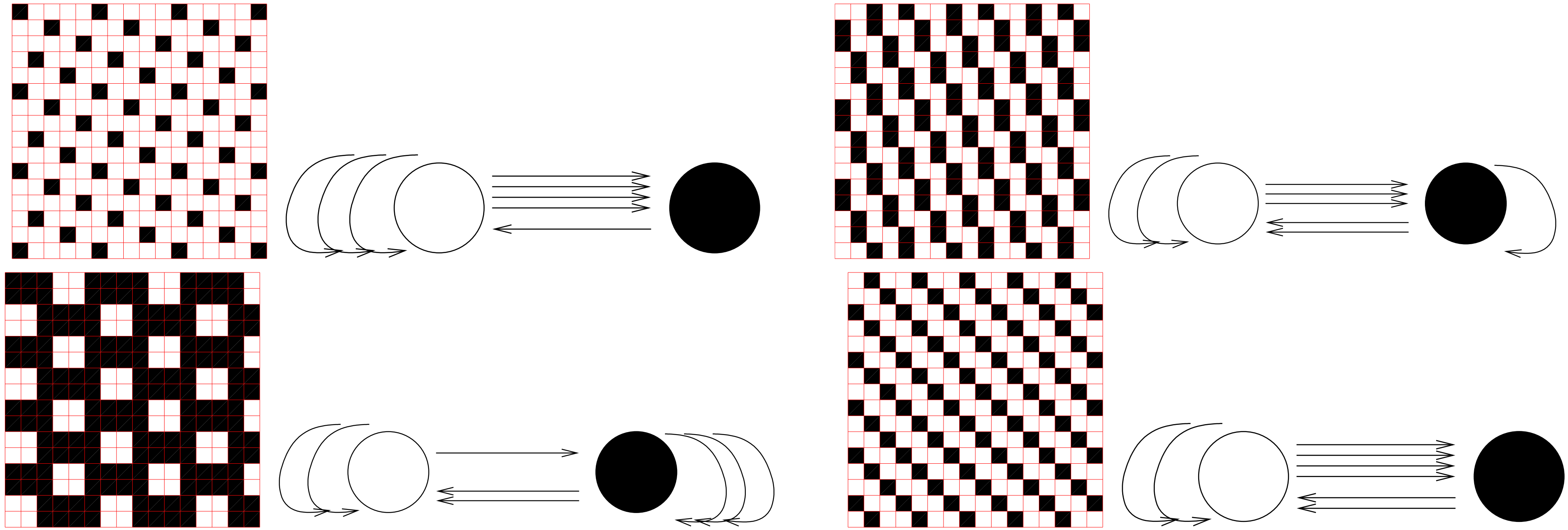
(c) Arrows in quotient are projection of arrows in original network $\quad \mathcal{E}_{\bowtie} = \{(\overline{\tau(i)}, i) : (\tau(i), i) \in \mathcal{E}\}$

(d) Quotient arrows are $\sim_{\mathcal{E}_{\bowtie}}$ when original arrows are $\sim_E$

Thm: $\mathcal{G}$-admissible ODE restricted to $\Delta_{\bowtie}$ is $\mathcal{G}_{\bowtie}$-admissible

Every $\mathcal{G}_{\bowtie}$-admissible ODE on $\Delta_{\bowtie}$ lifts to $\mathcal{G}$-admissible ODE

# Two Color Quotient Networks



- Balanced two coloring has two-cell quotient

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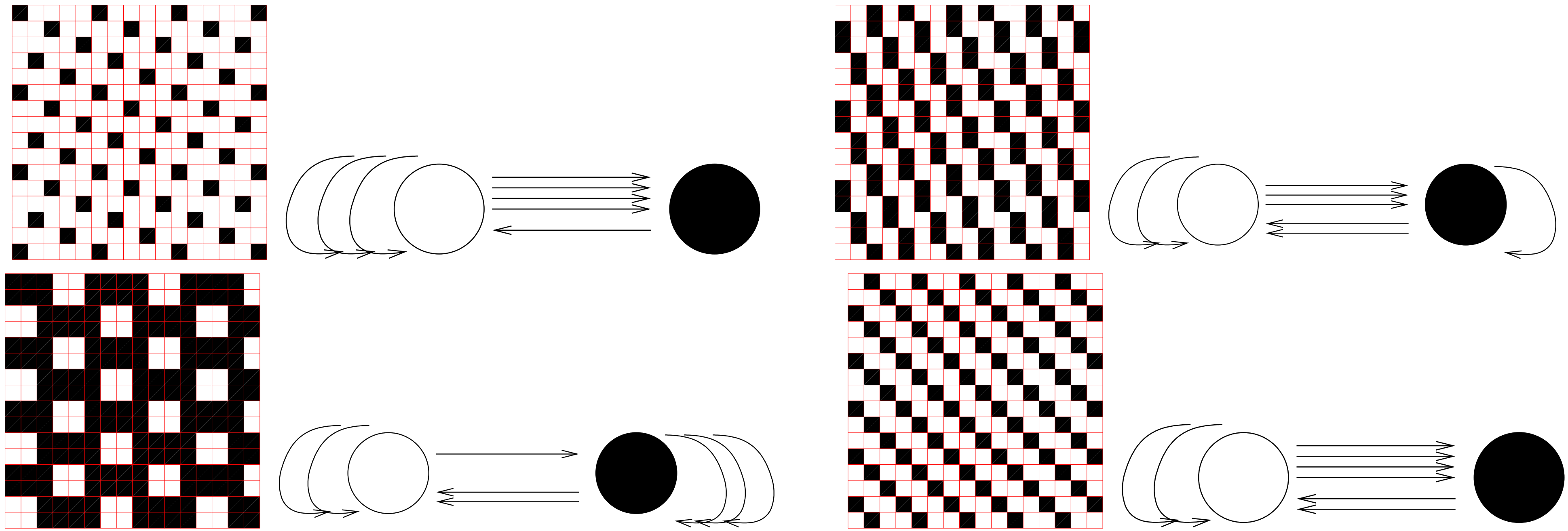
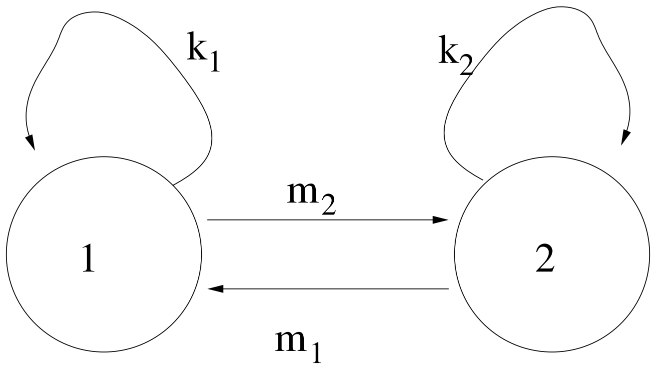


- Balanced two coloring has two-cell quotient
- Claim: Each balanced two coloring of square lattice leads to equilibria in codimension one bifurcations

# Homogeneous Two-Cell Networks




$$\ell = k_1 + m_1 = k_2 + m_2$$

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$x_1 = x_2$ is flow-invariant

- Jacobian = $\begin{bmatrix} \alpha + k_1\beta & m_1\beta \\ m_2\beta & \alpha + k_2\beta \end{bmatrix}$ where

  $\alpha =$ linear internal and $\beta =$ linear coupling

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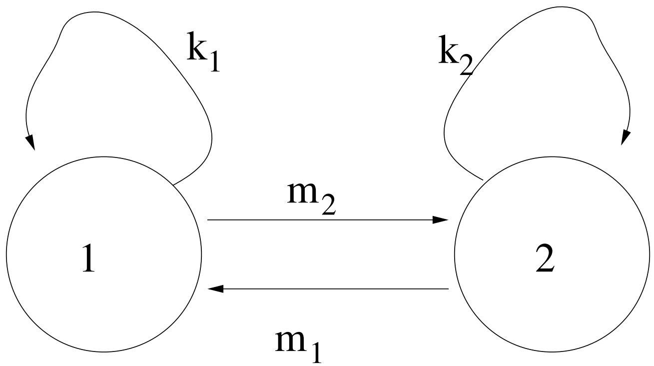


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- Eigenvalues are $\alpha + \ell\beta$ $((1,1))$ and $\alpha + (k_1 + k_2 - \ell)\beta$

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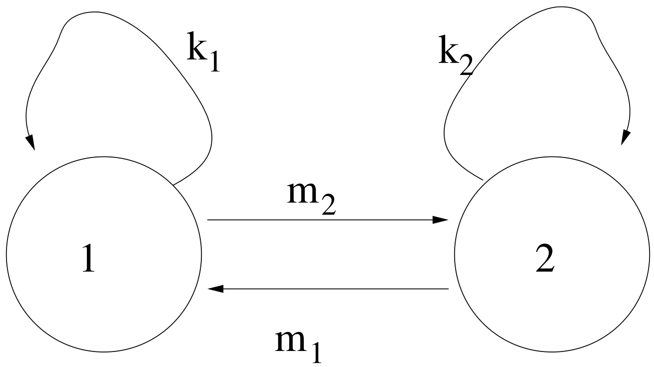


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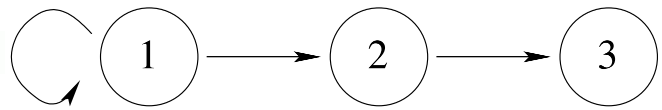
- Vary $\beta$: codimension 1 synchrony-breaking bifurcation

# Three-Cell Feed-Forward Network




$$\begin{aligned} \dot{x}_1 &= f(x_1, x_1) \\ \dot{x}_2 &= f(x_2, x_1) \\ \dot{x}_3 &= f(x_3, x_2) \end{aligned}$$

$$J = \begin{bmatrix} \alpha+\beta & 0 & 0 \\ \beta & \alpha & 0 \\ 0 & \beta & \alpha \end{bmatrix}$$

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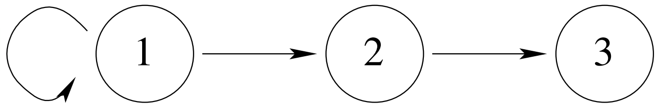
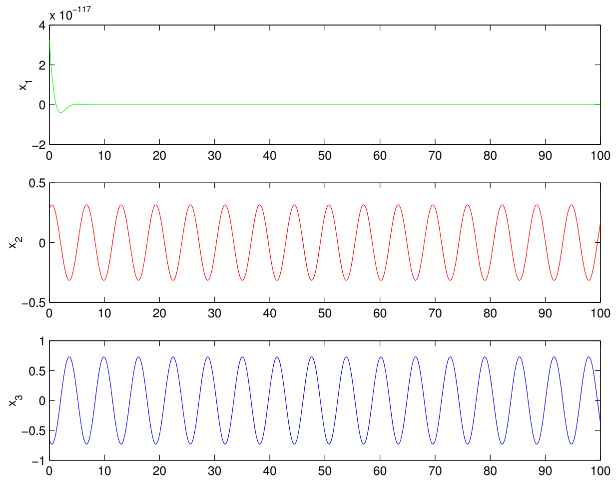
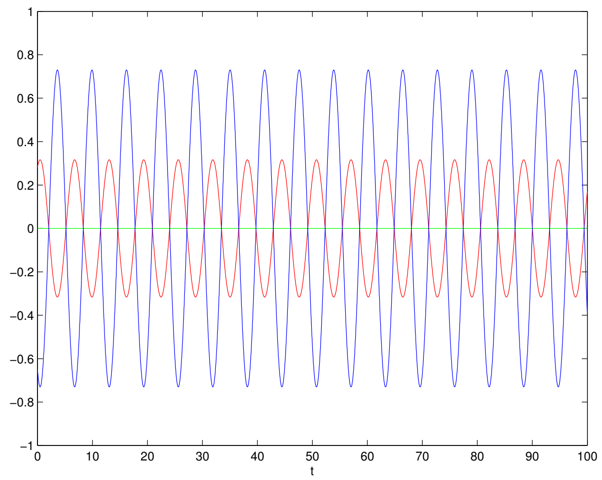
- 

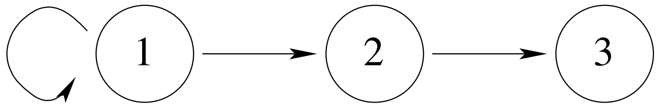
$$\begin{aligned} \dot{x}_1 &= f(x_1, x_1) \\ \dot{x}_2 &= f(x_2, x_1) \\ \dot{x}_3 &= f(x_3, x_2) \end{aligned} \qquad J = \begin{bmatrix} \alpha+\beta & 0 & 0 \\ \beta & \alpha & 0 \\ 0 & \beta & \alpha \end{bmatrix}$$

- Network supports solution by Hopf bifurcation where $x_1(t)$ equilibrium $\quad x_2(t), x_3(t)$ time periodic



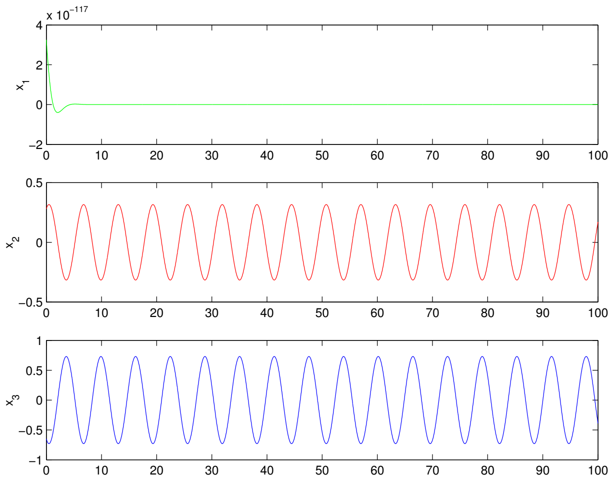


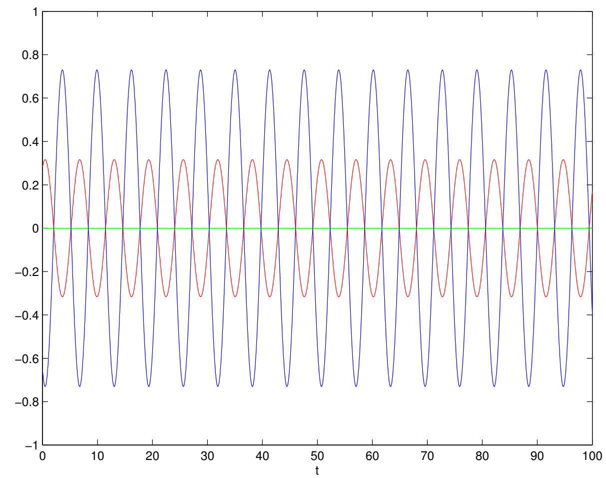
# Three-Cell Feed-Forward Network

- 


$$\begin{aligned} \dot{x}_1 &= f(x_1, x_1) \\ \dot{x}_2 &= f(x_2, x_1) \\ \dot{x}_3 &= f(x_3, x_2) \end{aligned} \qquad J = \begin{bmatrix} \alpha+\beta & 0 & 0 \\ \beta & \alpha & 0 \\ 0 & \beta & \alpha \end{bmatrix}$$

- Network supports solution by Hopf bifurcation where $x_1(t)$ equilibrium $\qquad x_2(t), x_3(t)$ time periodic



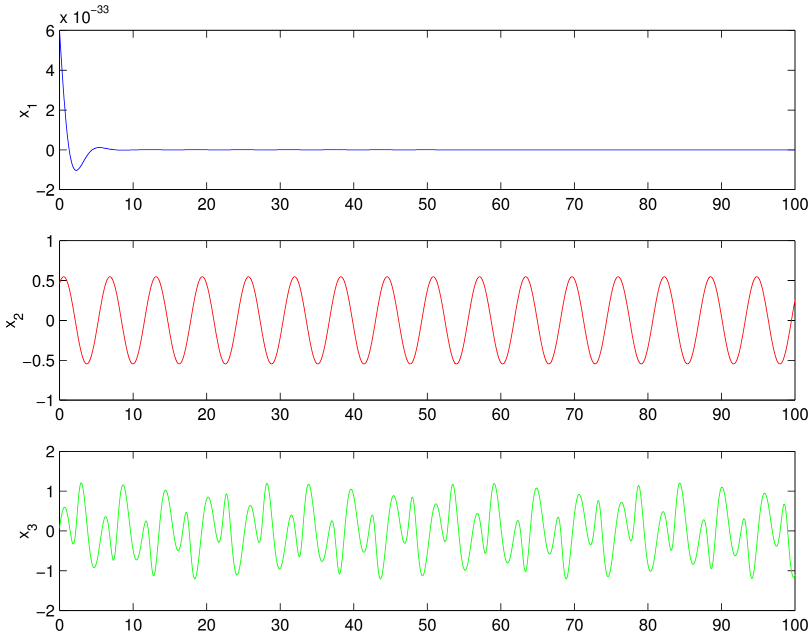



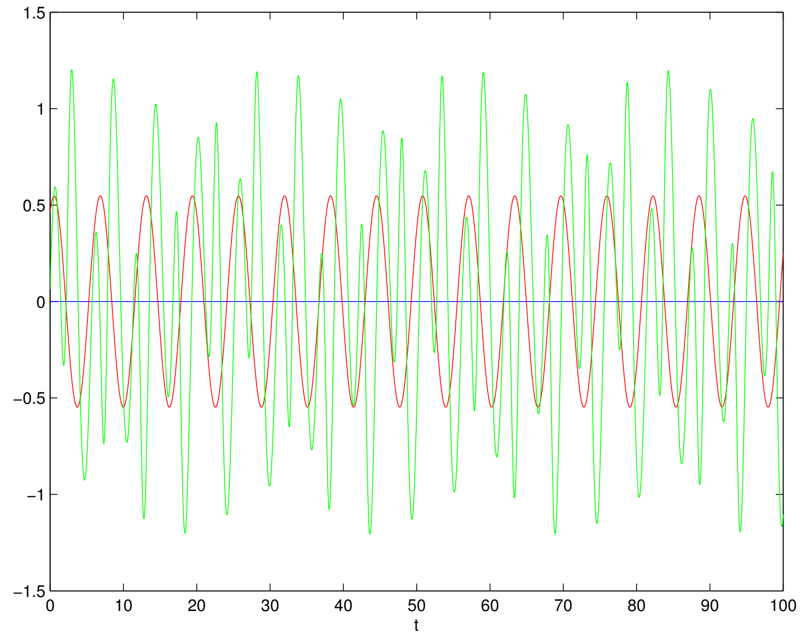

- $x_2(t) \approx \lambda^{1/2} \qquad x_3(t) \approx \lambda^{1/6}$

# Three-Cell Feed-Forward Network (2)

- Network supports solution where

$x_1(t)$ equilibrium, $x_2(t)$ time periodic, $x_3(t)$ quasiperiodic



# Something to think about

- In a far away land

# Something to think about

- In a far away land
- In a far away corner

# Something to think about

- In a far away land
- In a far away corner
- Near a big island (Hook Island)

# Something to think about

- In a far away land
- In a far away corner
- Near a big island (Hook Island)
- Near a small beach (Stonehaven)

# Something to think about

- In a far away land
- In a far away corner
- Near a big island (Hook Island)
- Near a small beach (Stonehaven)
- Is a beautiful small island