PRODUCT DYNAMICS

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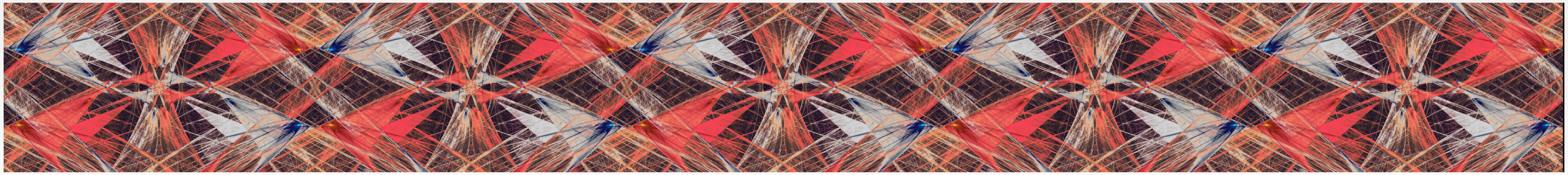
Workshop on Bifurcation Theory and Spatio-Temporal
Pattern Formation in PDE
in Honour of Professor William Langford

December 11–13, 2003.



Summary of Talk

- Heteroclinic networks:
  Templates for complex dynamics.
- Switching.
- Connection Selection.
- Product dynamics.
- Attractors.
- Results
- Indication of proofs.

Heteroclinic Networks

A feature of equivariant dynamics and, more generally, coupled cell systems is the existence of robust heteroclinic cycles and networks.

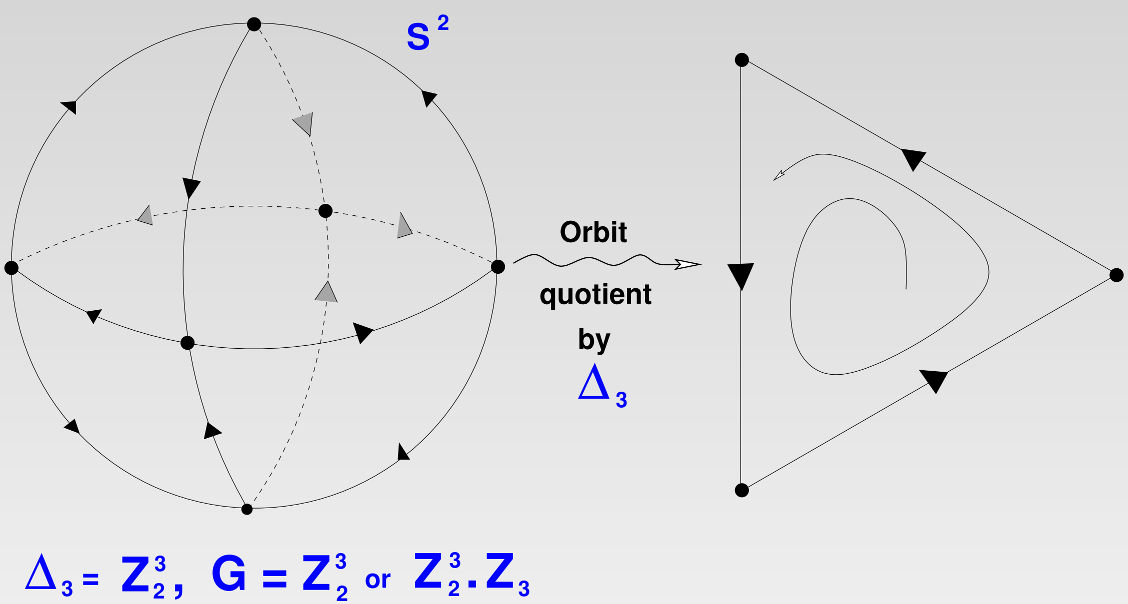


Figure 1: Network on S^2 and orbit quotient

1-dimensional connections

In simple situations, a heteroclinic network will be comprised of a number of *nodes*, A_j – typically, hyperbolic equilibria of index 1 – together with 1-dimensional connections between nodes (usually $W^u(A_j)$).

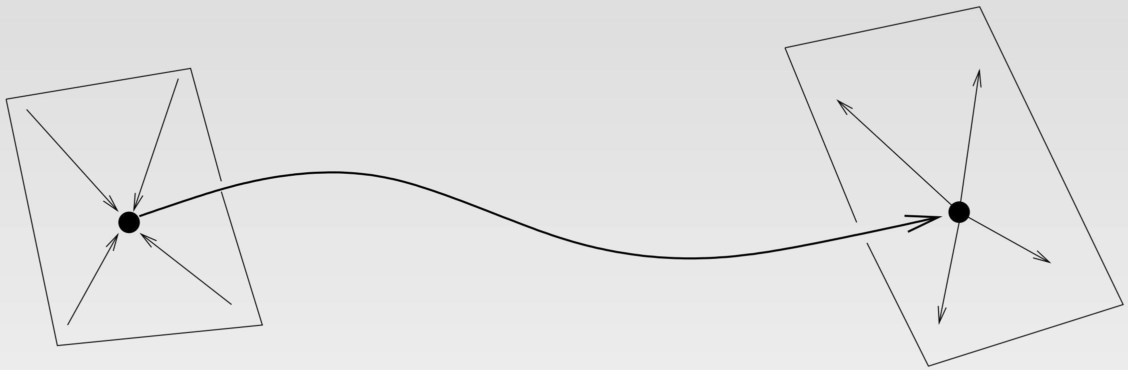


Figure 2: 1-dimensional connections

Even if there are only 1-dimensional connections, dynamics may be very complex near an attracting network. For example, in the recent thesis of Manuela Aguiar (Porto), Aguiar shows that random switching can occur between the nodes of a heteroclinic network, with switching precisely quantified by a subshift of finite type. This switching occurs arbitrarily close to the network (also work by Kirk & Silber, Guckenheimer & Worfolk).

Complex dynamics I

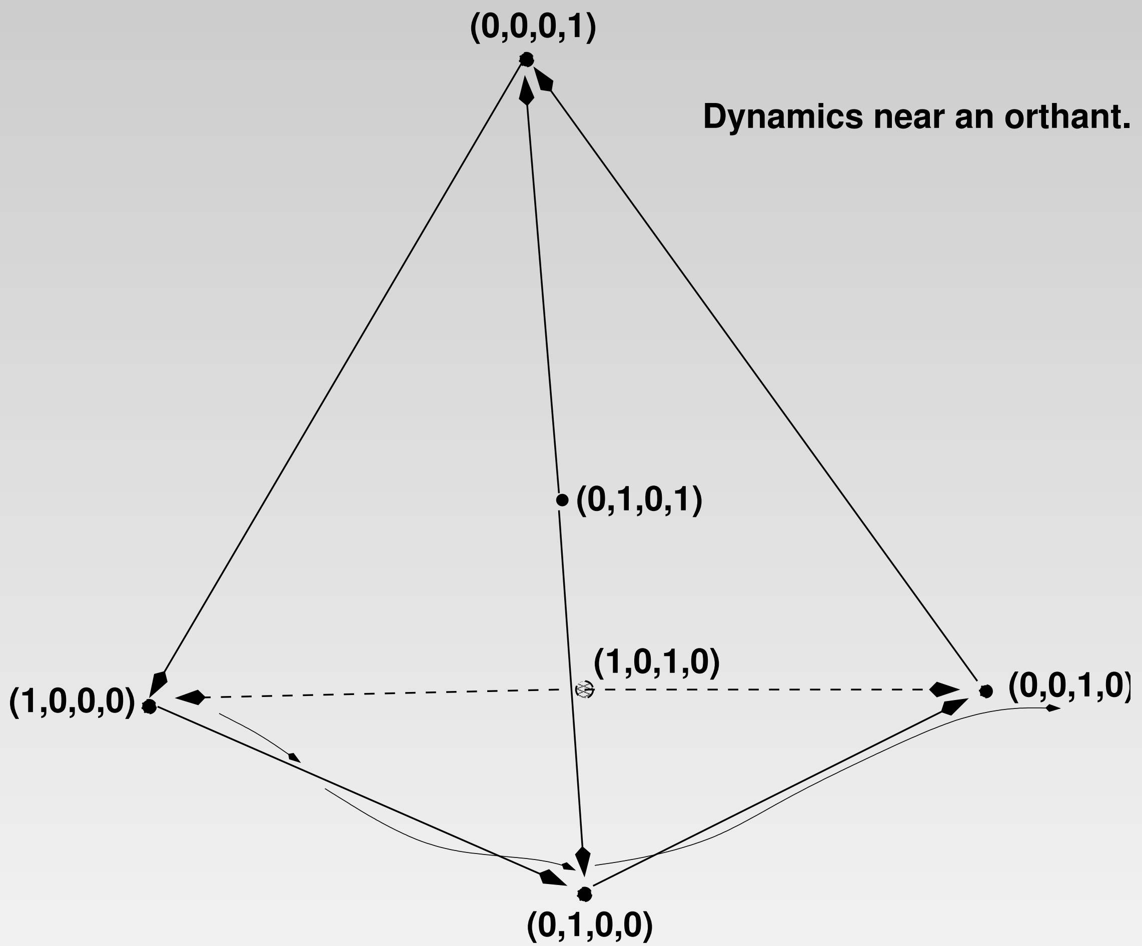


Figure 3: Switching

Complex Dynamics II

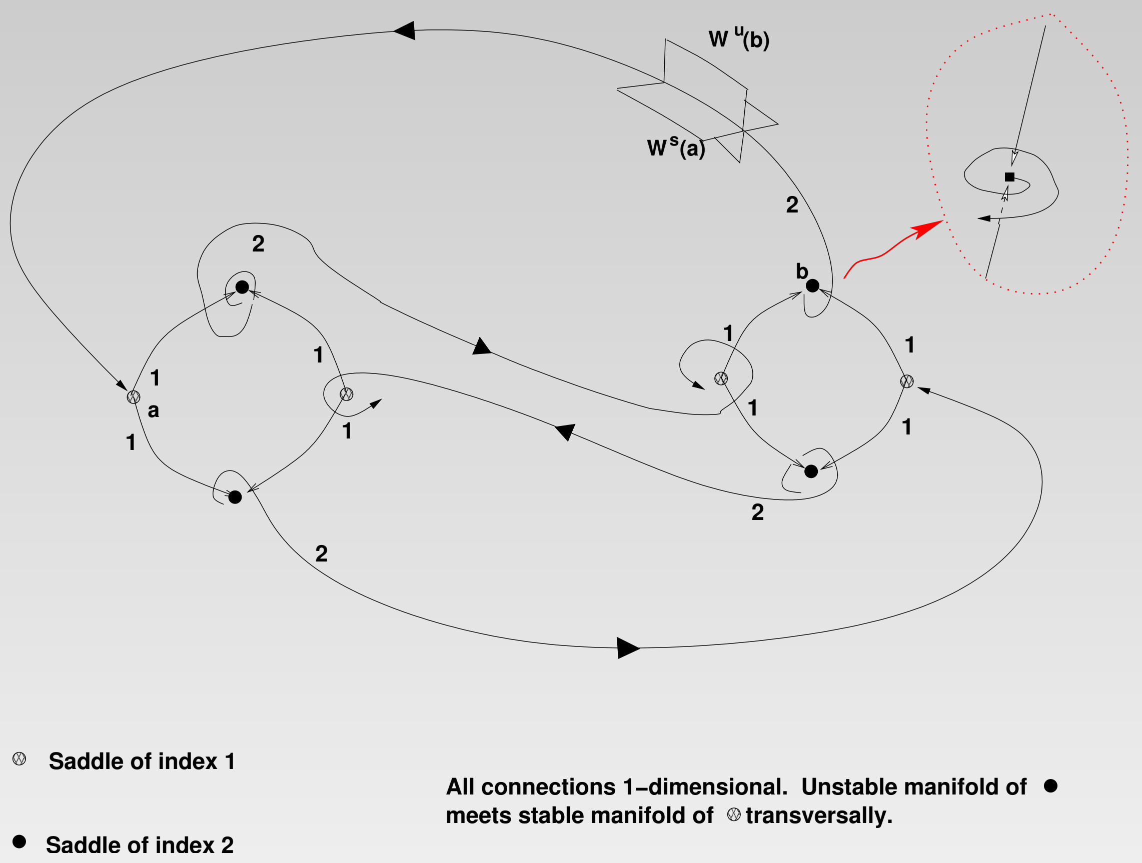


Figure 4: A more complicated network

Connection selection

More generally, nodes can be limit cycles or chaotic sets (“cycling chaos”). Also, there may be a continua of connections between nodes. The latter behaviour is typical of the kind of networks that can be expected in ‘coupled cell systems’ where invariant subspaces typically correspond to *synchronous* states.

In this talk we focus on the study of dynamics and asymptotics near a heteroclinic attractor. The main problem we consider is that of *connection selection*.

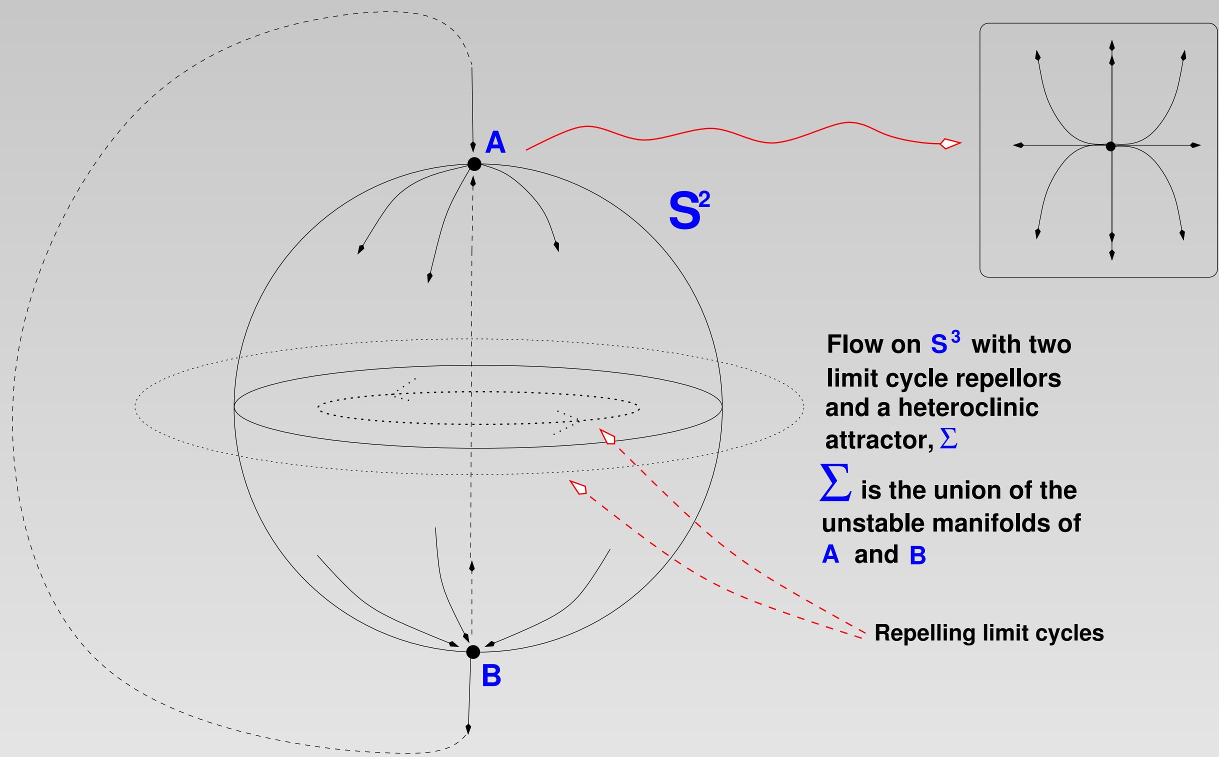


Figure 5: A heteroclinic attractor in S^3

Suppose $x \in \mathcal{B}(\Sigma)$ (basin of attraction). What, generically is $\omega(x)$? More precisely, what connections in $S^2 \subset \Sigma$ do we expect to 'see' in the dynamics.

History & Sources

The issue of connection selection was first raised in Ashwin & Chossat (Attractors for robust heteroclinic cycles with continua of connections, *J. Nonlinear Sci.*, 1997). Other relevant sources include Melbourne (Intermittency as a codimension three phenomenon, *J. Dyn. Diff. Eqn.* 1989), Kirk & Silber (A competition between heteroclinic cycles, *Nonlinearity* 1994). Ashwin & F (Heteroclinic networks in coupled cell systems, *Arch. Rat. Mech. & Anal. 1999*), Ashwin, F, Rucklidge & Sturman (Phase resetting effects for robust cycles between chaotic sets, *Chaos*, 2003), Guckenheimer & Worfolk ('Instant Chaos', *Nonlinearity*, 1992).

Product dynamics

The problem of connection selection is tricky.

In this talk, we focus on a simple situation (one where we can obtain results. . .).

A prerequisite for understanding the behaviour of coupled systems is understanding the dynamics of uncoupled (product) systems. For example, the product of periodic attractors.

We look at flows that are the product of a homoclinic attractor with either an attracting limit cycle, or a chaotic set or another homoclinic attractor.

Attractors

Let Φ_t be a continuous flow (or semiflow) defined on a compact region $M \subset \mathbb{R}^n$. We assume that M is forward invariant under Φ_t. Denote Lebesgue measure on $\mathbb{R}^n$ by ℓ. If $X \subset M$, let

$$\mathcal{B}(X) = \{x \in M \mid \omega(x) \subset X\}$$

denote the *basin of attraction* of X.

Definitions

A compact invariant set $X \subset M$ is a *(Milnor) Attractor* if

1. $\ell(\mathcal{B}(X)) > 0$.
2. For any proper compact invariant $Y \subset X$, $\ell(\mathcal{B}(X) \setminus \mathcal{B}(Y)) > 0$.

X is a *minimal* attractor if for all proper compact invariant $Y \subset X$, $\ell(\mathcal{B}(Y)) = 0$.

X is minimal iff $\exists$ a full measure $B \subset \mathcal{B}(X)$ such that $\omega(x) = X$, all $x \in B$.

Definitions

If Z is an invariant measurable set with $\ell(Z) > 0$, then the *likely limit set* $\Lambda(Z)$ of Z is the smallest closed invariant set that contains all ω-limit sets except for a zero measure subset of Z.

If X is a Milnor attractor then $\Lambda(\mathcal{B}(X)) = X$.

A model homoclinic attractor

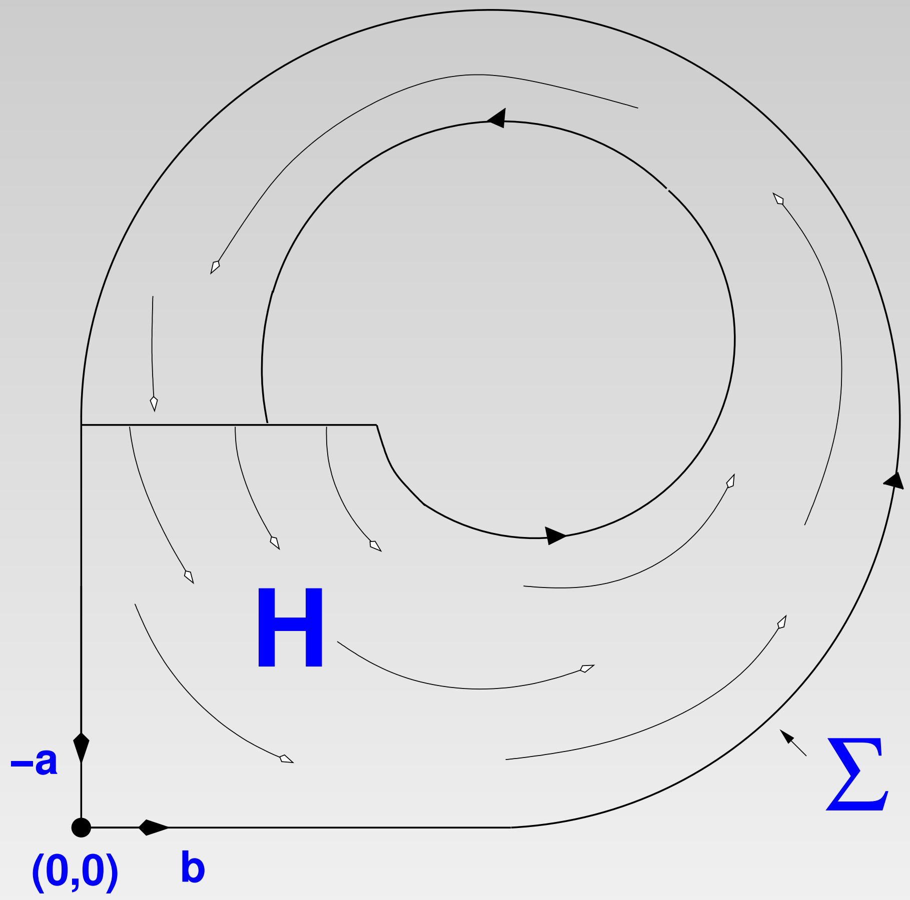


Figure 6: A model homoclinic attractor in $\mathbb{R}^2$

We consider a simple – but general – 2-dimensional model for an attracting homoclinic cycle. We assume that we are given a smooth (at least C^7) flow ϕ_t on $\mathbb{R}^2$ with an attracting homoclinic cycle Σ connecting the origin. We assume the origin is a hyperbolic saddle with associated eigenvalues $-a < 0 < b$ where

$$a > b > 0, \quad 4b,\ 3b,\ 2b \neq a, \quad 2a \neq 3b$$

These conditions imply that ϕ_t is C^3-linearizable at the origin (Samoval, 1972, Belickii, 1973).
Let $H \subset \mathcal{B}(\Sigma)$ be a forward-invariant, one sided neighbourhood of Σ.

Restricting to $\phi_t : H \to H$, it is well-known that Σ is a minimal attractor and, of course, $\Lambda(H) = \Sigma$

Results

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- Let $\psi_t(x) = x + \varpi t$ be a periodic flow on S^1 (or a hyperbolic attracting limit cycle). Then $\Sigma \times S^1$ is a minimal attractor for the product system $\phi_t \times \psi_t : H \times S^1 {\rightarrow} H \times S^1$.

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- Let $\psi_t : X \rightarrow X$ be the suspension of an SSFT (or basic set or hyperbolic attractor). Then $\Sigma \times X$ is a Milnor attractor for the product system $\phi_t \times \psi_t$.

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- Let $\psi_t : X \rightarrow X$ be the suspension of an SSFT (or basic set or hyperbolic attractor). Then $\Sigma \times X$ is a Milnor attractor for the product system $\phi_t \times \psi_t$.
- Let $\psi_t : H_i \rightarrow H_i$ be homoclinic attractors in $\mathbb{R}^2$ and $q_i \in \Sigma_i$ denote the equilibrium point on Σ_i, $i = 1, 2$. Set $\Sigma = (\{q_1\} \times \Sigma_2) \cup (\Sigma_1 \times \{q_2\})$. Then *either* Σ *or* $\Sigma_1 \times \Sigma_2$ is a (maximal) Milnor attractor for the product system,

For a class of model homoclinic attractors, Σ is a minimal attractor.

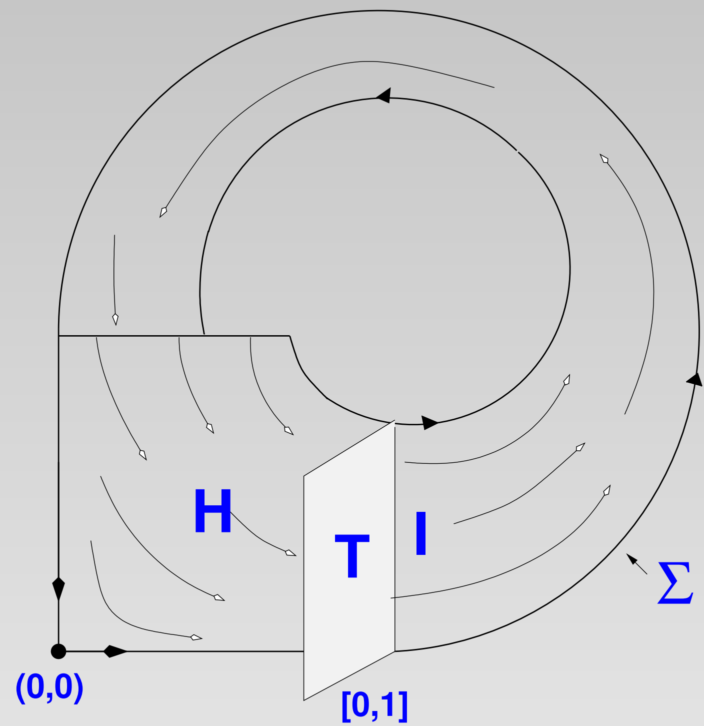
Product with limit cycle

Regard $S^1 = [0,1]/\sim,\ 1 \sim 0$.
Minimality of $\Sigma \times S^1$ follows from

THEOREM
For almost all initial conditions $(z,\theta) \in H \times S^1$,

$$\omega(z,\theta) = \Sigma \times S^1.$$

Proof of theorem. (Sketch) Let I be a cross section for the flow on H and $T = I \times [0,1]$ be a section for the product flow (see figure). We define a return map $P : T \to T$ in the obvious way.



T = I x [0,1]

Given $(z, \theta) \in T$, we obtain a sequence of iterates $(z_n, \theta_n) \subset T$ and so a sequence $(\theta_n) \subset S^1$.
It suffices to prove that for almost all $(z, \theta) \in T$, (θ_n) is uniformly distributed in S^1.

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If we let $S_n(z)$ denote the return time to the nth hit then, after some work, it can be shown that $\forall n > m$ $S'_n(z) - S'_m(z)$ is monotonic and

$$|S'_n(z) - S'_m(z)|$$

is uniformly bounded above zero.

It then follows by a corollary of a theorem of Davenport, Erdös & LeVeque, that $\theta + \varpi S_n(z)$ is uniformly distributed in $[0,1]$, almost all z (*not* all $z \neq 0$). ∎

THEOREM
Let $f_n(x)$ be a sequence of functions on $[\alpha, \beta]$ such that $f_n'(x) - f_m'(x)$ is monotonic for all $n > m$. Then if

$$|f_n'(x) - f_m'(x)| \geq \delta, \quad m \neq n,$$

some $\delta > 0$ then the sequence $f_n(x)$ is uniformly distributed modulo one for almost all $x \in [\alpha, \beta]$.

(A proof may be found in Glen Harman, Metric number Theory, Chapter V.)

Product with a chaotic set

The proof that the product of a homoclinic attractor and a suspended SSFT $\mathcal{S}$ is a Milnor attractor depends on showing that the likely limit set of $H \times \mathcal{S}$ equals $\Sigma \times \mathcal{S}$. This requires a fairly detailed knowledge of SSFTs and the use of the Borel-Cantelli lemma. It is not clear that the product is a minimal attractor.

This type of result is relevant for the study of cycling chaos.

Product of homoclinic attractors

Let $\psi_t : H_i \to H_i$ be homoclinic attractors in $\mathbb{R}^2$ and $q_i \in \Sigma_i$ denote the equilibrium point on Σ_i, $i = 1, 2$. Set

$$\Sigma = (\{q_1\} \times \Sigma_2) \cup (\Sigma_1 \times \{q_2\})$$

THEOREM

Either Σ or $\Sigma_1 \times \Sigma_2$ is a (maximal) Milnor attractor for the product system,

This result follows from a result about the likely limit set for product flows. First, a tautological Lemma.

Lemma

Let Z be a (forward) invariant measurable set with $\ell(Z) > 0$. Then $x \in \Lambda(Z)$ iff $\forall \epsilon > 0$, $\forall$ full measure subsets H of Z, $\exists a \in H$ such that

$$B_\epsilon(x) \cap \omega(a) \neq \emptyset.$$

Lemma

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Proof. $x \notin \Lambda(Z)$ iff $\exists \epsilon > 0$, and $\exists H \subset Z$, $\ell(Z \setminus H) = 0$, such that

$$B_\epsilon(x) \cap \omega(a) = \emptyset, \ \forall a \in H. \qquad \blacksquare$$

Theorem

Let $\phi_t : X \rightarrow X$, $\psi_s : Y \rightarrow Y$ be C^1-flows defined on compact regions $X \subset \mathbb{R}^m, \;\; Y \subset \mathbb{R}^n$. The likely limit set Λ of $X \times Y$ is invariant under the $\mathbb{R}^2$-action defined by $(\phi_t, \psi_s), \; (t, s) \in \mathbb{R}^2$.

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Proof. Write $\Phi_{t,s} = (\phi_t, \psi_s)$. Fix $(t, s) \in \mathbb{R}^2$, $(x, y) \in \Lambda$ and set $\Phi_{t,s}(x, y) = (x', y')$. Given $\epsilon > 0$, $\exists \delta > 0$ such that $\Phi_{t,s}(B_\delta(x, y)) \subseteq B_\epsilon(x', y')$.

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It follows easily from the theorem that

$$\Lambda(H_1 \times H_2) = \Sigma \text{ or } \Sigma_1 \times \Sigma_2.$$

The expectation is that we *always* have $\Lambda(H_1 \times H_2) = \Sigma$ *and* Σ is a minimal attractor.

This is true if we make very strong assumptions on the 'return' maps. Even then, methods depend on using restricted Diophantine conditions and Liouville type estimates to quantify the 'bad' initial conditions. Roughly speaking, the issue is to find all limit points of the double sequence $(\alpha\lambda_1^m - \beta\lambda_2^n)$, where $\lambda_1, \lambda_2 > 1$ and α, β depend on initial conditions.

Generalizations & Questions

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- If the vector field is only C^1 or fails to be C^2 linearizable, can the likely limit set of the product with a limit cycle have complex structure – for example, be locally the product of an interval with a Cantor set?

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- Can the results be generalized to general heteroclinic attractors?
- Can the results be extended to skew products (weak coupling)?