Invariant tori for Hamiltonian PDE

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Hamiltonian PDE

(1) $\partial_t v = J\, \delta_u H(v)$

$v(x,0) = v_0(x)$

phase space $\mathcal{H}$ a Hilbert space

symplectic form $\omega(X,Y) = \langle X, J^{-1} Y\rangle_{\mathcal{H}}$

$J^T = -J$

- flow of the dynamical system

$$v(x,t) = \varphi_t(v_0(x)),$$

tracing a curve in $\mathcal{H}$ through v_0

Contents:

- Hamiltonian PDE - examples
- Invariant tori - a variational problem
- Results
- Estimates of the linearized problem

1) Principal examples

(I) nonlinear wave equations

$$\partial_t^2 u - \Delta u \quad g(x\ u) \quad 0$$

$t \quad d \quad \mathbb{R}^d / \Gamma$ periodic boundary

else

$x \in \Omega \subseteq \mathbb{R}^d \quad \partial\Omega \Rightarrow \quad t) \quad 0$ Dirichlet

The Hamiltonian functional

$$H(\quad p) = \int_{\mathbb{T}^d} \quad p^2 \quad \tfrac{1}{2}|\nabla u \quad + G(x \quad dx$$

then

$$\partial_t u \quad p \qquad \delta_p H$$

$$\partial_t p \quad \Delta u \ - \partial_u G(x \quad) \qquad \delta_u H$$

$$g(x \quad) \quad \partial_u G(\quad)$$

The system has the for

$$\partial_t \begin{pmatrix} u \\ p \end{pmatrix} = J\, \delta H$$

th

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$ Darboux coordinates

Assume that

$$G(x, u) = \frac{1}{2} g_1(x) u^2 + \frac{1}{3} g_2(x) u^3 + \ldots$$

Then $H = H^{(2)} + H^{(3)} +$ Taylor expansion about $v = \binom{u}{p} = 0$

- The quadratic Hamiltonian is

$$H^{(2)} = \int_{\mathbb{T}^d} \frac{1}{2} p^2 + \frac{1}{2} |\nabla u|^2 + \frac{1}{2} g_1(x) u^2 \, dx$$

$$= \sum_k \frac{1}{2} |p_k|^2 + \frac{\omega_k^2}{2} |u_k|^2 ,$$

expansion in terms of eigenfunctions

$$\begin{pmatrix} u(x) \\ p(x) \end{pmatrix} = \sum_k \begin{pmatrix} u_k \\ p_k \end{pmatrix} \psi_k(x)$$

satisfying

$$L(g_1) \psi_k = (-\Delta + g_1(x)) \psi_k = \omega_k^2 \psi_k$$

$(\psi_k(x), \omega_k^2)$ eigenfunction, eigenvalue pair

- This is a harmonic oscillator, with frequencies ω_k

Solutions of the linearized equations

$$(3) \qquad \partial_t \begin{pmatrix} u \\ p \end{pmatrix} = J\, \delta H^{(2)} \begin{pmatrix} u \\ p \end{pmatrix}$$

are of the form

$$\begin{pmatrix} u(x,t) \\ p(x,t) \end{pmatrix} = \sum_k \varphi_k \begin{pmatrix} \cos(\omega_k t + \xi_k) & \frac{1}{\omega_k} \sin(\omega_k t + \xi_k) \\ -\omega_k \sin(\omega_k t + \xi_k) & \cos(\omega_k t + \xi_k) \end{pmatrix}$$

$$= \Phi_t \begin{pmatrix} u^0 \\ p^0 \end{pmatrix} \qquad \text{the linear flow.}$$

- Facts:
  - The Hamiltonian is preserved along the flow

    $$H^{(2)}(\Phi_t(v)) = H^{(2)}(v)$$

  - Actions are preserved along the flow

    $$I_k = \frac{1}{2}\left(\omega_k |u_k|^2 + \frac{1}{\omega_k} |p_k|^2\right)$$

    such that

    $$I_k(\Phi_t(v)) = I_k(v)$$

- Angles evolve linearly in time: $\xi_k \mapsto \omega_k t + \xi_k$
- All solutions are
  - periodic — all active $\omega_k = j_k \omega_0$, j_k
  - quasi-periodic — or — $\omega_k = \langle j_k, \vec{\omega}_0 \rangle$, $j_k \in \mathbb{Z}$
  - almost-periodic, — no finite # suffice

- Basic questions :

  + (1) Whether some solutions of the nonlinear problem have the same properties:

    + periodic
    + quasi-periodic
    + almost-periodic

    (KAM theory)

  + (2) Whether all solutions with $v_0 \in B_R(0) \subseteq \mathcal{H}$ remain in $B_{2R}(0)$

    (well-posedness)

    Whether action variables are preserved for long time intervals

    $$| I_k(v(t)) - I_k(v_0) | < \varepsilon^{\alpha}$$

    for

    $$|t| < T(\varepsilon) \sim \exp(1/\varepsilon^{\beta})$$

    (Nekhoroshev stability)

  + (3) Upper and lower bounds on growth of the action variables, or on higher Sobolev norms

    (Arnold diffusion)

III) nonlinear Schrodinger equation

$$\partial_t u \quad i \Delta u + Q(\quad u \quad) \quad 0$$

$x \in \mathbb{R}^d/\Gamma \quad \mathbb{T}^d$ periodic boundary conditions

else Dirichlet boundary ditions

$x \quad \Omega \quad \mathbb{R}^d \qquad 0$ for $\partial\Omega$

Initial data

$u_0 \qquad \partial\Omega$

flow of the dyn ical , tem

$t \qquad \varphi_t \; u_0 \;)$

The H tonian is

$$H \;) \qquad \int_{\mathbb{T}^d} \frac{1}{\;} \nabla \; |^2 \qquad G \qquad) \, dx$$

where

$G \quad z \quad)$ real valued when w w

$d \quad \partial_{\bar z} G \quad Q$

Then (4) can be expressed as

$$\partial_t u \qquad \delta_{\bar u} H(\;)$$

here J I plex symplectic coordinat

(III) Korteweg deVries equation

$$(5) \qquad \partial_t q = \frac{1}{6}\partial_x^3 q - \partial_x \left(\partial_q G(x,q) \right)$$

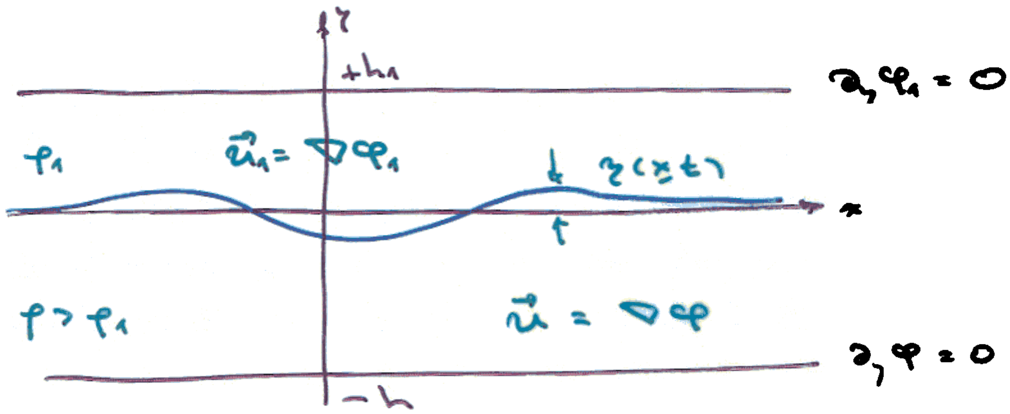
$$x \in \mathbb{R}^1 / \Gamma = \mathbb{T}^1$$

The Hamiltonian

$$H = \int_{\mathbb{T}^1} \frac{1}{12}(\partial_x q)^2 + G(x,q) \, dx$$

symplectic form given by $J = -\partial_x$

(IV) Large amplitude long waves in an interface



Equations for the interface

$$(6) \quad \partial_t \begin{pmatrix} \eta \\ u \end{pmatrix} = \begin{pmatrix} 0 & -\partial_x \\ -\partial_x & 0 \end{pmatrix} \begin{pmatrix} \delta_\eta H \\ \delta_u H \end{pmatrix}$$

symplectic form

$$\omega(X,Y) = \int (\partial_x^{-1} X_\eta) Y_u + (\partial_x^{-1} X_u) Y_\eta \, dx$$

same as for the Boussinesq equation

Hamiltonian

$$H \quad \quad R_0(\eta)\, u \quad \quad g\, \rho \, \frac{(\rho_1)}{2} \, \eta^2$$

$$+ R(\eta)(\partial_x u)^2 + \frac{1}{4}(\partial_x u) R_2(\eta)\, \partial_x \eta \quad R_3(\eta)\, u^2 (\partial_x$$

with tional coefficients

$$R(\eta) = \frac{(h+\eta)(h_1-\eta)}{\rho_1(h+\eta) + \rho(h\ \eta)} \qquad \text{nonlinear propagation velocity}$$

$$R(\eta) \quad \frac{1}{} \frac{h+\eta)^2 (h_1\ \eta)^2 \quad \rho_1(h_1\ \eta) + \rho(h+\eta\ ]}{\rho_1(h+\eta) \quad \rho(h\ \eta)\ ]^2}$$

$$R_2(\eta) \quad \frac{1}{3} \frac{\rho\rho_1 (h+h_1)(h+\eta)(h_1-\eta)\,[(h_1-\eta)^2 - (h+\eta)^2]}{[\rho\ (h\ \eta) \quad \rho(h\ \eta)]}$$

$$R_3(\eta) \quad 3\ \rho\rho \frac{(h+h_1)^2\,[\rho_1(h+\eta)^3 + \rho(h_1-\eta)^3]}{[\rho_1(h+\eta) + \rho(h_1-\eta)]^4}$$

linear dispersion coefficients

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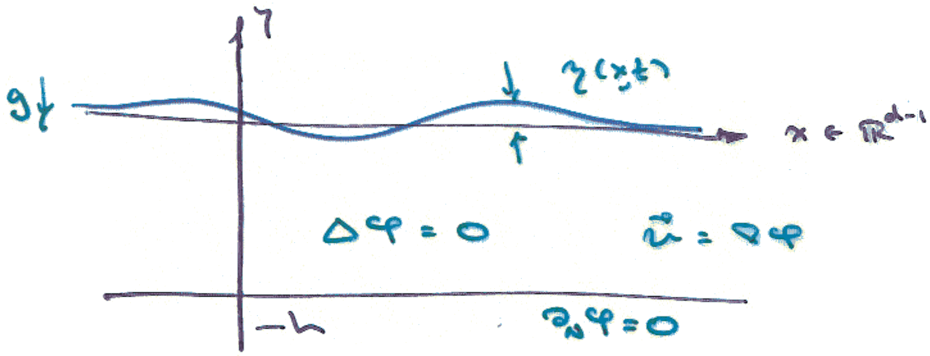
(V) surface water waves



$\xi(x) = \varphi(x, \eta(x))$

Euler's equations are equivalent to

$$(7) \qquad \partial_t \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \delta H$$

where the Hamiltonian is given by

$$H(\eta, \xi) = \frac{1}{2} \xi \, G(\eta) \xi + \frac{g}{2} \eta^2 \, dx$$

The Dirichlet-Neumann operator $G(\eta)$ satisfies

$\xi(x) \longmapsto \varphi(x, y)$ harmonic extension

$\longmapsto N \cdot \nabla \varphi \, dS(\eta)$ normal derivative

$:= \underline{G(\eta) \xi \, dx}$

2) An ... ant torus

T ... S ... T ... ... linear pa...

$S(\xi + t\Omega) \quad \varphi_t(\;\;)$ flow invariant

The freq ... Ω ... $\mathbb{R}$

This ... ... lies that

$\partial_t S \quad J \, \delta H(\;) \;$ and $\; \partial_t S \quad \Omega \, \partial_\xi S$

Problem ... ve this equation ... for $(S \;... \; \Omega)$

where Ω ... a bifur... ... paramete...

This ... generally a ... small divisor problem

Rewrite ... as

$$J \; \Omega \, \partial_\xi S \quad \delta H(S) \quad 0$$

* A variational problem

Consider the pace of mappings

$$S\quad X \quad = \{ S\ \xi) \quad \mathbb{T}^m \quad x \}$$

Define from functionals

$$+\quad I(\ ) \qquad \int_{\mathbb{T}^m} \langle S \quad J^{-1} \partial_{\xi_0} S \rangle \, d\xi$$

$$\delta_w I \qquad J \, \partial_{\xi_w}$$

and the average Hamiltonian

$$+\quad H(S \quad = \quad \int_{\mathbb{T}^m} H(S(\xi)) \, d\xi$$

$$\delta_w H \qquad \partial_w H$$

Consider the subvariety of X defined by

$$\Gamma = \{ S \quad X \quad I \quad a_1 \qquad I_m\ \xi) \quad a_m \}$$

$$\subseteq$$

V iational p blem critical points of H s the variety Γ_a correspond to solutions of equation a) with Lagrange multiplier Ω

NB invariance under the action f $\mathbb{T}$

Two questions:

Do critical points exist?

The functional $\Omega \cdot \partial_J S$ is degenerate

How to understand multiplicity?

Topology of $M_a / \mathbb{T}^m$ and its equivariant cohomology.

Answers — in some cases

(i) study the linearized problem

Fröhlich - Spencer estimates

Nash-Moser method.

(ii) Morse-Bott theory of critical points / orbits.

(3) The linearized problem

The linearized problem and v = [illegible] given by the quadratic Hamiltonian

$$H = \sum \frac{\omega_n}{2} \left(\omega_n p \qquad q_{\xi n}^{2} \right)$$

$$\sum \qquad I_n$$

where $v \begin{pmatrix} q \\ p \end{pmatrix}$

Representing mapping S $\mathbb{T}$ $\mathcal{H}$ of a torus into phase space

$$S \qquad S(x, \xi) \qquad \sum_k S_k(\ ) \phi_k$$

$$\sum_{j,k} S_{jk} \phi_k(x) \quad e^{\xi} \qquad j \quad \mathbb{Z}^m$$

The linearized equation be then

$$\left(\delta_v^2 H_0 \quad \Omega \, \delta_v \mathbf{I} \right) S$$

$$\sum_k \begin{pmatrix} \omega & \Omega j \\ \Omega & \end{pmatrix} \begin{pmatrix} q_{jk} \\ p_{jk} \end{pmatrix} \phi_k(x) \, e^{ij\xi}$$

Eigenvalues of this 2×2 block diagonal problem

$$\mu(j,k) \qquad \omega_k \qquad \Omega \, j$$

Null space

Choose $\Omega^0 = (\Omega_1, \dots \Omega_m) \in \mathbb{R}^m$ a frequency vector which is a solution of the m-many resonance re

$$(10) \qquad \omega_{k_\ell} - \Omega^0 \cdot j_\ell = 0 \qquad \ell = 1, \dots m$$

This identifies an eigenspace $X_1 \subseteq X$ within the space of mappings X_1 spanned by $\{ \phi_k(x) e^{i j \cdot \theta}$

Proposition $X_1 \subseteq X$ is even dimensional $\geq 2m$:
(it could be infinite dimensional, but it is symplectic)

The non-resonant case is when $\dim(X_1) = 2m$.
Otherwise $\dim(X_1) > 2m$, the case is resonant.

The other eigenvalues are

$$\{ \mu_{jk} = \omega_k \pm \Omega \cdot j \neq 0 \}$$

which typically forms a dense set in $\mathbb{R}$, these are the small divisors.

Lyapunov Schmidt decomposition

In the space X of mappings S π' we decompose the equations $X = X_1 \oplus X_2$

$Q X_1$ X null space

$P X$ $(I - Q) X$ X_2

The equations to solve are therefore

$Q [J \quad \Omega \partial S \quad H \quad] = 0$ bifurcation eq

$P [J \quad \Omega \partial_S \quad \delta H (\quad)] = 0$ well defined

D compute mapping: do. S S S

If con for S_2 $S (S \Omega)$ there is a reduced v fixed blows Ulei tein Moser)

$I (\quad)$ $I (\quad S)$

(S_1) $\bar{H} (\quad)$

$\in X$ I $) \quad a \quad j$ $\}$

critical pts of $\bar{H}$ $\cap$ are solutions of ($^\circ$)

The linearized operator about approximate torus embedding S_0

(11) $\left(\delta_v H(S_0) - \Omega\, \delta_v^2 I \right) V$

$$\left[\mathrm{diag}_{2\times 2} \begin{pmatrix} \omega & \Omega \\ \Omega & \omega \end{pmatrix} + W(\ \ w) \right] V$$

here the index $j, k) \in \mathbb{Z} \oplus \mathbb{Z}^d$

<u>Definition</u> A lattice site $(j, k) \in \mathbb{Z}^m \oplus \mathbb{Z}^d$ is <u>d_0 - singular</u> for Ω when

$$d_0 > \omega_k \pm \Omega$$

& <u>regular</u> otherwise

<u>Proposition</u> F $A \subseteq \mathbb{Z}^m \oplus \mathbb{Z}^d$ having by regular sites & for $\| W \|_{op} < \frac{d}{2}$ then

$$\left\| \left(\delta_v^2 H(S_0) \quad \Omega\, \delta_v^2 I \right)_A \right\|_{op} < \frac{4}{d}$$

Fröhlich Spencer estimates are used to add the regular site

Fröhlich - Spencer estimates:

Depend upon two properties of the operator

$$(\delta_v^2 H(s_0) - \Omega \cdot \delta_v^2 I) = D(\Omega) + W$$

(i) nonresonance: if $(j,k) = y \in \mathbb{Z}^m \otimes \mathbb{Z}^d$, $(j',k') = y'$

then

$d_n < |\omega_k \pm \Omega \cdot j| < d_0$

(12) $|\omega_{k'} \pm \Omega \cdot j'|$ singular sites

$R_n < |y|, |y'|$

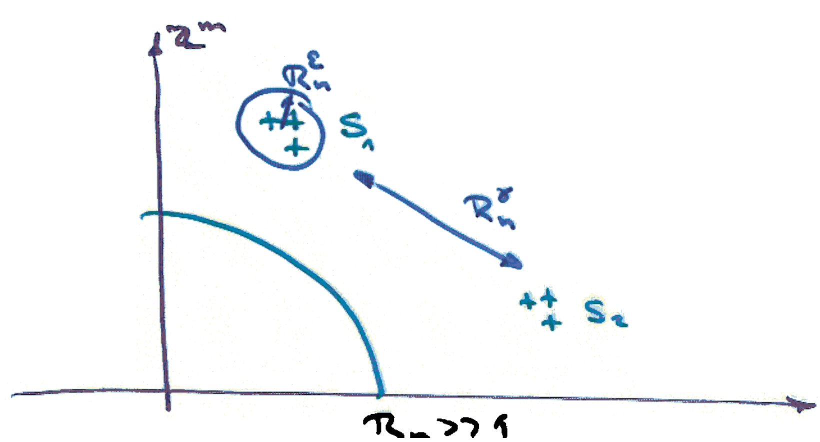
(ii) separation if y y' are singular sites

with $R_n < |y|, |y'|$, then either

$\mathrm{dist}(y, y') < R_n^{\varepsilon}$ $0 < \varepsilon << 1$

or else

$\mathrm{dist}(y, y') >> R_n^{\gamma}$ $0 << \gamma$



S_1, S_2 singular regions

(4) Results

nonlinear wave equation

S Kuks (19) E Way (19 9) Dirichlet
W C & E Wayne (1993) p. solution b.c.
J Bourg (199) d >

separation imposed by diophantine conditions

n. linear Schrödinger equation

Kuks. c.
W C & E Way (1994)
Kuks & J Pöschel (199
J Bourgain (1996) d >

separation imposed by the dispersion relation

perturbations of KdV
Kuks)
T Kappeler & Pöschel 2.

Birkhoff law b for the linear Schrödinger eq'

W C J Delort & E Szeftel 2(3)

(v) Standing water waves
Plotnikov & Toland h ∞) 2001)
I Plotnikov & Toland (h) 200 AS)