

ON THE STABILITY  
OF SOLITARY WAVES

Frédéric DIAS      Ecole Normale Supérieure  
de Cachan

IN COLLABORATION WITH

Thomas BRIDGES      University of Surrey (UK)  
Frédéric CHARDARD      graduate student  
                                ENS - Cachan

# THE PROBLEM UNDER CONSIDERATION

$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} + \frac{\partial}{\partial x}(u^{p+1}) + P \frac{\partial^3 u}{\partial x^3} - \frac{\partial^5 u}{\partial x^5} = 0$$

$$P \geq 1, \quad P \in \mathbb{R}$$

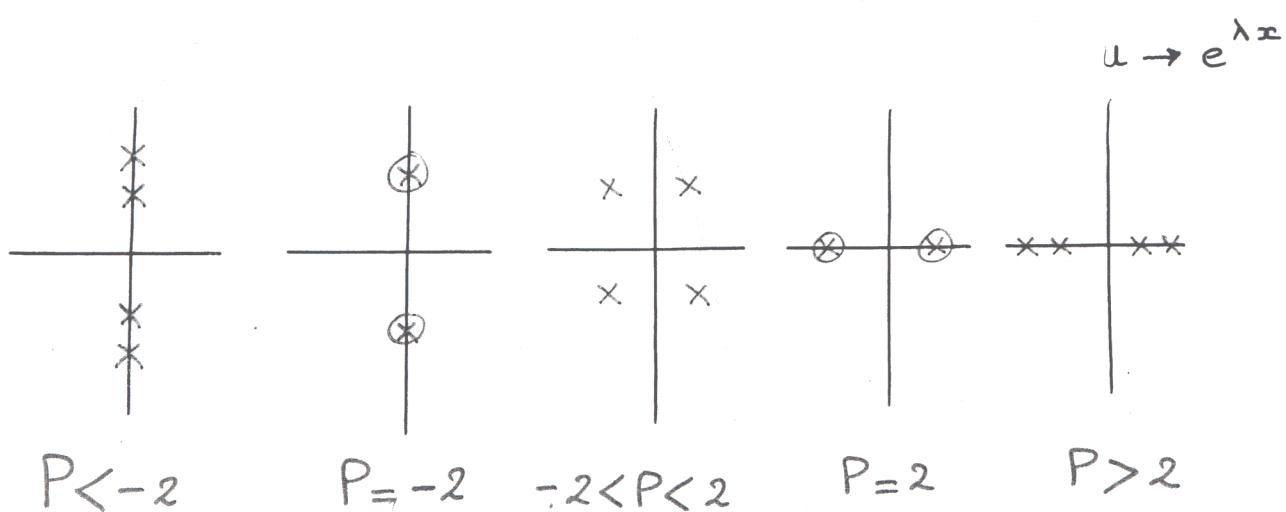
5th order KdV eq.

We are interested in solitary wave solutions and in their linear stability.

## STATIONARY PROBLEM

$$\frac{\partial^4 u}{\partial x^4} - P \frac{\partial^2 u}{\partial x^2} + u - u^{p+1} = 0$$

This problem was studied by Amick & Toland (1992), Champneys & Toland (1993), ... (existence of s.w. solutions)



# Amick & Toland (1992)

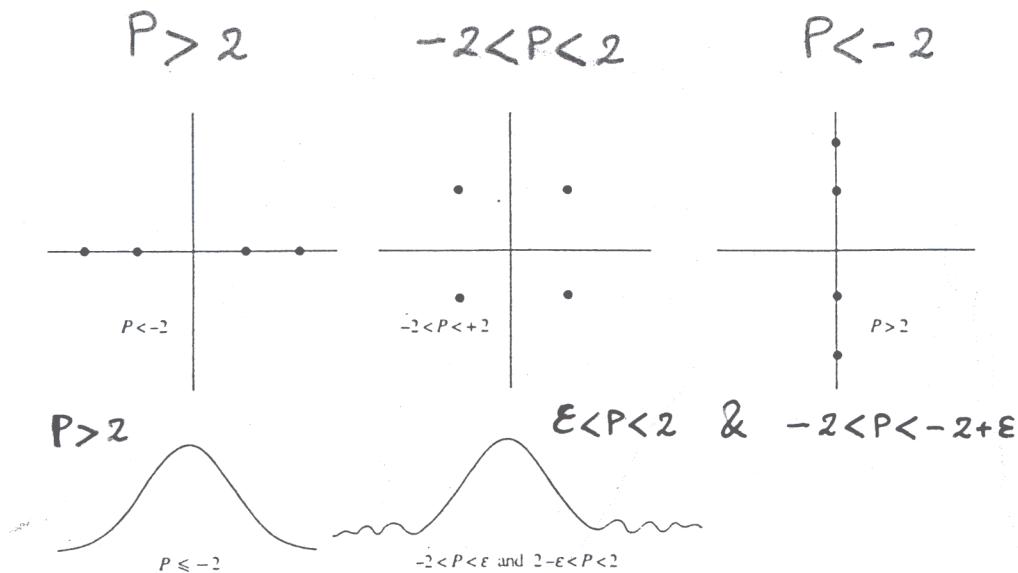


FIGURE 1. The upper part of the figure illustrates the eigenvalues of the problem linearized about 0 for different ranges of  $P$ . The lower part shows schematically the type of solutions which are known to exist for the values of  $P$  indicated. (See the Conclusion for a discussion.)

Types de solutions dont l'existence a été  
prouvée  
(types of solutions for which an existence  
proof was given)

# NUMERICAL COMPUTATION OF SOLITARY WAVES

- periodic domain (length of domain  $L \gg 1$ )
- spectral method + Newton's method
- comparison with explicit solution

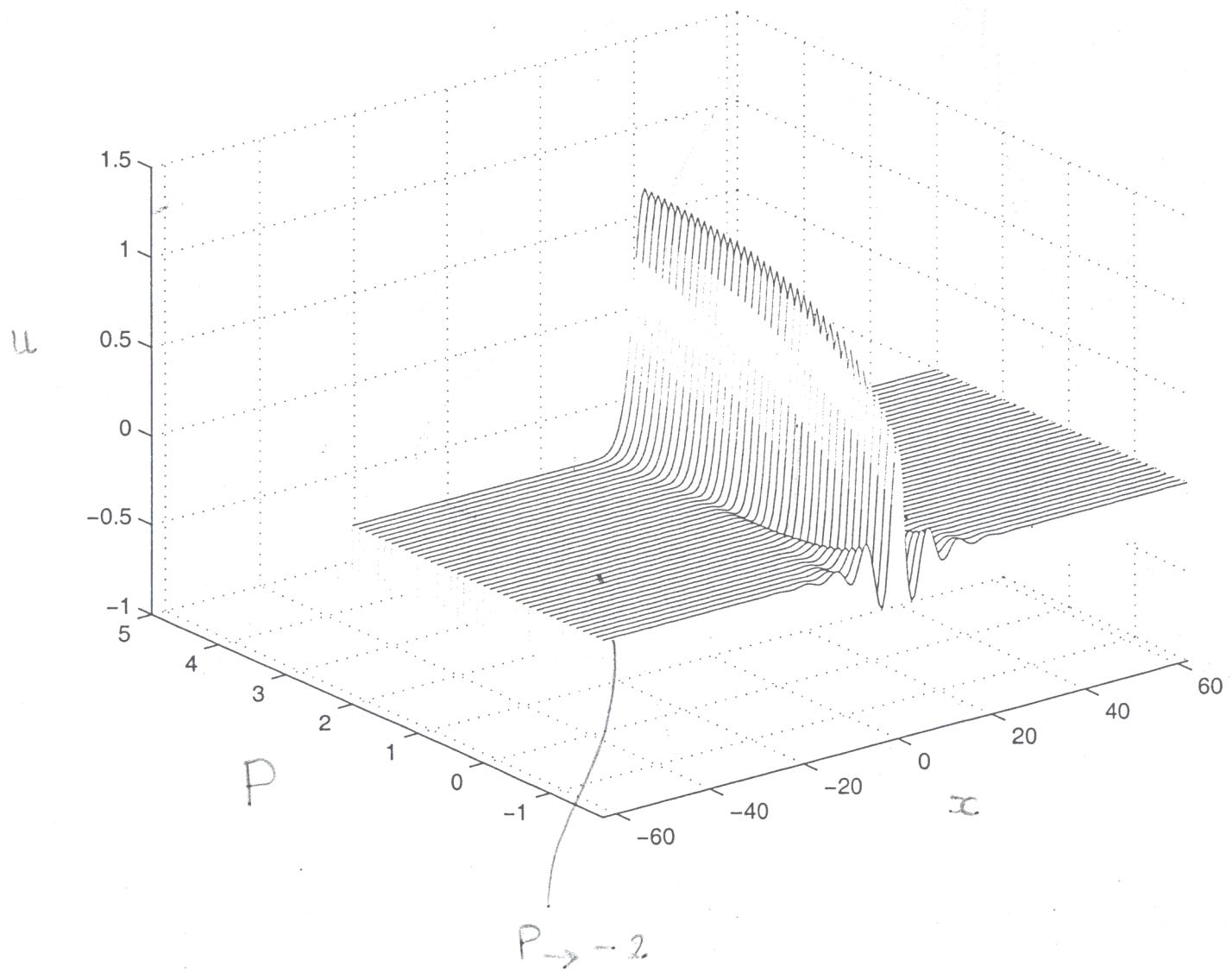
$$u(x) = \left[ \frac{(p+4)(3p+4)}{8(p+2)} \right]^{1/p} \operatorname{sech}^4 \left[ p \sqrt{\frac{1}{8(p+2)}} x \right]$$

Obtained when  $P = \frac{p^2 + 4p + 8}{2(p+2)}$

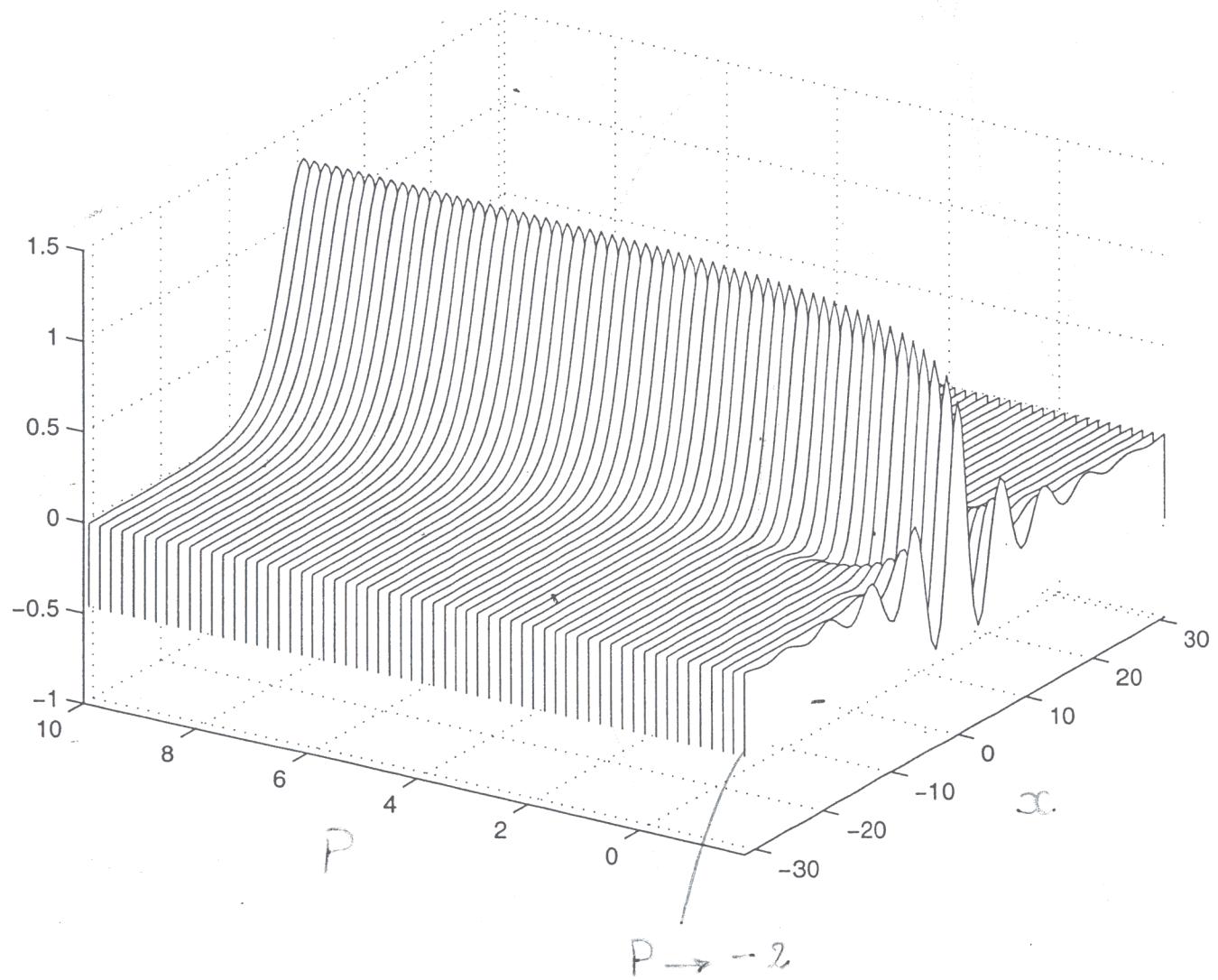
for example, for  $p=1$ ,  $P = \frac{13}{6}$

$$u(x) = \frac{35}{24} \operatorname{sech}^4 \left( \frac{x}{2\sqrt{6}} \right)$$

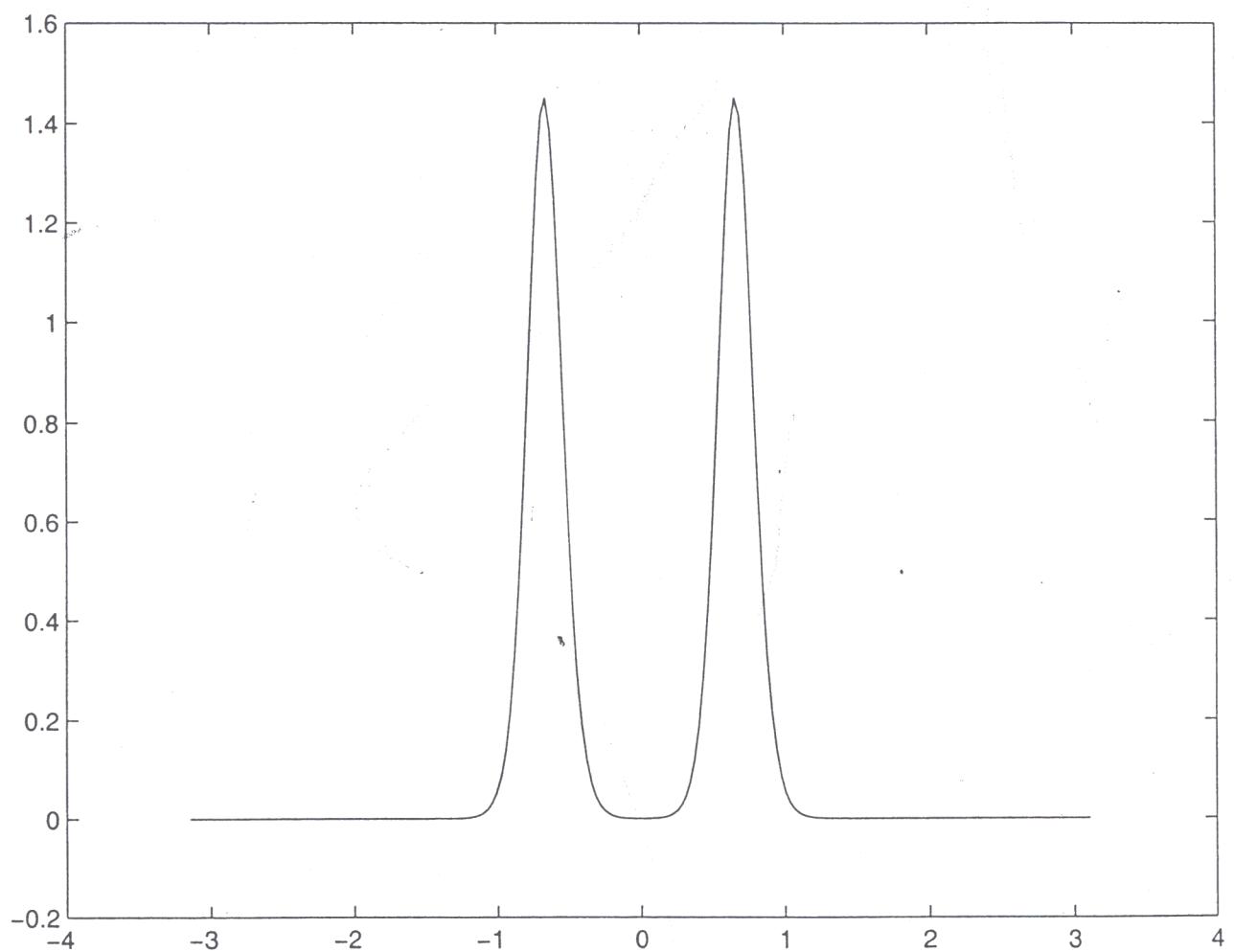
# ONDES SOLITAIRES ( $p=1$ ) SOLITARY WAVES



ONDES SOLITAIRES ( $p=3$ )  
SOLITARY WAVES

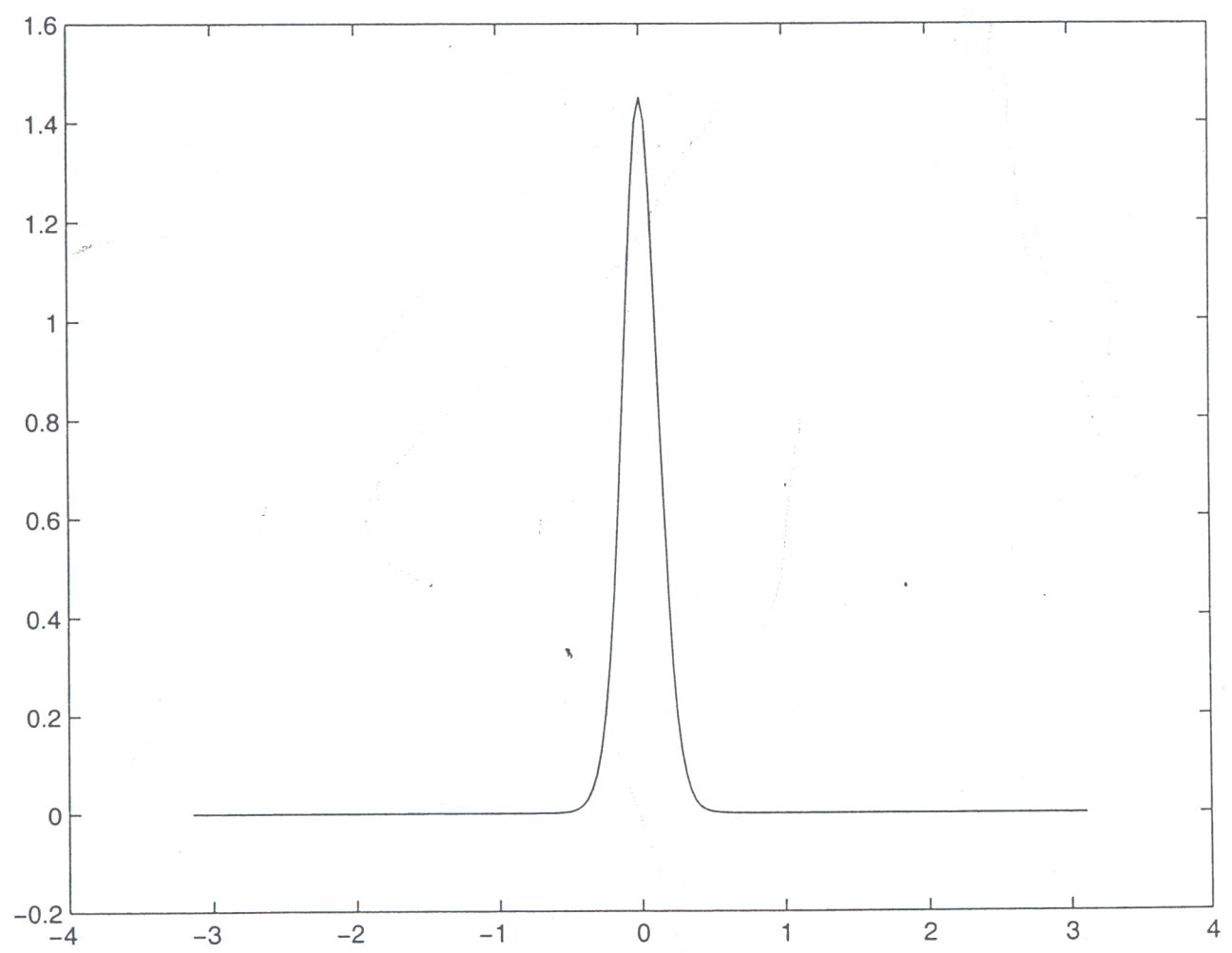


# DOUBLING OF THE PRIMARY HOMOCLINIC ORBIT



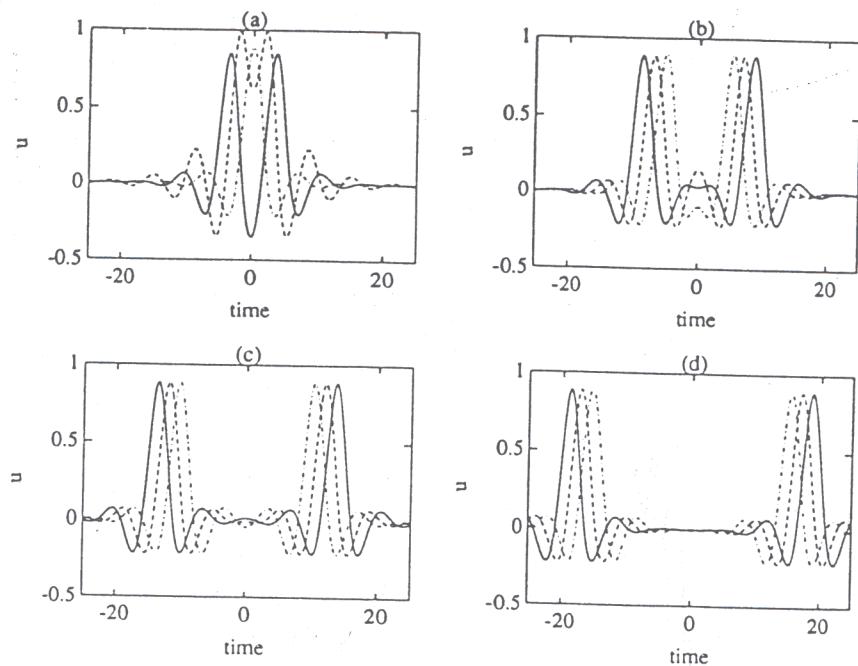
$$P=1$$

$$P=1.9$$



Champneys  
Toland  
(1993)

$$P = -1.5$$

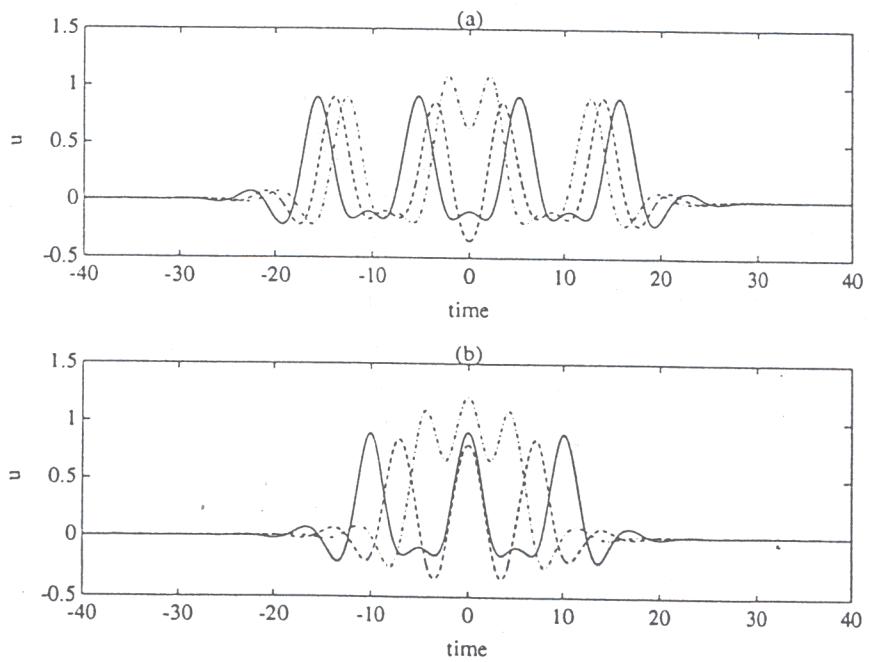


bimodal solitary waves

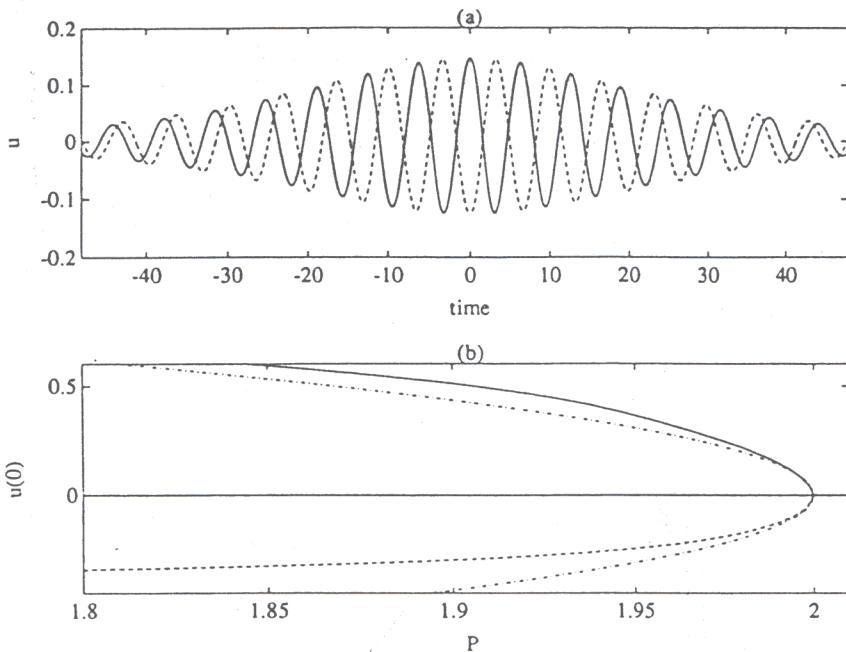
4-modal

$$P = -1.5$$

3-modal

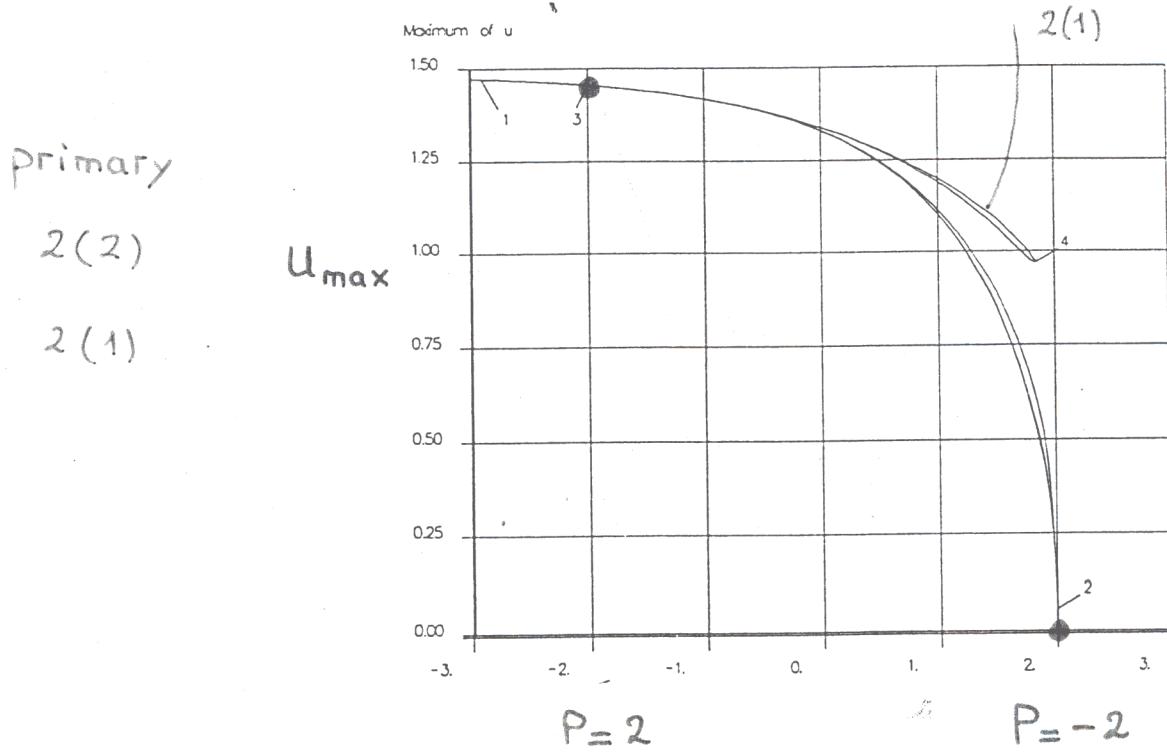


Comparison  
analytical/numerical  
Results



Primary orbit (—) and bi-modal orbit (---)

## BIFURCATION DIAGRAM



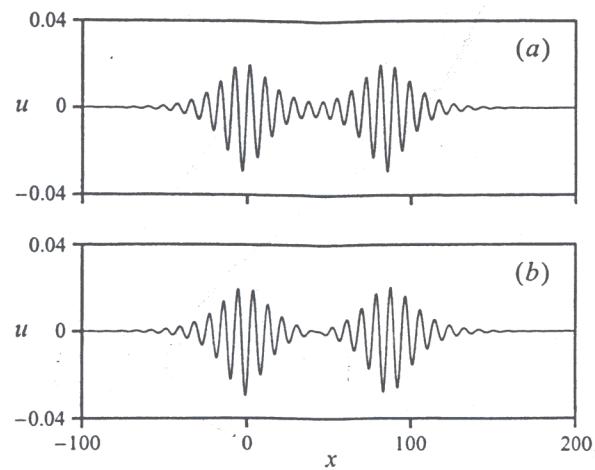
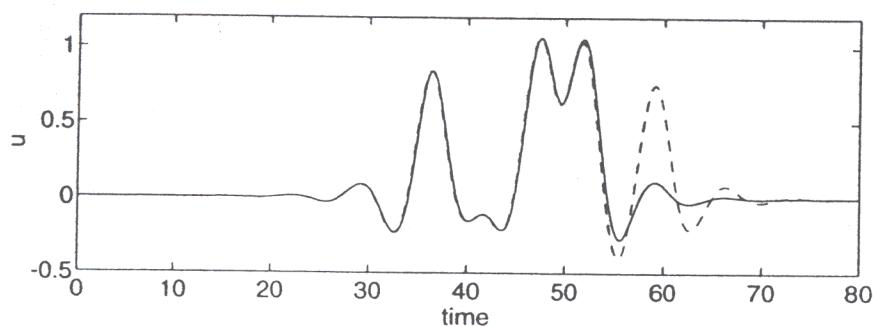


FIGURE 3. Profiles of numerically computed two-packet solitary waves. (a) Symmetric wave corresponding to  $m = 19$ ,  $c = -0.2615$ ,  $\phi_- = -0.303\pi$ ; (b) asymmetric wave corresponding to  $m = 20$ ,  $c = -0.2614$ ,  $\phi_- = 1.24\pi$ .

Yang, Akytas [1997]  
Journal of Fluid  
Mechanics

# ASYMMETRIC HOMOCLINIC ORBITS

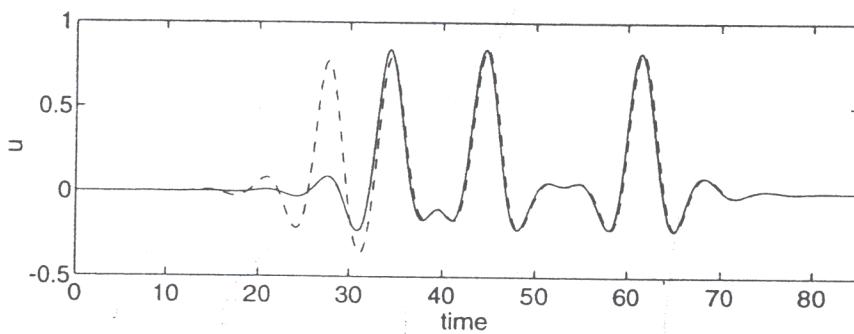
Champneys  
et al.



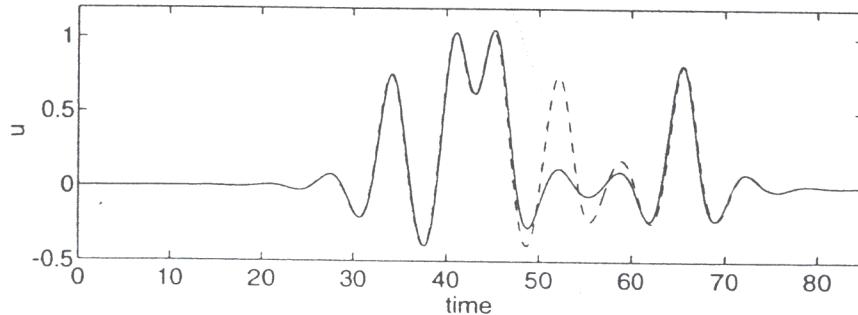
4th order

Graphs of the orbits  $3(3,1)$  and  $4(3,1,2)$  at  $P = 1.6$

ODE



Graphs of the orbits  $3(3,5)$  and  $4(2,3,5)$  at  $P = 1.6$



Graphs of the orbits  $4(2,1,6)$  and  $5(2,1,2,4)$  at  $P = 1.6$

# VARIOUS APPROACHES FOR STUDYING THE STABILITY OF SOLITARY WAVES

- Energy-momentum methods

Benjamin, Bona ... 1970's (+ wide range of examples)

Il'ichev, Semenov 1992 (5th order KdV)

Levandosky 1999 (5th order KdV)

D., Kuznetsov 1999 (5th order KdV)

Requires the second variation of the main functional  
to have a precise eigenvalue structure which is often  
violated

- Spectral problem for the linearization about a solitary wave without consideration of the Evans function

Beyn, Lorenz (1999)

Barashenkov, Pelinovsky D., ... (1998)

use finite differences, collocation or spectral methods

$x \in [-L_\infty, L_\infty]$ ; exact asymptotic boundary conditions

at  $x = \pm L_\infty$  depend on  $\lambda$  in a nonlinear way;

approximate boundary conditions  $\Rightarrow$  spurious discrete

eigenvalues generated from the fractured continuous spectrum

- Numerical computation of Evans function

Evans 1977

Pego, Smerkia, Weinstein 1993

Bridges, Derkso, Gottwald 2002

naive approach: take the  $k$  eigenvectors associated with the  $k$  eigenvalues of negative real part as starting vectors for the integration of  $v_x = A(x, \lambda)v$  from  $x = L_\infty$  to  $0$ , with similar strategy for  $x < 0$ .

not so good: the  $k$  solutions will not maintain linear independence numerically (all vectors  $\rightarrow$  eigenvector with largest growth rate) — stiffness problem  
orthogonalization has problems: with analyticity and transforms linear equation into nonlinear equation  
also: the starting eigenvectors will not in general be analytic for all  $\lambda$  in a given open set

All these problems are eliminated by using exterior algebra

*Nerve Axon Equations: IV  
The Stable and the Unstable Impulse*

JOHN W. EVANS

*Communicated by MURRAY ROSENBLATT*

**1. Introduction.** This is the fourth in a series of papers treating the nerve impulse in nerve axon equations. In the preceding papers [2, 3, 4], which we call in the sequel I, II and III, respectively, we have shown that the stability of a nerve impulse under small perturbations of the initial conditions is determined by the absence of bounded solutions to a related family of ordinary differential equations.

In this paper we show that there is a complex analytic function whose zeros serve to determine the relevant properties of solutions to the above family of ordinary differential equations, and through this function we relate the stability of the impulse to the behavior of certain wave forms that have velocities near the velocity of the impulse. This enables us to see why there is generally an unstable impulse as well as a stable impulse.

In Sections 2 and 3 some parametrized solutions of the ordinary differential equations arising in I, II and III are defined, and these solutions are used in Section 4 to define a complex valued function  $D(\lambda)$  of a complex variable  $\lambda$ . The main results that relate the properties of  $D(\lambda)$  to the stability of the impulse are given in Section 5. Section 6 is devoted to a discussion of the main results and of the bearing of these results on the existence of an unstable as well as a stable impulse in existing models. In Sections 7, 8, 9 and 10, asymptotic properties of the functions defined in Sections 2 and 3 are developed. These properties are used in the final section, Section 11, in the proof of the main results.

Indiana Univ.

Math. Journal

(1975)

J. reine angew. Math. 410 (1990), 167–212

Journal für die reine und  
angewandte Mathematik  
© Walter de Gruyter  
Berlin · New York 1990

J. reine angew. Math. (1990)

A topological invariant arising in the stability  
analysis of travelling waves

By J. Alexander\*) at College Park, R. Gardner\*\*) at Amherst  
and C. Jones\*\*\*) at College Park

## THE EVANS FUNCTION (example)

Let us consider the PDE  $u_t = u_{xx} - u + u^2$

It admits  $\phi = \frac{3}{2} \operatorname{sech}^2 \frac{x}{2}$  as stationary solution.

let us linearize about  $\phi$ :  $u_t = u_{xx} - u + 2\phi u = Lu$

Spectral ansatz:  $u_t \rightarrow \lambda u$  ( $u \rightarrow e^{\lambda t} u$ )

$$\lambda u = Lu \quad \text{or} \quad v_x = A(\lambda, x)v \quad \text{with} \quad v = \begin{pmatrix} u \\ u_x \end{pmatrix}$$

$$A(\lambda, x) = \begin{pmatrix} 0 & 1 \\ 1+\lambda-2\phi & 0 \end{pmatrix} \quad A_\infty(\lambda) = \begin{pmatrix} 0 & 1 \\ 1+\lambda & 0 \end{pmatrix}$$

↓

one eigenvalue with negative real part

$$\operatorname{Re} \lambda < 0$$

one eigenvalue with positive real part

$\alpha_+ e^{\sigma_+(\lambda)x}$   $\xi_+(\lambda) \rightarrow U_+(x)$  solution which does not grow exp.  
as  $x \rightarrow +\infty$

$\alpha_- e^{\sigma_-(\lambda)x}$   $\xi_-(\lambda) \rightarrow U_-(x)$  solution which does not grow exp.  
as  $x \rightarrow -\infty$

A value  $\lambda$  is an eigenvalue if these two solutions are proportional. The Evans function is defined by

$$E(\lambda) = \det [U_+(x), U_-(x)]$$

$U_+(x) \wedge U_-(x)$   
wedge  
product

## WEAK SPECTRAL STABILITY

The linearisation around a solitary wave  $\phi(x)$  gives

$$u_t - u_x + ((p+1)\phi^p u)_x + P u_{xxx} - u_{xxxxx} = 0$$

With the spectral ansatz  $u \rightarrow e^{\lambda t} u$ , we obtain

the spectral problem  $Lu = \lambda u$  with

$$Lu = \frac{d}{dx} (u_{xxxx} - Pu_{xx} - (p+1)\phi^p u + u)$$

It can be rewritten as a 1st order system

$$v_x = A(x, \lambda)v, \quad v \in \mathbb{C}^5$$

$$v = (u, u_x, u_{xx}, u_{xxx}, u_{xxxx} - Pu_{xx} - (p+1)\phi^p u + u)^T$$

$$A(x, \lambda) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ (p+1)\phi(x)^p - 1 & 0 & P & 0 & 1 \\ \lambda & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A_\infty(\lambda) = \lim_{x \rightarrow \pm\infty} A(x, \lambda) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & P & 0 & 1 \\ \lambda & 0 & 0 & 0 & 0 \end{pmatrix}$$

The solitary wave  $\phi$  is weakly spectrally stable if it has no eigenvalues in the half-plane  $\operatorname{Re}(\lambda) > 0$

Why weakly? The spectral activity on the imaginary axis is not considered.

Bridges, Derk & Gottwald (2002) introduced a numerical scheme that can exhibit spectral instability or weak spectral stability. It is based on the Evans function.

Characteristic polynomial of  $A_\infty(\lambda)$ :

$$\Delta(\mu, \lambda) = \mu^5 - P\mu^3 + \mu - \lambda$$

Let's take (for example)  $P > -2$ . Then, when  $\operatorname{Re}(\lambda) > 0$ , there are 3 roots of  $\Delta(\mu, \lambda)$  with positive real part, and 2 with negative real part.

Let  $U^+(x, \lambda)$  be the 2-dimensional space of solutions of  $v_x = A(\lambda, x)v$  which don't grow exp. as  $x \rightarrow \infty$ .

Let  $U^-(x, \lambda)$  be the 3-dimensional

as  $x \rightarrow -\infty$ .

A value of  $\lambda \in \Lambda$  (subspace of  $\mathbb{C}^+$ ) is an eigenvalue if these two subspaces have a non-trivial intersection.

The Evans function tells if there is an intersection.

$$E(\lambda) = U^-(x, \lambda) \cap U^+(x, \lambda)$$

$U^+(x, \lambda)$  and  $U^-(x, \lambda)$  must be constructed

↓  
exterior algebra

Can be considered as paths in  $\Lambda^k(\mathbb{C}^n)$  and  $\Lambda^{(n-k)}(\mathbb{C}^n)$  ( $k=2$  in our case). To calculate these paths, integrate the induced linear systems on  $\Lambda^k(\mathbb{C}^n)$  and  $\Lambda^{(n-k)}(\mathbb{C}^n)$ . The Hodge star operator (isomorphism between  $\Lambda^k(\mathbb{C}^n)$  and  $\Lambda^{(n-k)}(\mathbb{C}^n)$ ) can be used to relate the system on  $\Lambda^{(n-k)}(\mathbb{C}^n)$  to the adjoint system on  $\Lambda^k(\mathbb{C}^n)$ .

## NUMERICAL SCHEME

The system  $U_x = Au$ ,  $u \in \mathbb{C}^5$ , induces a system on

$$\wedge^2(\mathbb{C}^5) : U_x = A^{(2)} U, \quad U \in \wedge^2(\mathbb{C}^5)$$

$$A^{(2)}(x, \lambda) = \left\{ \begin{array}{ccccccccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & P & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -\gamma(x) & 0 & 0 & 0 & P & 0 & 1 & 1 & 0 \\ -\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -\gamma(x) & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\lambda & \gamma(x) & 0 & 0 & 0 & 0 & P \end{array} \right\}$$

where  $\gamma(x) = (p+1)\phi(x)^P - 1$ . The system at infinity  $A_\infty^{(2)}(\lambda)$  is obtained by replacing  $\gamma(x)$  by  $-1$ .

Let  $\sigma_-(\lambda)$  the eigenvalue of  $A_\infty^{(2)}(\lambda)$  with largest negative real part ( let  $\sigma_+(\lambda) \dots$  )

For  $\lambda$  fixed, the algorithm has three steps :

- integration of induced ODE for  $x > 0$
- integration of induced ODE for  $x < 0$
- matching at 0

# INTEGRATION FROM $x = L_\infty$ TO $x = 0$

- factorization:  $U^+(x, \lambda) = e^{\sigma_+(\lambda)x} \tilde{U}^+(x, \lambda)$

$$\hookrightarrow \frac{d}{dx} \tilde{U}^+ = [A^{(2)}(x, \lambda) - \sigma_+(\lambda) \mathbb{1}] \tilde{U}^+$$

$$\tilde{U}^+(x, \lambda) \Big|_{x=L_\infty} = \mathcal{J}^+(\lambda)$$

- implicit method (Gauss-Legendre Runge-Kutta 2<sup>nd</sup> order)  
(implicit mid-point method)

$U_x = B(x, \lambda)U$  is discretized as

$$U^{n+1} = \left[ I - \frac{1}{2} \Delta x B_{n+\frac{1}{2}} \right]^{-1} \left[ I + \frac{1}{2} \Delta x B_{n+\frac{1}{2}} \right] U^n$$

$$\text{with } B_{n+\frac{1}{2}} = B(x_{n+\frac{1}{2}}, \lambda)$$

# INTEGRATION FROM $x = -L_\infty$ TO $x = 0$

$$\sigma_-(\lambda) = -\sigma_+(\lambda) \quad \dots \quad \frac{d}{dx} \tilde{V}^- = \dots$$

$$\begin{aligned} \text{AT } x=0 : \quad E(\lambda) &= \langle \overline{V^-(0, \lambda)}, U^+(0, \lambda) \rangle_{\mathcal{H}} \\ &= \langle \overline{\tilde{V}^-(0, \lambda)}, \tilde{U}^+(0, \lambda) \rangle_{\mathcal{H}} \end{aligned}$$

One computes the Evans function along the real  $\lambda$ -axis

$E(\lambda)$  is real when  $\lambda \in \mathbb{R}$ .

When there are no unstable real eigenvalues,  
one uses Cauchy's theorem to numerically count  
the eigenvalues in the positive half-plane.

Many points are needed for the numerical computation  
of the Evans function (typically:  $2 \times 10^4$  points  
 $\Delta x \sim 10^{-3}$ )

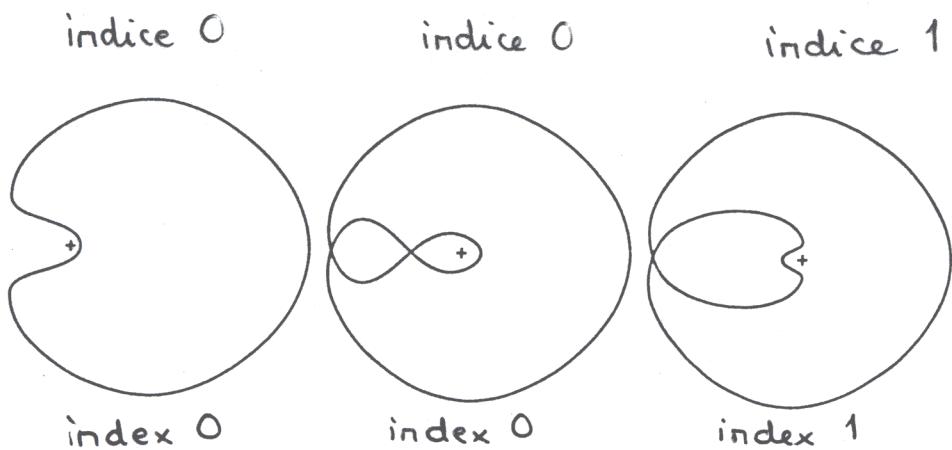
$E(\lambda)$  has a double zero at the origin.

## RESULTS :

P	-2	10
P		
1	STABLE	
2	STABLE	
3	STABLE	
4	STABLE	
5	STABLE / UNSTABLE	

Stability of primary wave

3 3,5



Schématisation des trois types de courbes observées lors du tracé de l' image des droites d' équation  $\operatorname{Re}(\lambda) = \varepsilon$   
 $(\varepsilon \text{ petit})$

Qualitative behavior of the three types of curves observed when plotting the image of the straight lines  $\operatorname{Re}(\lambda) = \varepsilon$  ( $\varepsilon$  small)

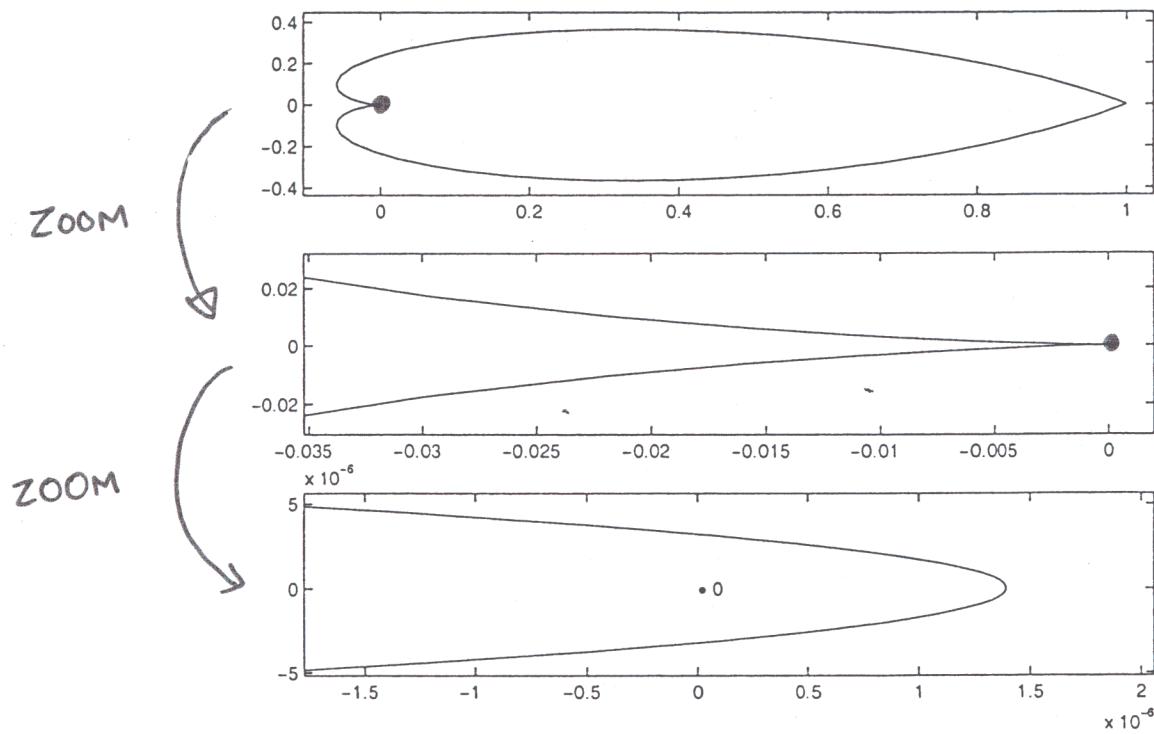
$E(\varepsilon + iy) \quad -Y_\infty < y < Y_\infty, \varepsilon \ll 1$   
 (small offset needed to circumvent the 2nd order pole of  $E(\lambda)$  at  $\lambda=0$ )

$$Y_\infty \sim 10^8$$

$$p=5$$

$$\varepsilon = 0.002$$

$$P=0$$



La solution est stable : l'indice de  
la courbe par rapport à 0 est 0

Stable solution : index is 0

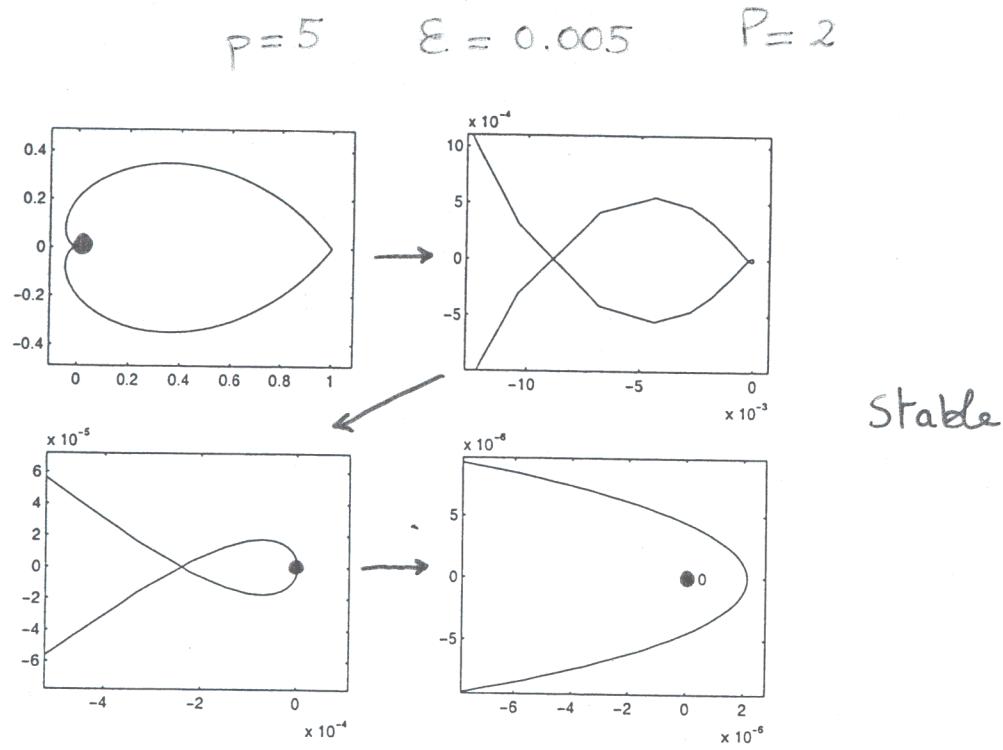


FIG. 4:  $p = 5, \varepsilon = 0.005, P = 2$ . La solution est stable : l'indice de la courbe par rapport à 0 est 0.

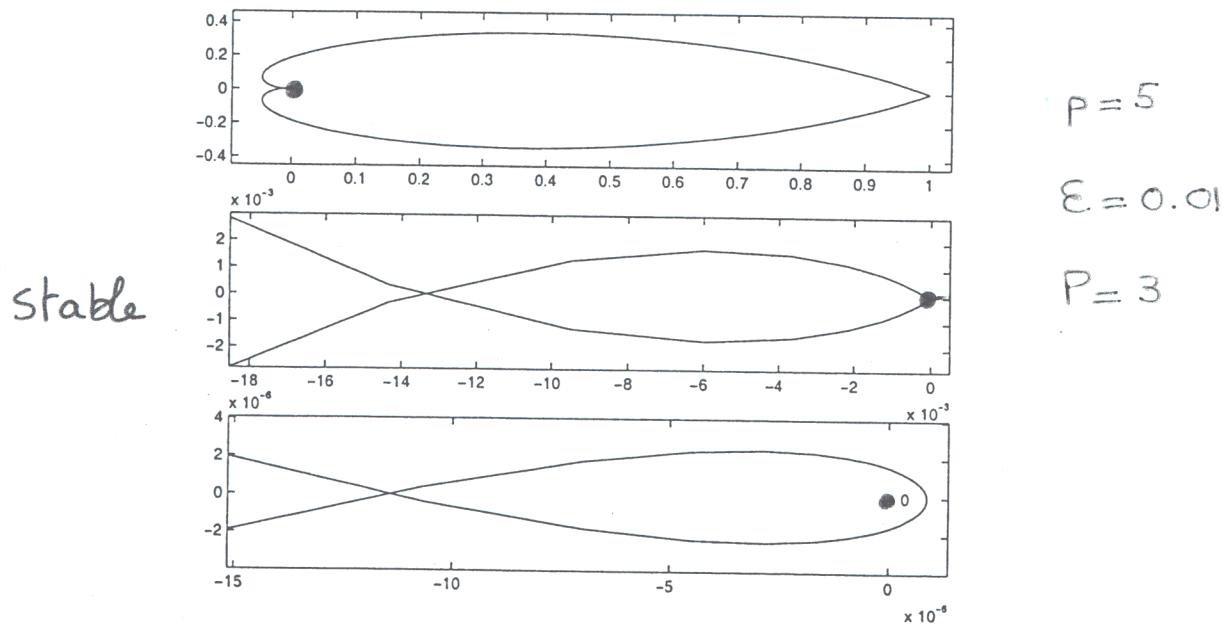
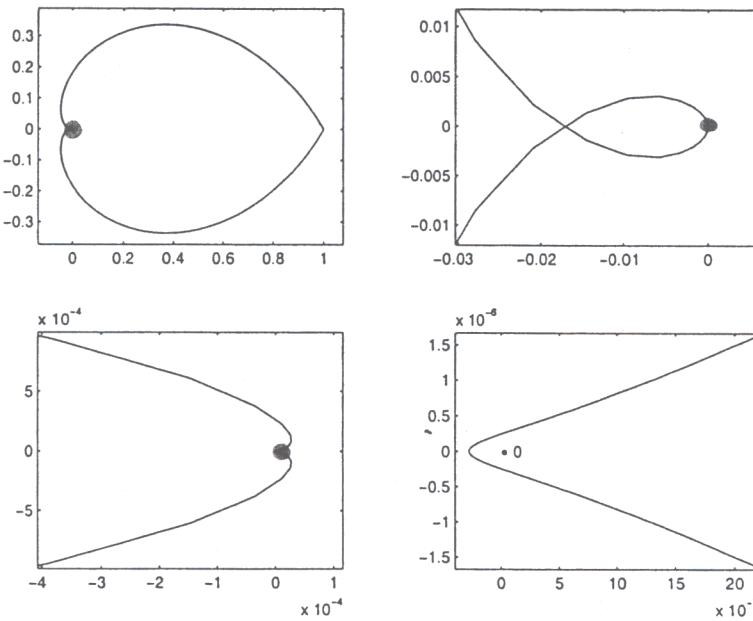


FIG. 5:  $p = 5, \varepsilon = 0.01, P = 3$ . La solution est stable : l'indice de la courbe par rapport à 0 est 0.

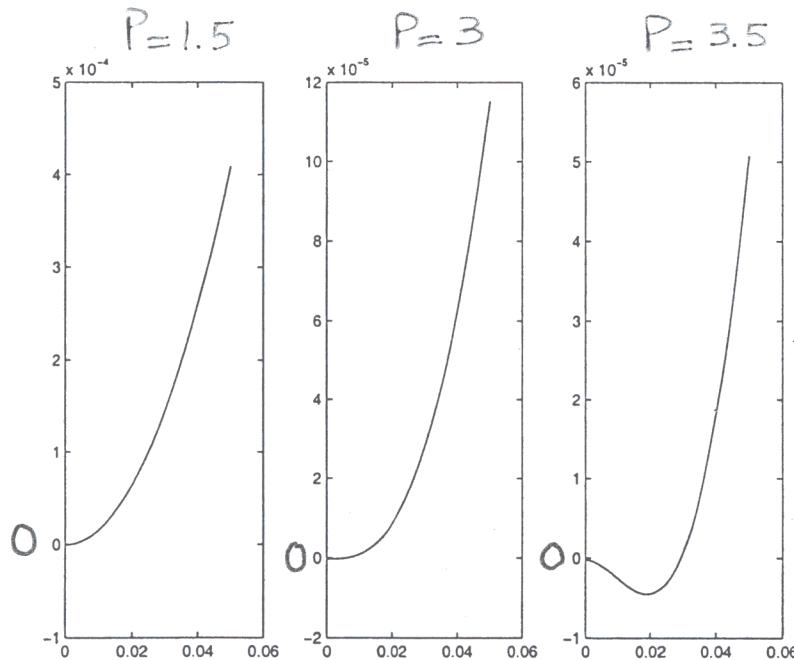
$$p = 5 \quad \varepsilon = 0.001 \quad P = 3.5$$



Solution  
instable

Unstable

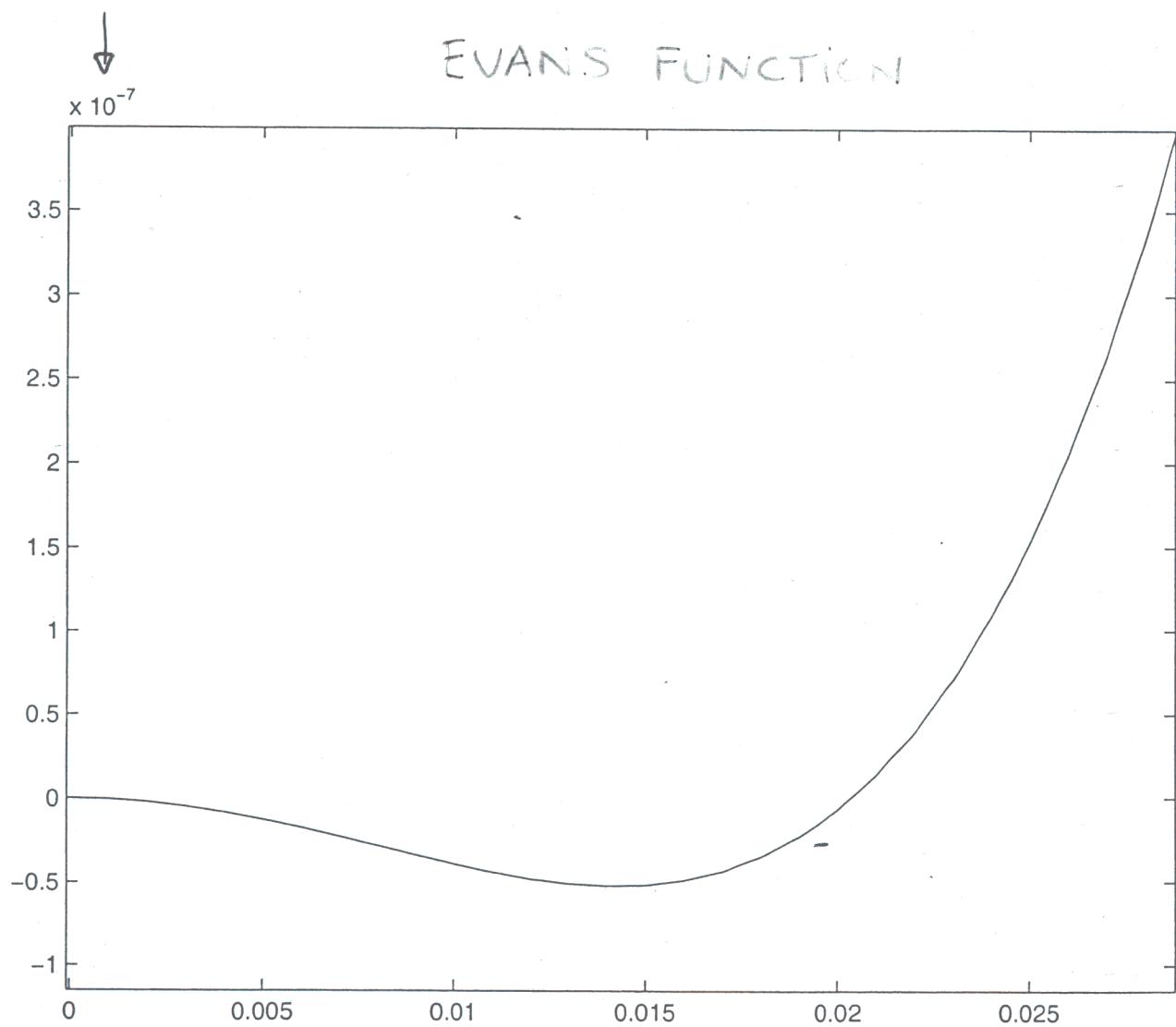
FIG. 6:  $p = 5, \varepsilon = 0.001, P = 3.5$ . La solution est instable : l'indice de la courbe par rapport à 0 est 1.



fonctions  
d' Evans  
Evans function  
 $\lambda \in \mathbb{R}$

FIG. 7: Graphe des fonctions d'Evans des solutions pour  $p = 5$  et  $P = 1.5, 3, 3.5$  le long de l'axe réel.

# UNSTABLE Bi-MODAL SOLUTION



$$\rho = 1 \quad P = 0.4$$