Neumann Eigenfunctions and Brownian Couplings

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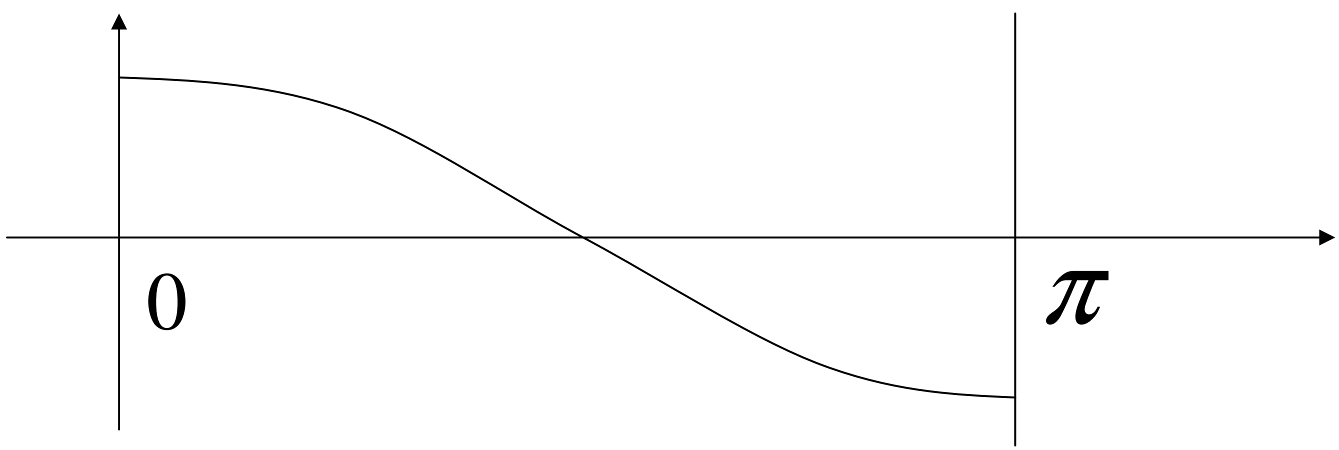
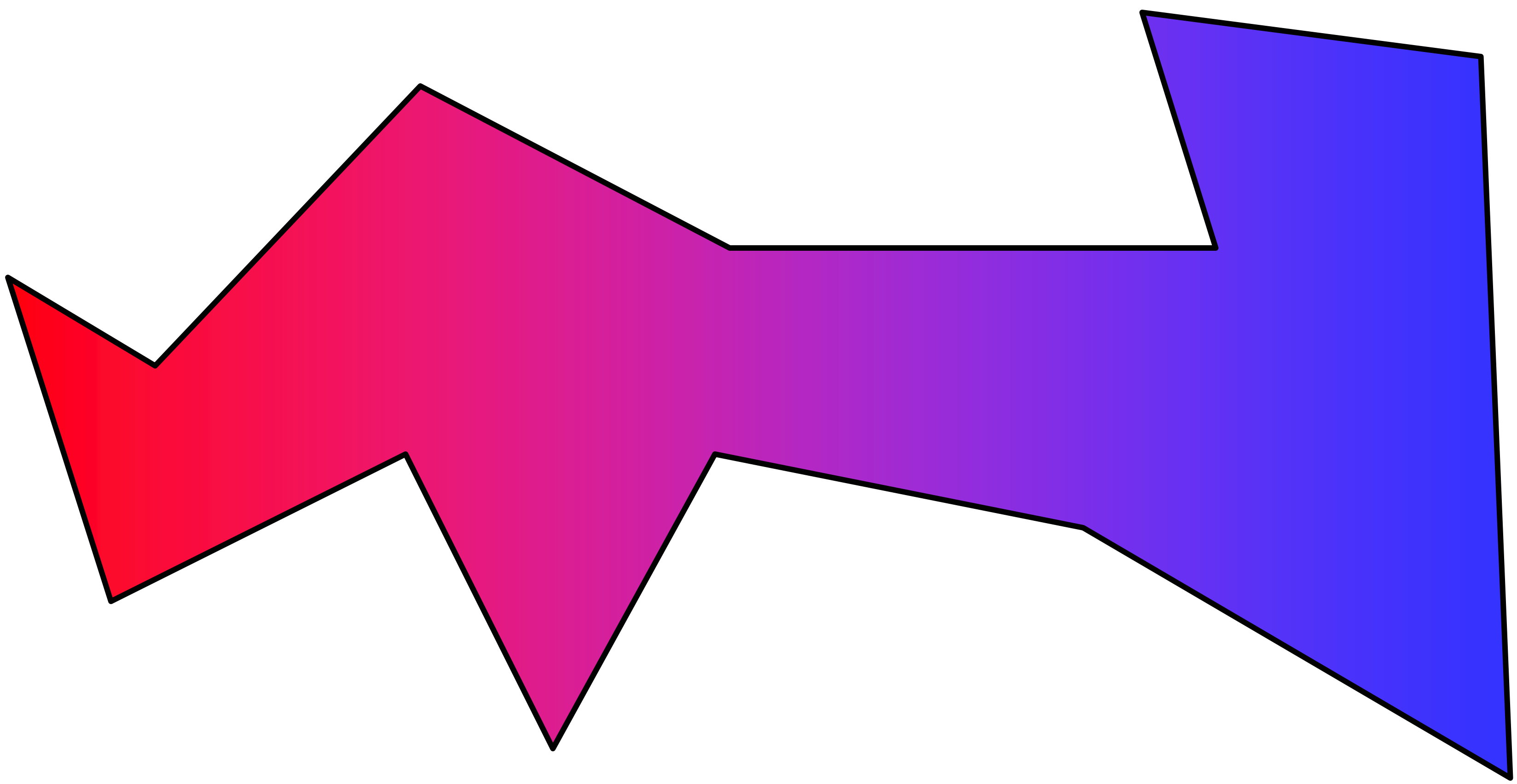
Hot Spots Conjecture

Rauch (1974) : In Euclidean domains, the second Neumann Laplacian eigenfunction attains its maximum at the boundary.



Multidimensional domains



Earliest results

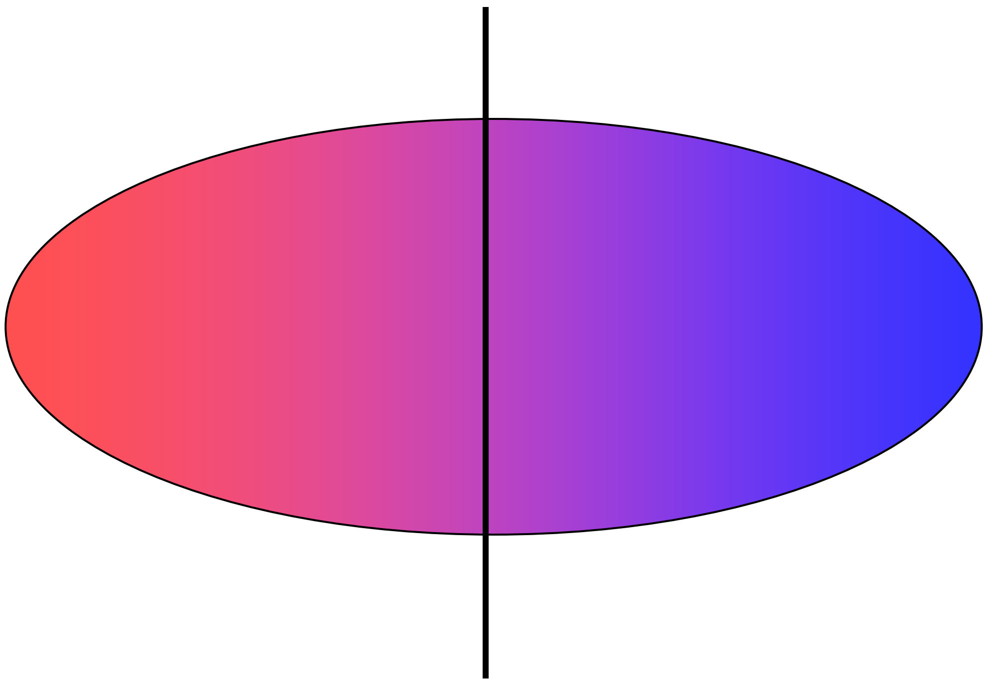
Kawohl (1985) : Conjecture holds for cylindrical domains $(0,1)\times D$

Bañuelos and B (1999) : Conjecture holds for

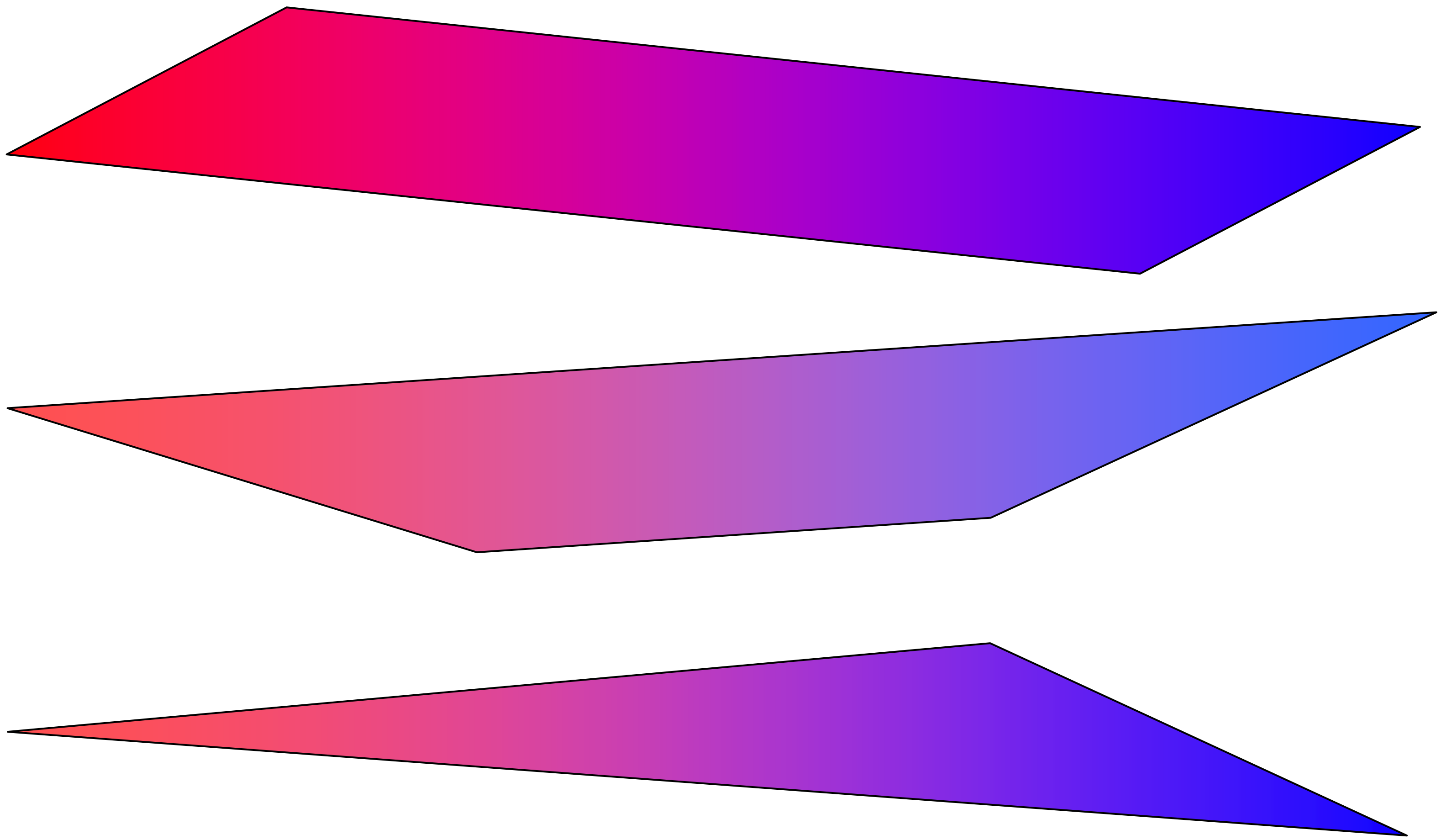
(i) some convex planar domains with a line of symmetry, and

(ii) “lip” domains.

Symmetric domains



Lip domains

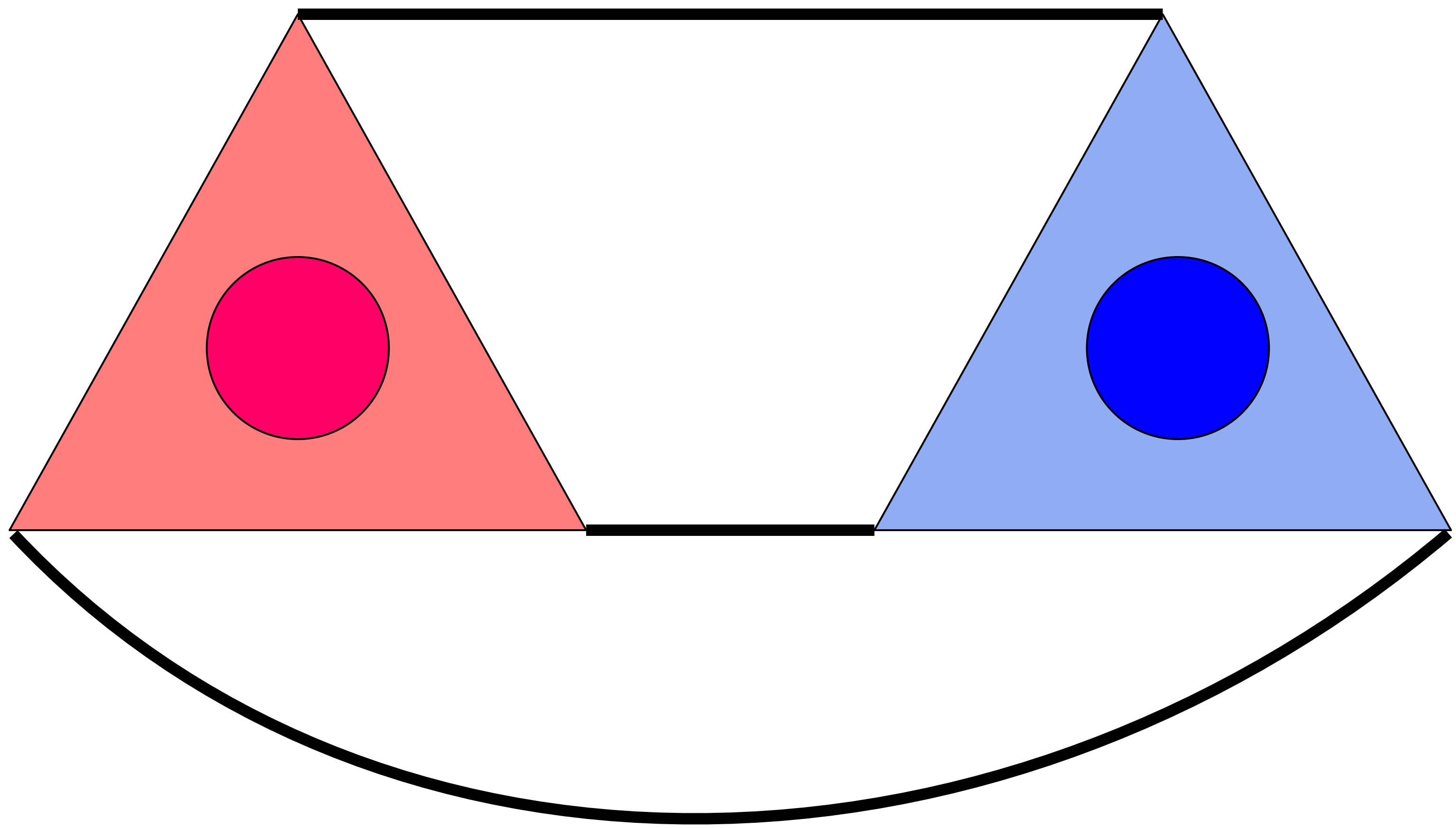


Counterexamples

B and Werner (1999) : There exists a domain where the maximum of second Neumann eigenfunction is attained in the interior.

Bass and B (2000) : There exists a domain where the maximum and minimum of second Neumann eigenfunction are attained in the interior.

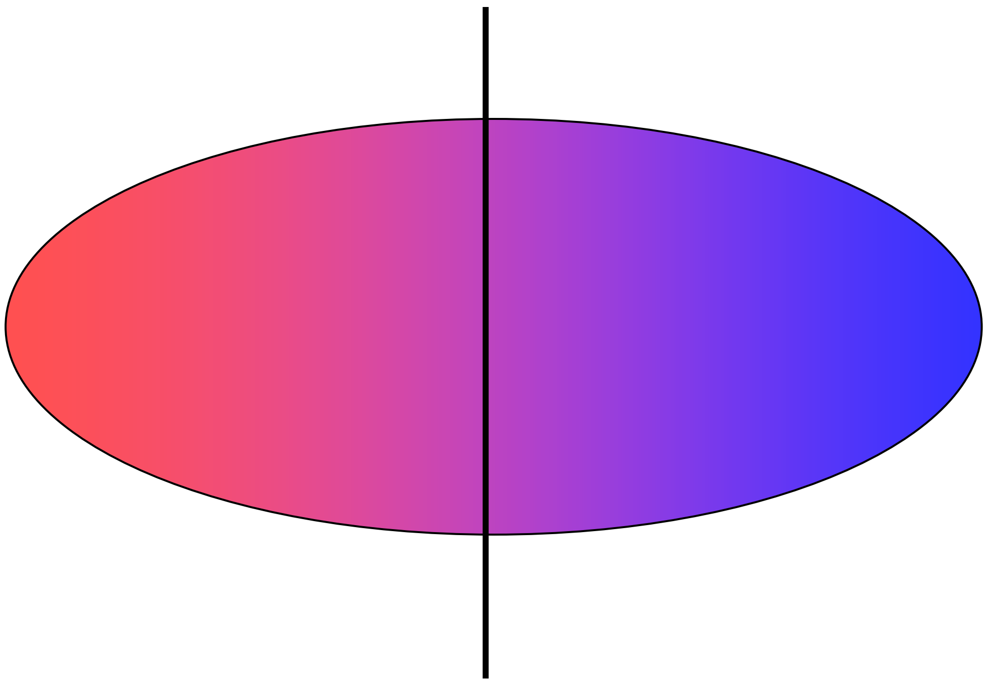
Counterexample idea



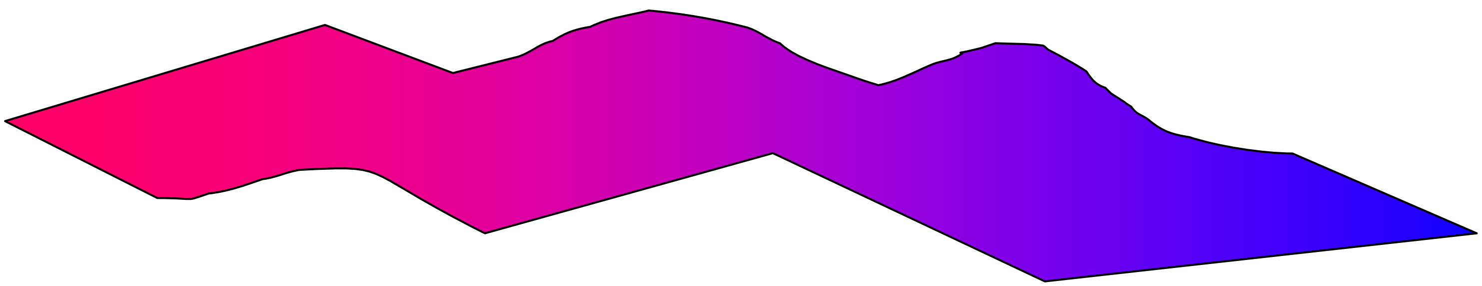
Positive direction

Pascu (2002) : Conjecture holds for planar convex domains with a line of symmetry.
Atar and B (2004) : Conjecture holds for all lip domains.

Symmetric domains



Lip domains



Other papers

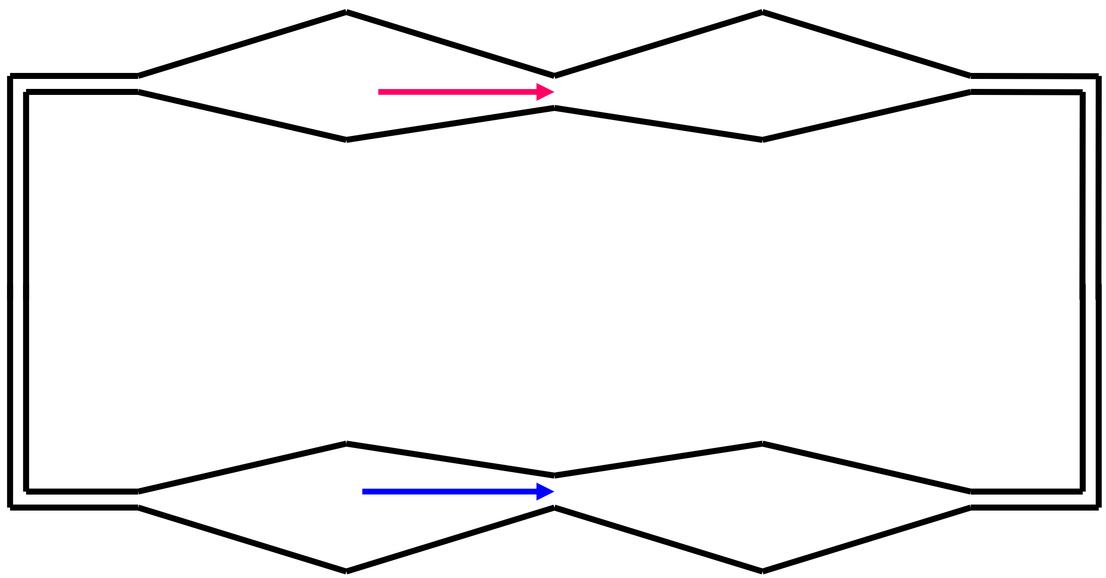
Jerison and Nadirashvili (2000)
Atar (2001)
……….

Open problems

(i) Hot spots conjecture for convex domains.

(ii) Hot spots conjecture for simply connected planar domains.

Counterexample with one hole



(Work in progress)

Thin domain families

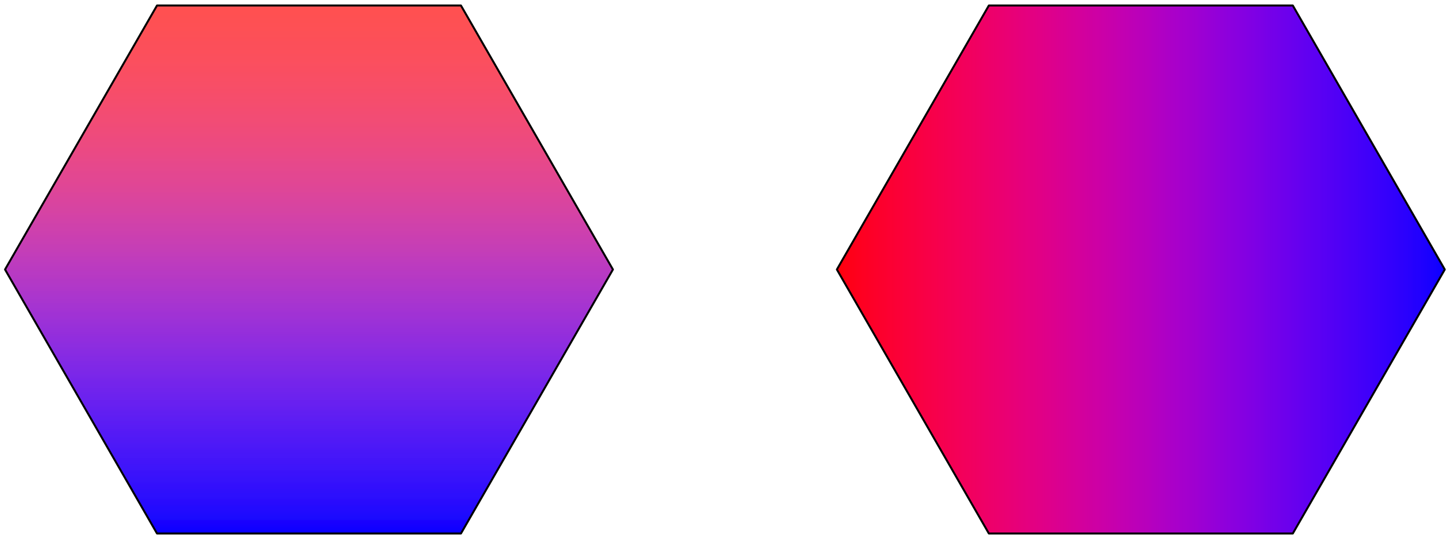
The family of **symmetric** domains is nowhere dense in the family of all convex domains.

The same holds for **lip** domains.

Large family of convex domains

Theorem (B, work in progress**)** :
There exists a family of planar convex domains with a non-empty interior such that the hot spots conjecture holds for all domains in this family.

Eigenvalue multiplicity

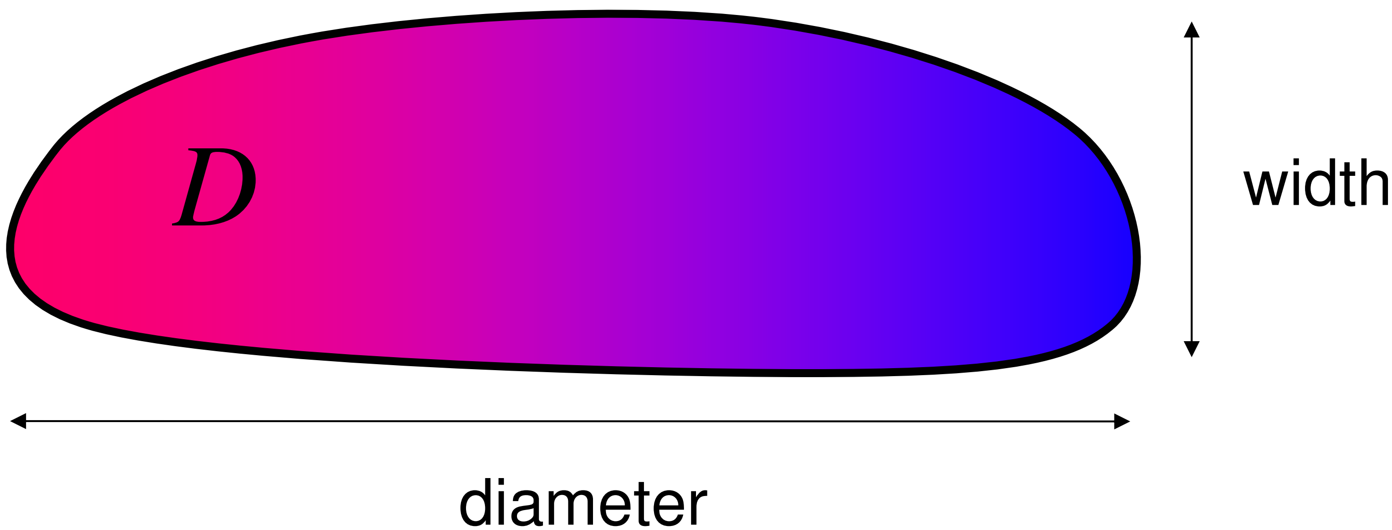


Maximum eigenvalue multiplicity

Nadirashvili (1986, 1988) :
The maximum multiplicity of the second Neumann eigenvalue for a simply connected planar domain is 2.

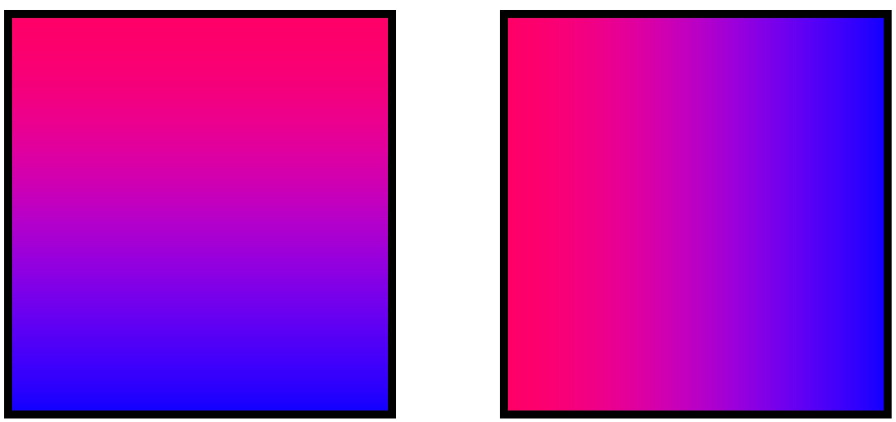
Long convex domains

Bañuelos and B (1999) : If D is convex and diameter/width is greater than 3.07 then the second eigenvalue is simple.



Convex domains - conjecture

Bañuelos and B (1999) : (Conjecture) If D is convex and diameter/width is greater than 1.41 then the second eigenvalue is simple.



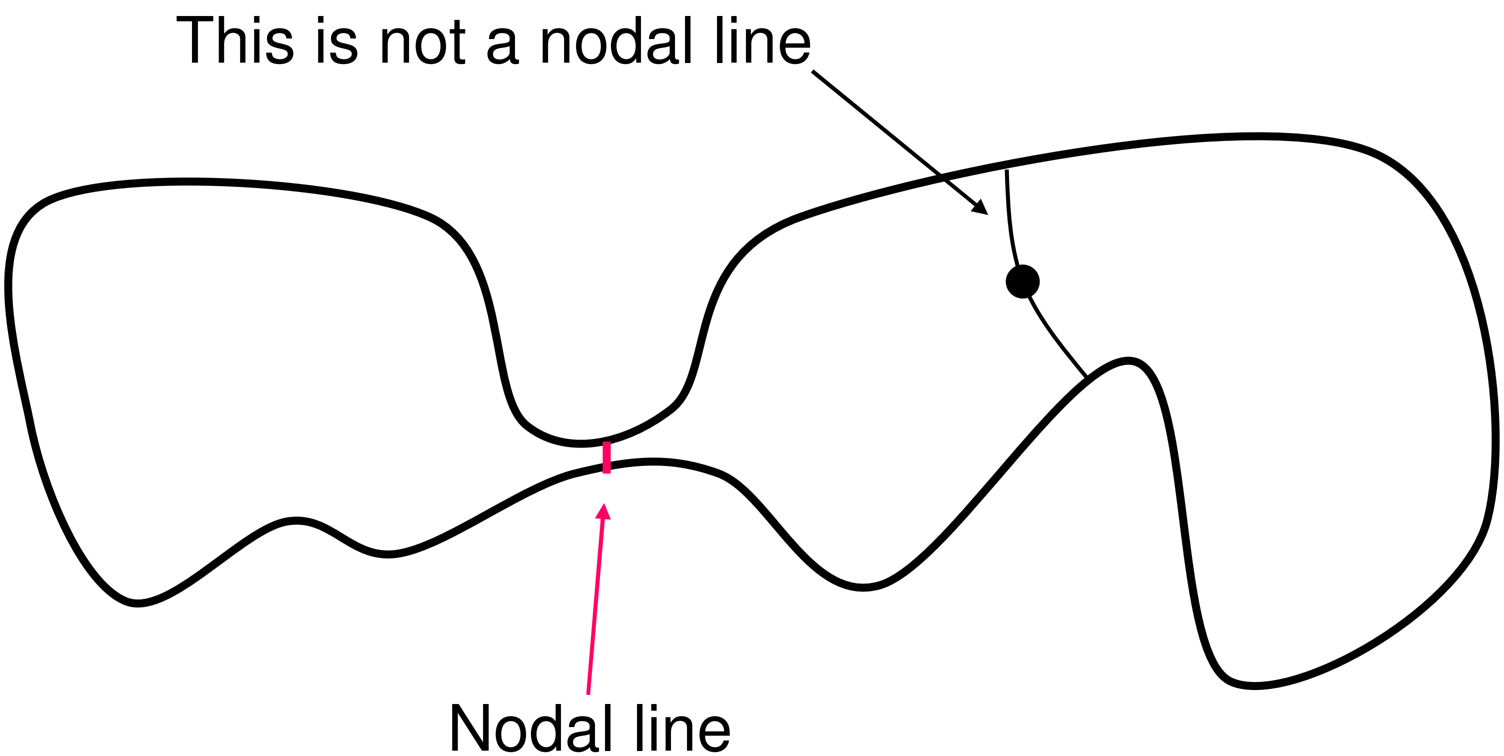
width = 1

diameter = $\sqrt{2}$

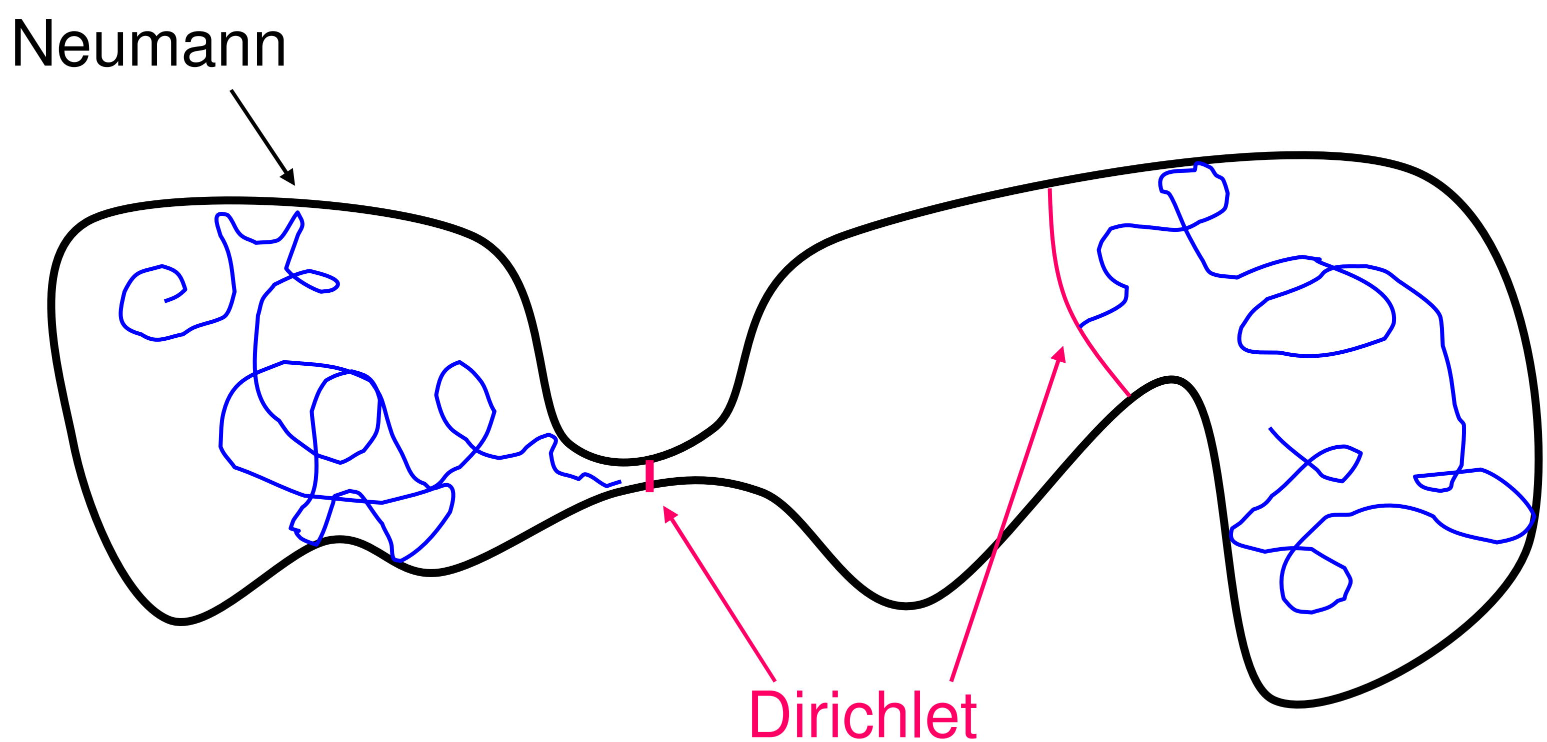
Points outside nodal lines

Bañuelos and B (1999) : If for some point the nodal (zero) line for any second eigenfunction does not pass through this point, then the second eigenvalue is simple.

Bottleneck domains

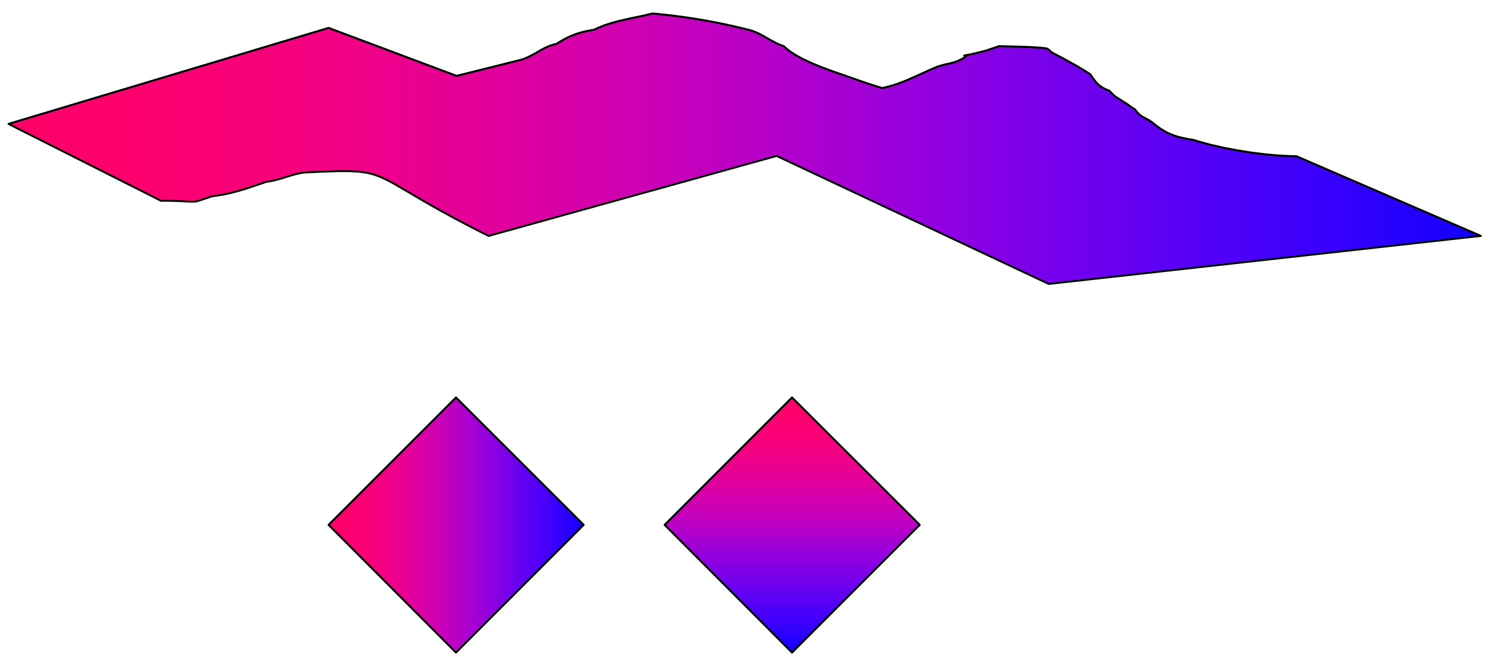


Bottleneck domains
(idea of proof)

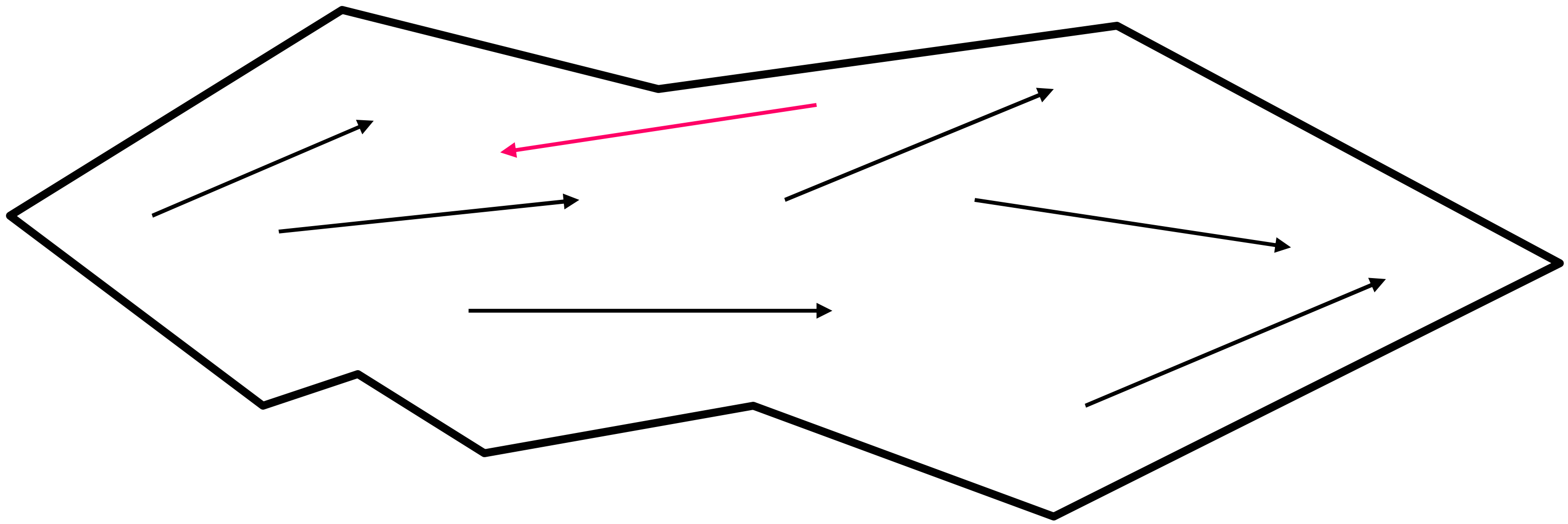


Lip domains

Atar and B (2004) : Second Neumann eigenvalue is simple in lip domains (except squares).



Lip domains
(idea of proof)



Probabilistic approach

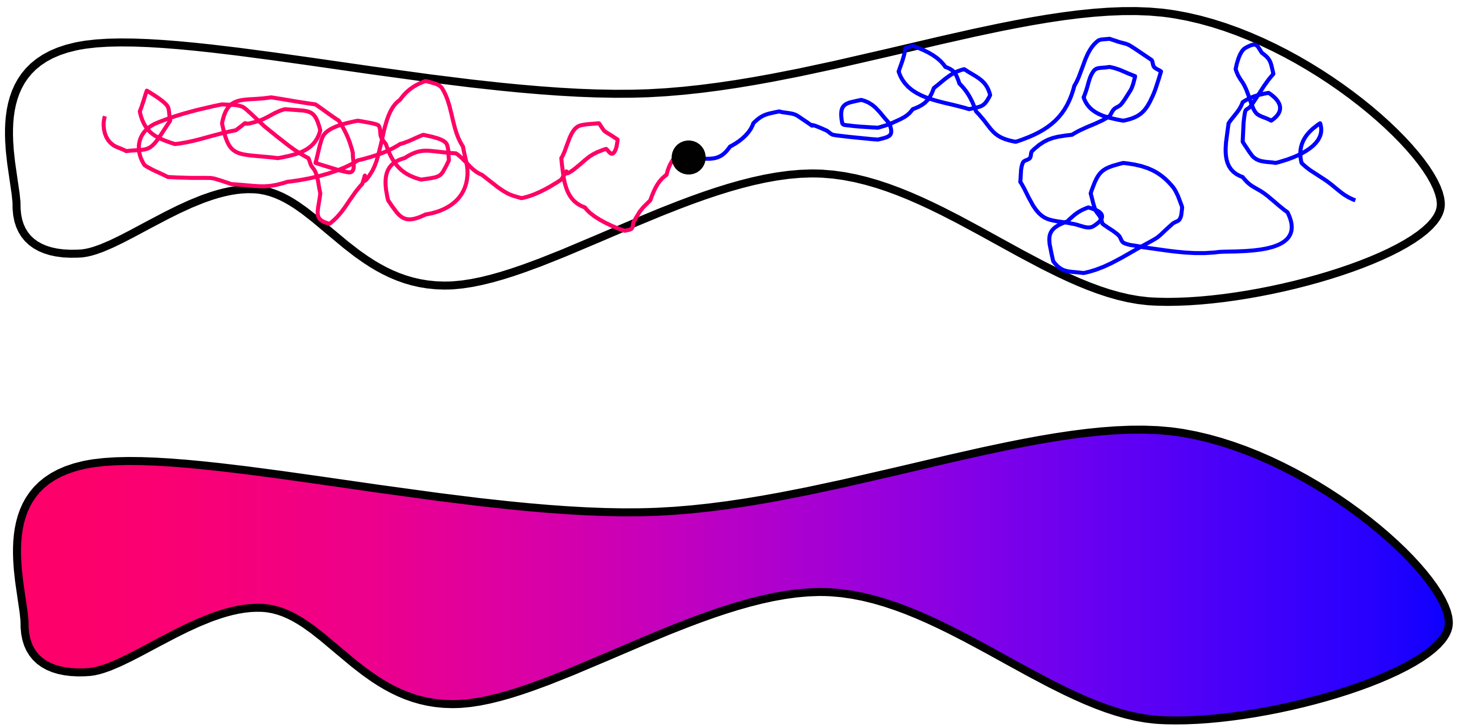
Stationary distribution is unique

$$\Downarrow$$

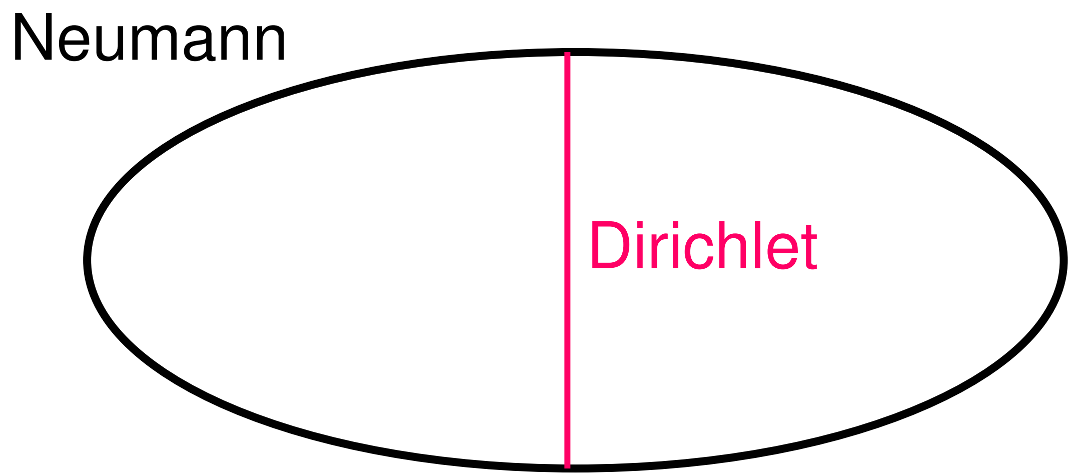
Second Neumann eigenfunction is unique

(work in progress)

Probabilistic approach (idea of proof)



Nodal line location
Motivation



Research on hot spots problem:

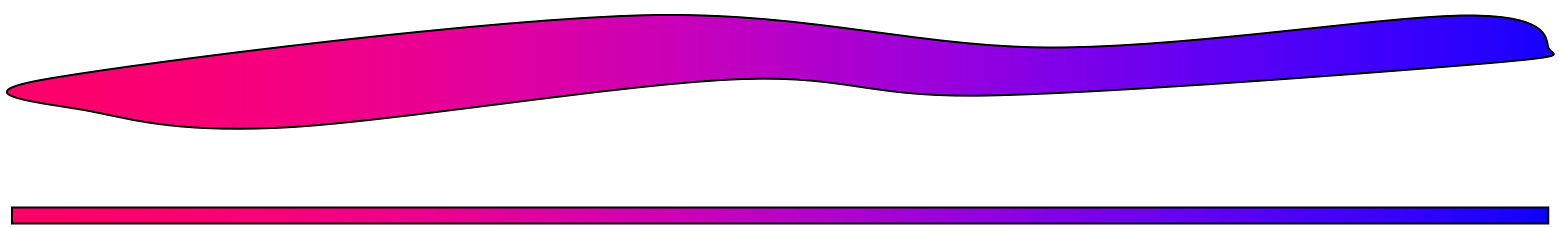
Bañuelos and B (1999)

Jerison and Nadirashvili (2000)

Pascu (2002)

Nodal line location
Old results

(i) Rectangles, ellipses, etc.
(ii) Domains with symmetry
(iii) **Jerison (2000)** Long and thin domains



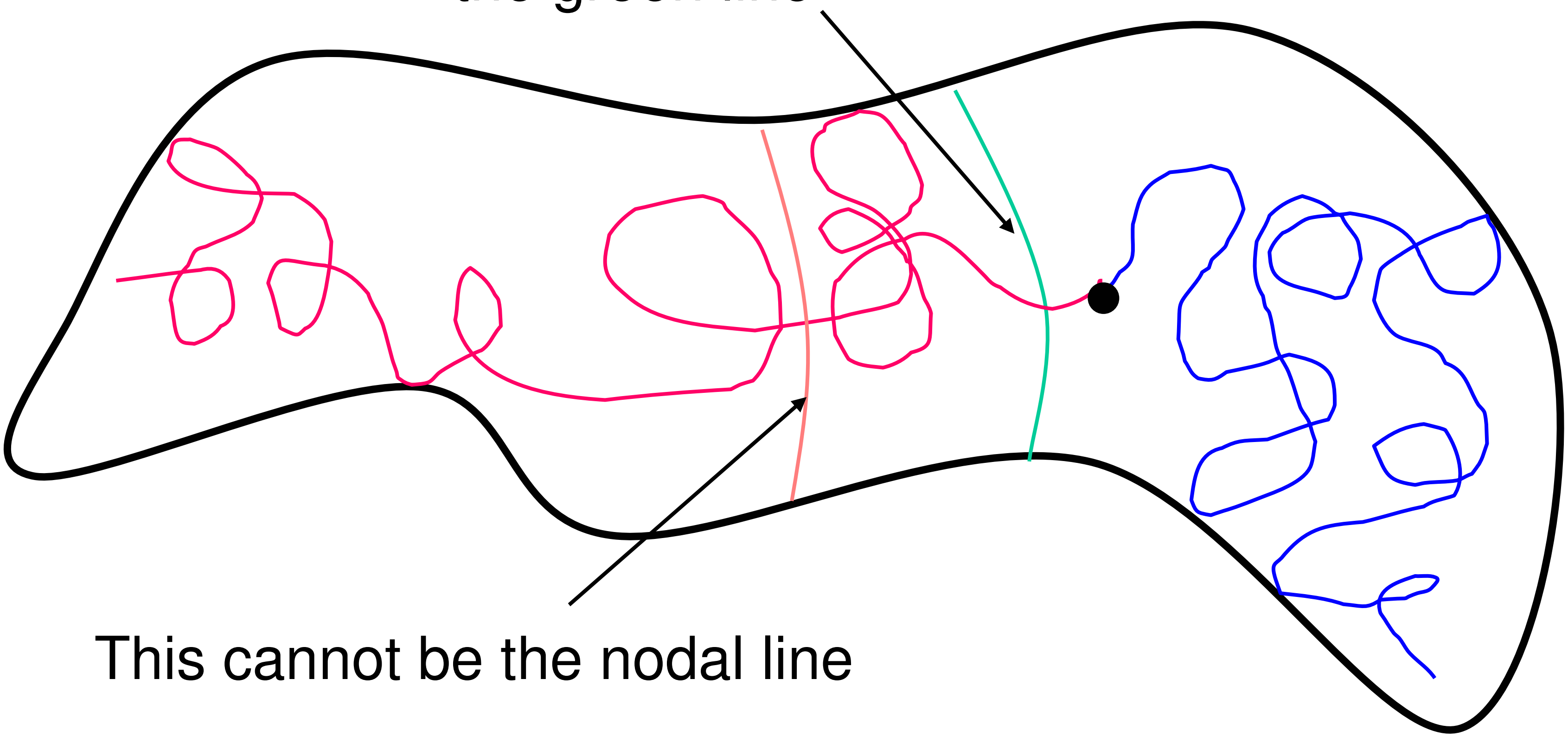
(**Melas (1992)** : Dirichlet nodal lines)

Nodal lines and couplings

Atar and B (2002) : Consider a coupling of reflected Brownian motions. If the particles cannot couple in a subset of the domain then this subset is too small to contain a nodal domain.

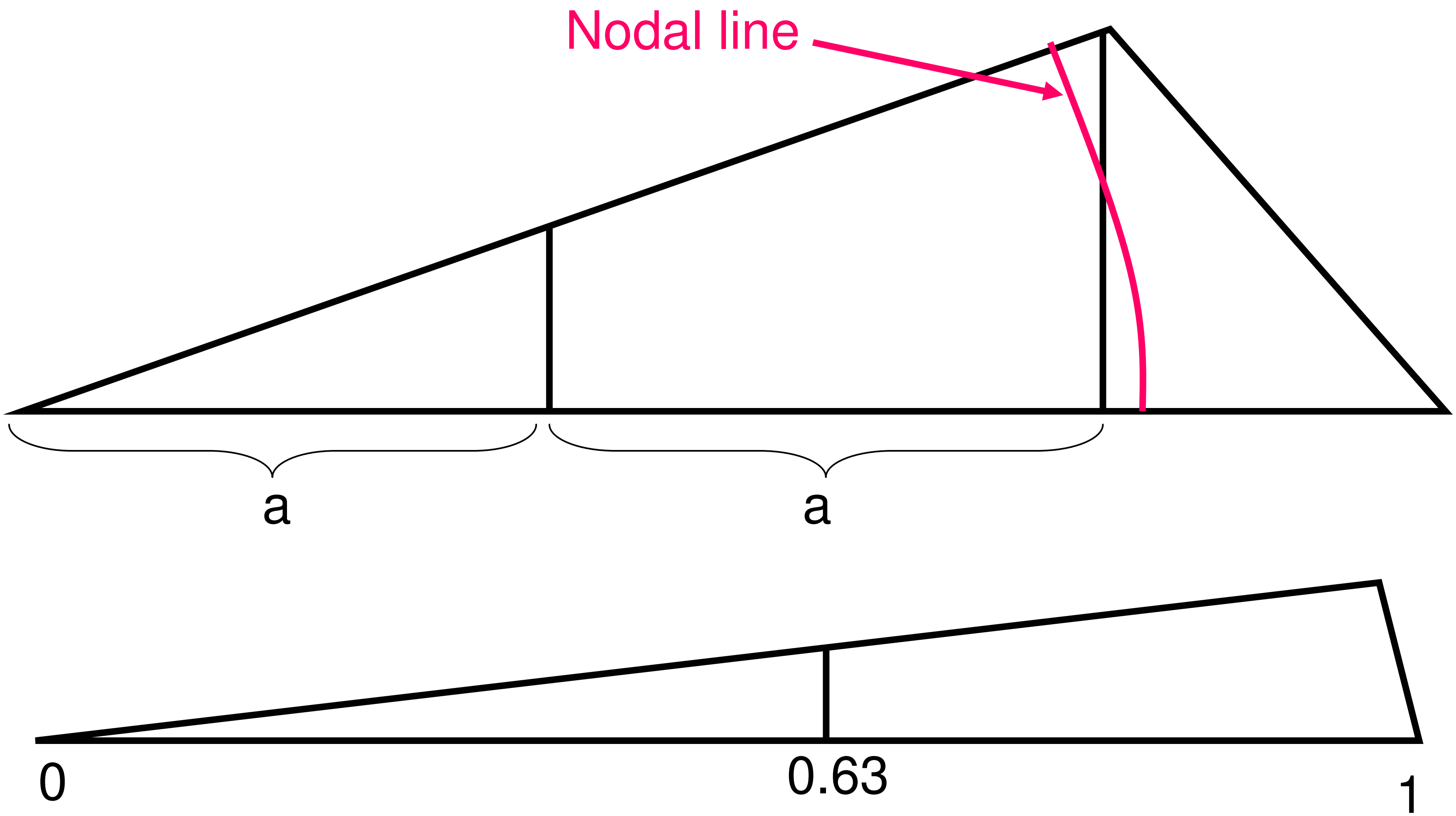
Nodal lines and couplings (idea of proof)

Coupling point cannot lie to the left of the green line

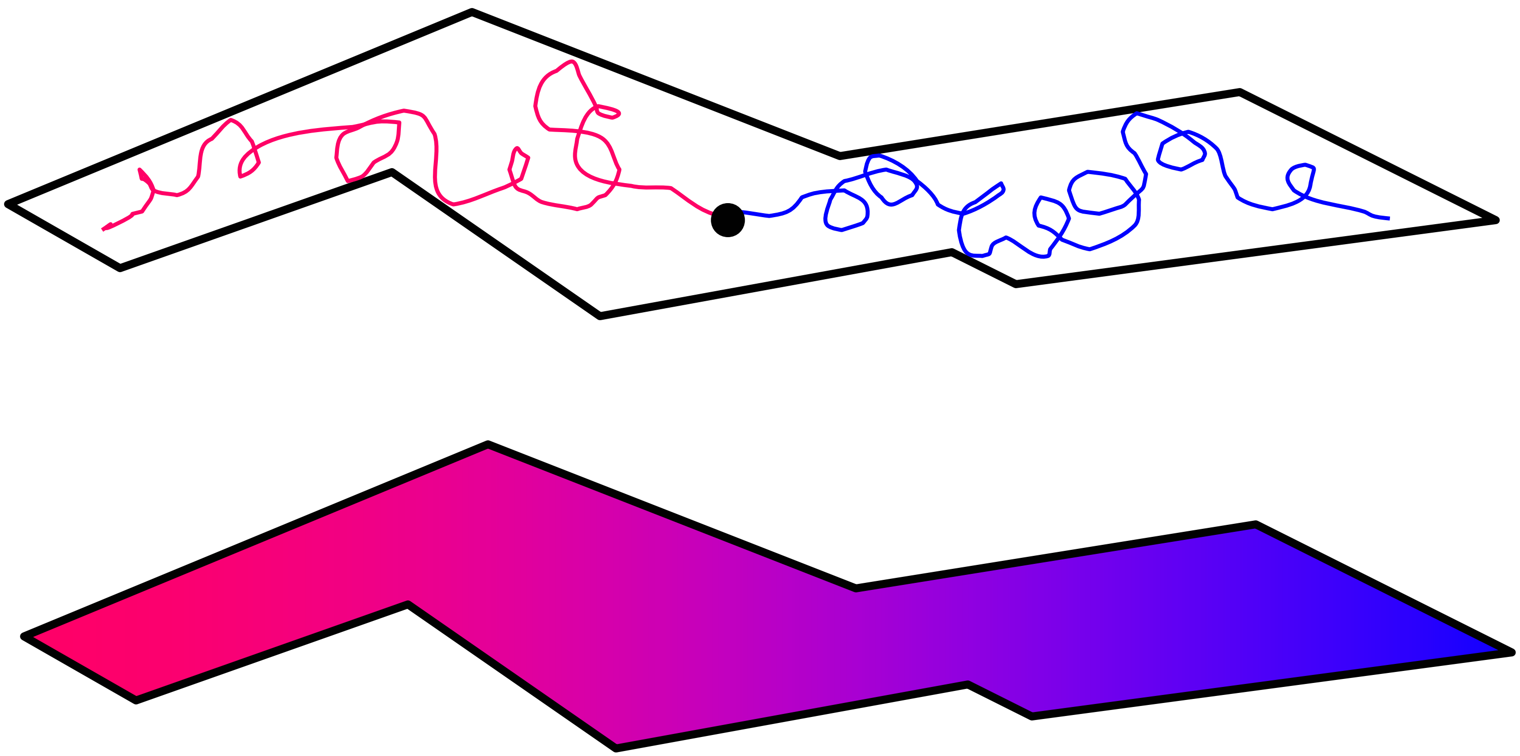


This cannot be the nodal line

Obtuse triangles

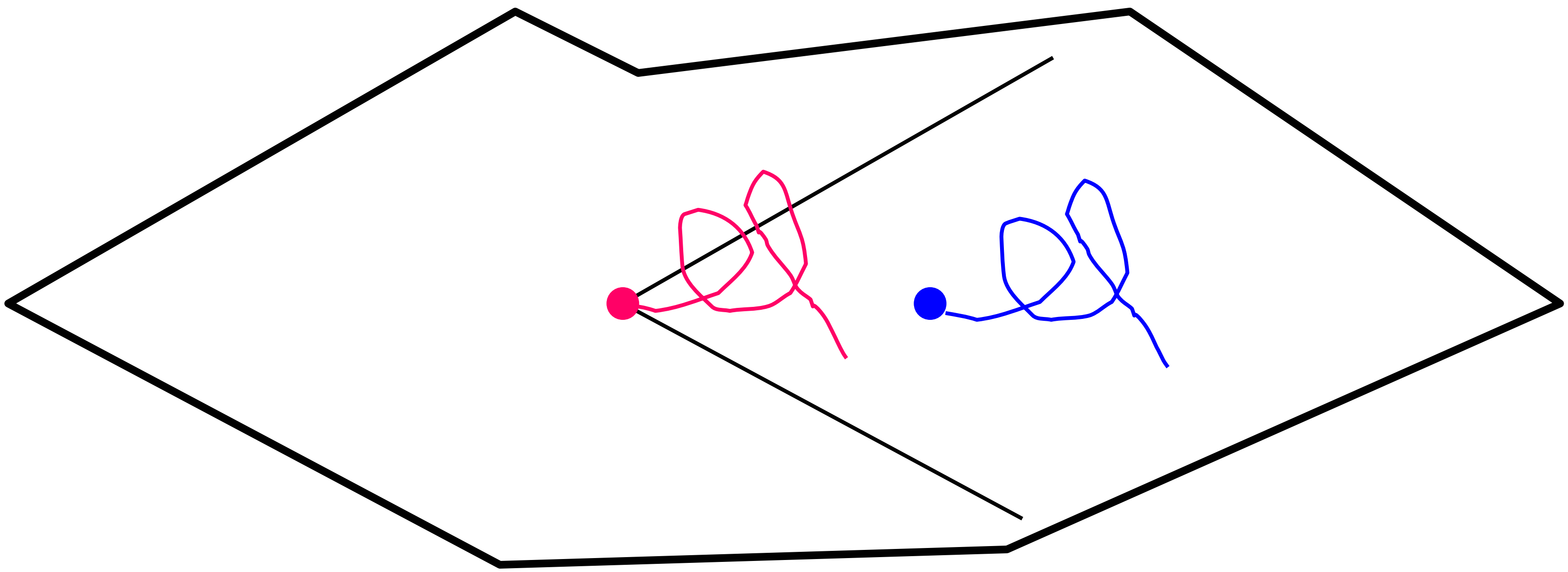


Brownian couplings



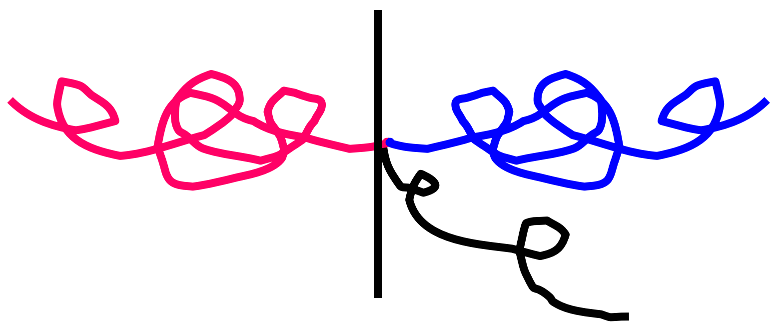
Monotone couplings

B and Kendall (2000)



Synchronous coupling

Mirror couplings: construction



Wang (1994)

B and Kendall (2000) : reflection on a single line

Atar and B (2004) : piecewise smooth domains

Synchronous couplings via unique strong solutions to Skorohod equation

$$X_t = x + B_t + \int_0^t N(X_s)\,dL_t^X$$

$$Y_t = y + B_t + \int_0^t N(Y_s)\,dL_t^Y$$

Lions and Sznitman (1984) :

C^2 - domains

Skorohod equation: unique strong solutions

$$X_t = x + B_t + \int_0^t N(X_s)\, dL_t^X$$

Bass, B and Chen (2004) : Unique strong solutions exist in planar Lipschitz domains with Lipschitz constant less than 1.

Skorohod equation - unique strong solutions : idea of proof

(i) Weak uniqueness for (B_t, X_t)
(ii)Strong existence
(iii)Start with RBM's with smooth reflection vector fields **(Dupuis and Ishii (1993))**
(iv)Use monotonicity to pass to the limit

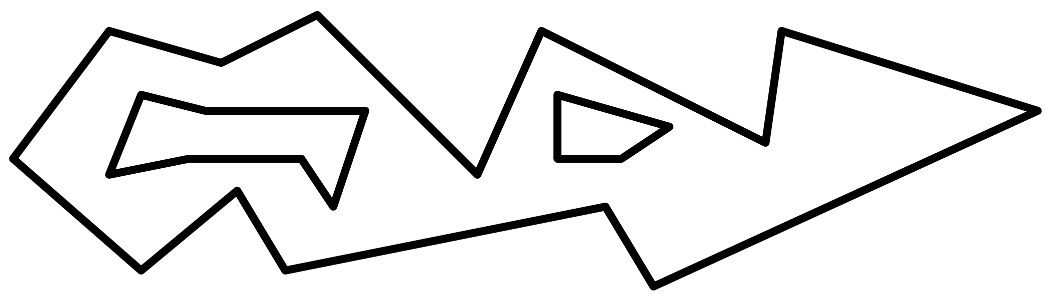
Convergence of synchronous couplings : motivation

B and Kendall (2000) : “efficient” couplings; rate of coupling = second Neumann eigenvalue

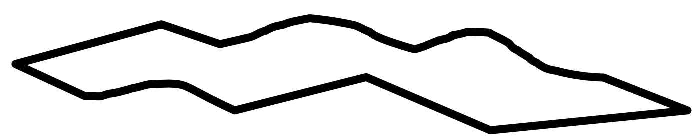
Convergence of synchronous couplings

B and Chen (2002) : Synchronous couplings converge in

(i) Polygonal domains



(ii)Lip domains



Synchronous couplings in smooth domains

B, Chen and Jones (work in progress)

If $\Lambda(D) > 0$ then, a.s.,

$$\lim_{t\to\infty} \frac{\log dist(X_t, Y_t)}{t} = -\frac{\Lambda(D)}{2|D|}$$

Lyapunov exponent

Angle between tangent lines - $\alpha(x, y)$

Curvature - $\upsilon(x)$

Harmonic measure - $\chi_x(dy)$

$$\Lambda(D) = \int_{\partial D} \upsilon(x)dx + \iint_{\partial D \times \partial D} |\log\cos\alpha(x, y)|\,\chi_x(dy)dx$$

By Gauss-Bonnet Theorem,

$(1/2\pi)\int_{\partial D} \upsilon(x)dx =$ 1 - number of holes

Domains with few holes

Corollary : If a smooth domain has at most one hole then synchronous couplings converge.

Lyapunov exponent
Open problems

Lemma. If D is the exterior of a disc then $\Lambda(D) = 0$

Conjecture. (numerical) If D is the exterior of an ellipse then $\Lambda(D) = 0$

Open problem. Are there any bounded domains D with $\Lambda(D) < 0$?

Shy couplings

X_t, Y_t - two copies of a Markov process defined on the same space

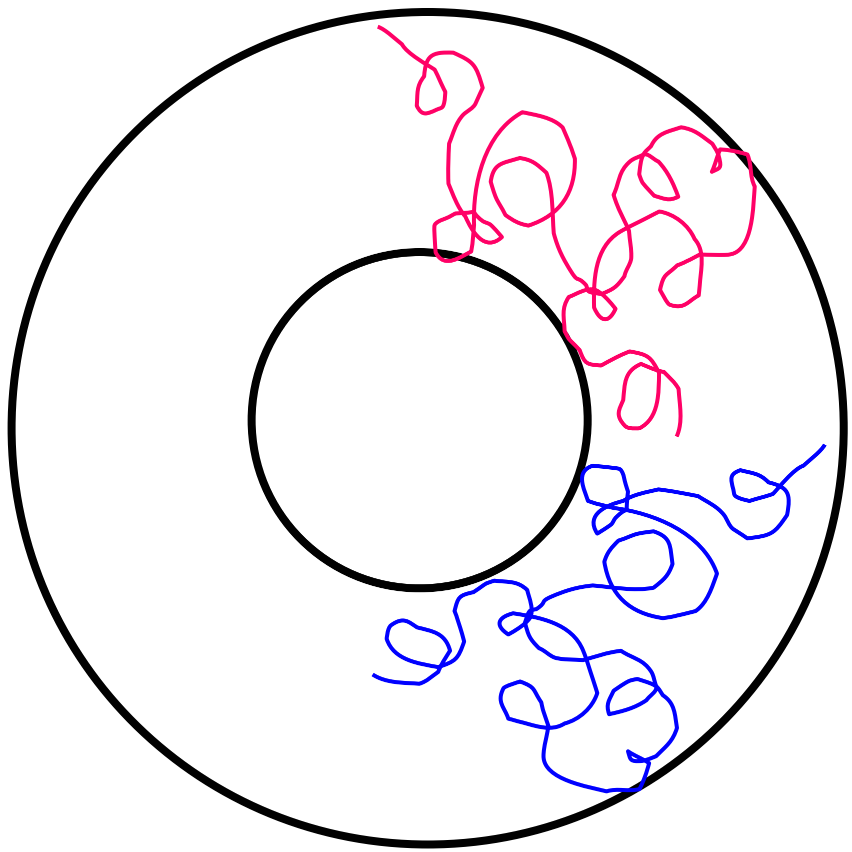
Definition. Shy coupling:

$$\inf_{0<t<\infty} dist(X_t, Y_t) > 0, a.s.$$

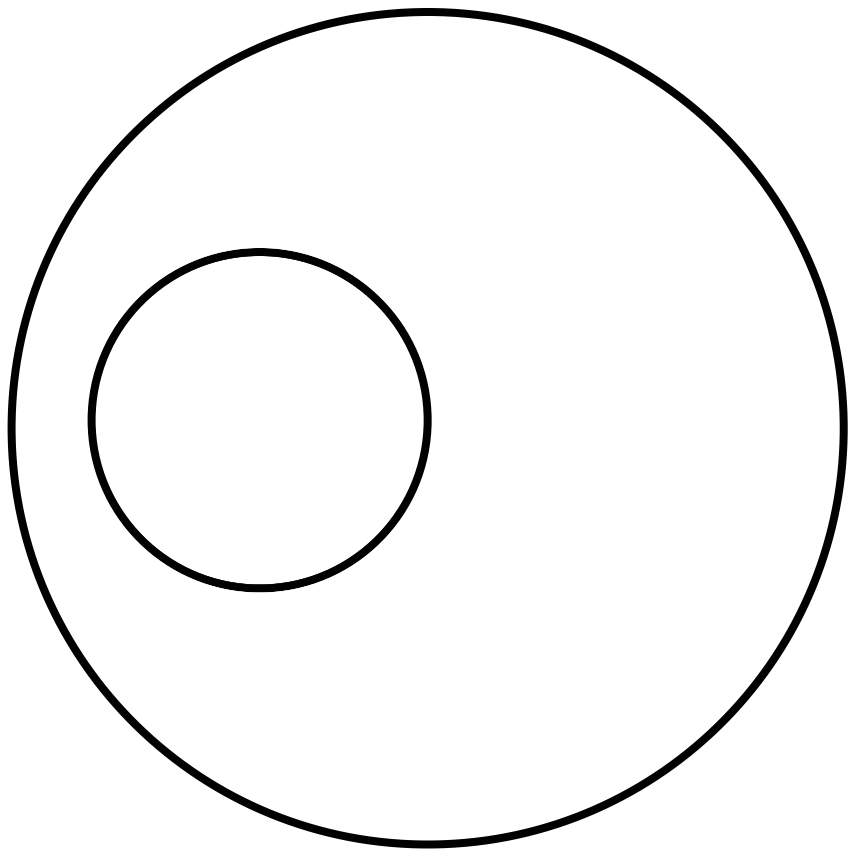
Shy couplings of RBM’s

Benjamini, B and Chen (in progress): There is no shy coupling of reflected Brownian motions in a convex bounded Euclidean domain.

Shy couplings in annuli



Shy couplings – open problem



Rigid couplings

Benjamini, B and Chen (in progress):
There exists a coupling of Brownian motions in the plane such that

$$dist(X_t, Y_t) = \sqrt{t}$$