9. THE KIM-SARNAK THEOREM

1. Preliminaries

In this lecture, our goal is to establish the best estimates on the Selberg eigenvalue conjecture and the Ramanujan conjecture for GL_n due to Kim and Sarnak [2]. Before we do so, let us examine the averaging idea assuming the Lindelöf hypothesis for automorphic L-functions. This conjecture predicts that

$$L(1/2+it,\pi) = O(f(\pi)^\epsilon(|t|+2)^\epsilon)$$

where $f(\pi)$ denotes the conductor of π.

Given a Dirichlet series

$$f(s) = \sum_{n=1}^\infty a_n n^{-s}$$

we can write by partial summation

$$f(s) = \sum_{n\le x} \frac{a_n}{n^s} + s\int_x^\infty \frac{S(t)dt}{t^{s+1}}$$

where

$$S(t) = \sum_{n<t} a_n.$$

If χ is a primitive Dirichlet character mod q, suppose that

$$f(s,\chi) = \sum_{n=1}^\infty a_n\chi(n)/n^s$$

extends to an entire function and satisfies a "Lindelöf hypothesis" of the form

$$f(1/2+it,\chi) = O(q^\epsilon(|t|+2)^\epsilon)$$

then standard methods of analytic number theory show that

$$S(t,\chi) := \sum_{n\le t} a_n\chi(n) \ll t^{1/2}q^\epsilon.$$

Thus, by what was said above, we find

$$f(\beta,\chi) = \sum_{n\le x} a_n\chi(n)n^{-\beta} + O(q^\epsilon x^{1/2-\beta}).$$

Now, let us consider an averaging

$$\sum_{\chi\ne\chi_0} f(\beta,\chi) = \sum_{n\le x} a_n n^{-\beta}\left(\sum_{\chi \text{even},\chi\ne\chi_0}\chi(n)\right) + O(q^{1+\epsilon}x^{1/2-\beta}).$$

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The inner sum is equal to $\phi(q)/2-1$ if $n \equiv \pm 1 \bmod q$ and -1 if $\not\equiv \pm 1 \bmod q$, so that if we choose $x=q$, we get

$$\sum_{\chi \text{even}, \chi \neq \chi_0} f(\beta,\chi) = \frac{\phi(q)-1}{2} + O(q^{1-\beta}) + O(q^{3/2-\beta}) + O(q^{1-\beta+\epsilon}).$$

If $f(\beta,\chi)=0$ for all $\chi \neq \chi_0$, we get a contradiction if $\beta > 1/2$.

Now let π be a cuspidal automorphic representation and let us apply this result to

$$f(s) = \prod_{p<\infty} L(s, \pi_p \times \tilde{\pi}_p).$$

By the method to be described below, we will get for the Ramanujan and Selberg conjectures the following estimates:

$$|\Re(\mu_{j,\infty})| \leq 1/4$$

as well as

$$|\alpha_j(p)| \leq p^{1/4}$$

for the Satake parameters. The challenge is to do this calculation without the Lindelöf hypothesis. This is the context of the paper by Luo, Rudnick and Sarnak [4].

2. Rankin-Selberg theory

Let π be a cuspidal automorphic representation of $GL_m(\mathbb{A}_{\mathbb{Q}})$. For π_∞ spherical (or unramified), the gamma factor of $L(s,\pi)$ is

$$L(s,\pi_\infty) = \prod_{j=1}^{m} \Gamma_{\mathbb{R}}(s-\mu_{j,\infty})$$

where

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2).$$

Selberg's conjecture is the assertion that $\Re(\mu_{j,\infty}) = 0$ for $j=1,...,m$.

If π corresponds to a Maass form of eigenvalue $\lambda = 1/4 + r^2$, then $\mu_{1,\infty} = ir$, $\mu_{2,\infty} = -ir$. Selberg's conjecture is then the statement that r is not purely imaginary. In other words, $\Re(\mu_{j,\infty}) = 0$. The gamma factor of $L(s, \pi \times \overline{\pi})$ is

$$L(s, \pi \times \tilde{\pi}_\infty) = \prod_{j,k=1}^{m} \Gamma_{\mathbb{R}}(s - \mu_{j,\infty} - \mu_{k,\infty}).$$

Let $\beta_0 = 2\max \Re(\mu_{j,\infty})$, then $L(s, \pi_\infty \times \tilde{\pi}_\infty)$ is holomorphic for $\Re(s) > \beta_0$. If χ is a primitive even Dirichlet character, then the same is true for $L(s, (\pi \times \chi)_\infty \times \tilde{\pi}_\infty)$. For χ even, primitive of sufficiently large prime conductor q, we have $\pi \times \chi \not\simeq \pi$ so that

$$L(s, \pi_\infty \times \tilde{\pi}_\infty) L(s, \pi \times \chi \times \tilde{\pi})$$

is entire. Hence, β_0 is a "trivial" zero of $L(s, \pi \times \chi)$ Thus,

$$L(\beta_0, \pi \times \chi \times \overline{\pi}) = 0$$

for all such χ. In this way, the problem becomes the familiar one of proving that certain twists of L-functions do not vanish at a given point. We will prove that

$$\sum_{q \sim Q} \sum_{\chi \neq \chi_0, \, \chi \, \text{even}} L(\beta, \pi \times \chi \times \overline{\pi}) \gg \frac{Q^2}{\log Q}$$

for $\Re(\beta) > 1 - \frac{2}{m^2+1}$. This is the basic strategy. The same strategy can be applied to improve estimates on the Ramanujan conjecture at the finite primes. Indeed, for p unramified, we have

$$L(s, \pi_p \times \tilde{\pi}_p) = \prod_{j,k=1}^{m} (1 - \alpha_j(p)\overline{\alpha_k(p)}p^{-s})^{-1}.$$

Suppose

$$p^{\beta_0} = \max_j |\alpha_j(p)|^2.$$

Then,

$$L(s, \pi_p \times \tilde{\pi}_p)$$

has a pole at $s = \beta_0$. Hence, the partial L-function

$$L^{(p)}(s, \pi \times \tilde{\pi}) = L(s, \pi_p \times \tilde{\pi}_p)^{-1} L(s, \pi \times \pi)$$

has a trivial zero at $s = \beta_0$. The same is true for all twists

$$L^{(p)}(s, \pi \times \chi \times \tilde{\pi})$$

for characters χ with $\chi(p) = 1$. By choosing special q's as in [6], one deduces the analogous theorem.

Thus, this argument puts both the finite and infinte versions of the Ramanujan conjectures on the same footing.

3. An application of the Duke-Iwaniec method

We begin by noting that if

$$L(s, \pi \times \tilde{\pi}) = \sum_{n=1}^{\infty} b(n)n^{-s}$$

and

$$L(s, \pi \times \chi \times \tilde{\pi}) = \sum_{n=1}^{\infty} b(n)\chi(n)n^{-s}$$

then the twisted L-function satisfies a functional equation of the form

$$\Lambda(s, \pi \times \chi \times \tilde{\pi}) = \epsilon(s, \pi \times \chi \times \tilde{\pi})\Lambda(1 - s, \pi \times \overline{\chi} \times \tilde{\pi})$$

where the global epsilon factor is given by

$$\epsilon(s, \pi \times \chi \times \tilde{\pi}) = \chi(f(\pi \times \tilde{\pi}))\epsilon(s, \pi \times \tilde{\pi})\epsilon(s, \chi)^{m^2}$$

and this can be shown to be equal to

$$\chi(f(\pi \times \tilde{\pi}))\tau(\chi)^{m^2} q^{-m^2 s}\epsilon(s, \pi \times \tilde{\pi})$$

which involves a bit of representation theory (see [4]).

We now apply the argument of Duke and Iwaniec [1]. Let $f \in C_c^\infty(0, \infty)$ with

$$\int_0^\infty f(x)dx = 1.$$

Set

$$k(s) = \int_0^\infty f(y)y^s \frac{dy}{y}.$$

Thus, $k(s)$ is entire, rapidly decreasing and $k(0) = 1$. For $x > 0$, let

$$F_1(x) = \frac{1}{2\pi i} \int_{(2)} k(s)x^{-s} \frac{ds}{s}$$

and

$$F_2(x) = \frac{1}{2\pi i}\int_{(2)} k(-s)G(-s+\beta)x^{-s}\frac{ds}{s}$$

where

$$G(s) = \frac{L(1-s,\pi_\infty \times \tilde{\pi}_\infty)}{L(s,\pi_\infty \times \tilde{\pi}_\infty)}.$$

Recall that

$$\beta_0 = 2\max_j \Re(\mu_\infty(j))$$

and we assume $0 < \Re(\beta) < 1$.

Lemma 1. (1) *$F_1(x)$ and $F_2(x)$ are rapidly decreasing as $x \to \infty$.*
(2) *As $x \to 0$,*

$$F_1(x) = 1 + O(x^{-N})$$

for all $N \geq 1$.
(3) *As $x \to 0$,*

$$F_2(x) \ll 1 + x^{1-\beta_0-\Re(\beta)-\epsilon}.$$

Proof. The asymptotics for $F_1(x)$ follow upon shifting the contour of integration to the right (for $x \to \infty$) and to the left for $x \to 0$). As for $F_2(x)$, we apply Stirling's formula to deduce that $G(s)$ is of moderate growth in vertical strips and so we may shift contours. To get the behaviour as $x \to \infty$, we shift the contour to the right. For the behaviour as $x \to 0$, we shift the contour to the left. If $\Re(\beta) + \beta_0 - 1 < 0$, we pick up a simple pole at $s = 0$ which gives $F_2(x) = O(1)$. Otherwise, we pick up the first pole at $s = \beta + \beta_0 - 1$ and there are none to its right. In this case, we get the bound

$$F_2(x) \ll x^{1-\beta_0-\Re(\beta)}(-\log x)^{d-1}$$

where $d \leq m^2$ is the maximal order of a pole of

$$L(s,\pi_\infty \times \tilde{\pi}_\infty)$$

on the line $\Re(s) = \beta_0$.
∎

The next step is to derive the "approximate functional equation" in the following form. With F_1 and F_2 defined as above, for $\chi \neq \chi_0 \bmod q$, with q coprime to the conductor of π, and $0 < \Re(\beta) < 1$, we have for $\Pi = \pi \times \tilde{\pi}$,

$$L(\beta,\Pi\times\chi) = \sum_{n=1}^{\infty}\frac{b(n)\chi(n)}{n^\beta}F_1(n/Y) + \tau(\pi\times\tilde{\pi})(q^{m^2}f)^{-\beta}\sum_{n=1}^{\infty}\frac{b(n)\tilde{\chi}(n)}{n^{1-\beta}}\chi(f)\tau(\chi)^{m^2}F_2(nY/fq^{m^2}).$$

To see this, consider the integral

$$\frac{1}{2\pi i}\int_{(2)} k(s)L(s+\beta,\Pi\times\chi)Y^s\frac{ds}{s} = \sum_{n=1}^{\infty}\frac{b(n)\chi(n)}{n^\beta}\left(\frac{1}{2\pi i}\int_{(2)} k(s)(Y/n)^s\frac{ds}{s}\right) = \sum_{n=1}^{\infty}\frac{b(n)\chi(n)}{n^\beta}F_1(n/Y).$$

By the lemma, this converges absolutely and again by the lemma, we may shift the contour to $\Re(s) = -1$. Thus,

$$\frac{1}{2\pi i}\int_{(2)} k(s)L(s+\beta,\Pi\times\chi)Y^s\frac{ds}{s} = L(\beta,\Pi\times\chi) + \frac{1}{2\pi i}\int_{(-1)} k(s)L(s+\beta,\Pi\times\chi)Y^s\frac{ds}{s}.$$

Applying the functional equation to the second integral, we get

$$\frac{1}{2\pi i}\int_{(-1)} k(s)\tau(\pi\times\tilde{\pi})\chi(f)\tau(\chi)^{m^2}(fq^{m^2})^{-s-\beta}G(s+\beta)L(1-s-\beta,\pi\times\overline{\chi})Y^s\frac{ds}{s}.$$

We now change s to $-s$ and integrate term by term to get

$$\tau(\pi\times\tilde{\pi})\chi(f)\tau(\chi)^{m^2}(fq^{m^2})^{-\beta}\sum_{n=1}^{\infty}\frac{b(n)\overline{\chi}(n)}{n^{1-\beta}}F_2(nY/fq^{m^2}).$$

We sum this over the non-trivial even characters mod q and apply the orthogonality relation noted before, to obtain several sums. The first sum to consider is

$$\sum_{q\sim Q}\sum_{\chi\neq\chi_0,\,\chi\text{ even}}\sum_{n}\frac{b(n)\chi(n)}{n^\beta}F_1(n/Y)=\sum_{q\sim Q}\frac{q-1}{2}\sum_{n\equiv\pm1(q)}\frac{b(n)}{n^\beta}F_1(n/Y)-\sum_{q\sim Q}\sum_{(n,q)=1}\frac{b(n)}{n^\beta}F_1(n/Y).$$

We single out the contribution from $n=1$:

$$\sum_{q\sim Q}\frac{q-1}{2}F_1(1/Y)=\sum_{q\sim Q}\frac{q-1}{2}(1+O(Y^{-N}))\sim\frac{cQ^2}{\log Q}$$

for some positive constant c as we will choose Y so that $Q\ll Y\ll Q^{m^2}$. In fact, we will choose

$$Y\asymp Q^{(m^2+1)/2}.$$

The sum over $n\equiv 1 \bmod q$ with $n\neq 1$ gives

$$\sum_{q\sim Q}\sum_{n\equiv 1 \bmod q,n\neq1}\frac{b(n)}{n^\beta}F_1(n/Y)=\sum_{m}\frac{b(m)}{m^{\Re(\beta)}}F_1(m/Y)\left(\sum_{q\sim Q,q|(m-1),\,m\neq1}\frac{q-1}{2}\right)$$

which is

$$\ll Q\sum_m\frac{b(m)m^\epsilon}{m^{\Re(\beta)}}|F_1(m/Y)|,$$

where we have used the fact that for $m\neq1$, the number of representations $n=1+dq=1+d_1q_1$ for fixed n is $O(n^\epsilon)$ for any $\epsilon>0$. Now use

$$F_1(x)\sim 1$$

as $x\to0$ to get that this is

$$\ll QY^{1-\Re(\beta)+\epsilon}.$$

Similarly, the same estimate holds for terms $n\equiv -1 \bmod q$. To treat the second terms arising from the approximate functional equation, we use

$$\sum_{\chi\neq\chi_0,\chi\text{ even}}\overline{\chi}(m)\chi(f)\tau(\chi)^{m^2}\ll q^{(m^2+1)/2}$$

by Deligne's bounds for hyperkloosterman sums. Thus, we get

$$\sum_{q\sim Q}(fq^{m^2})^{-\beta}\sum_{\chi\neq\chi_0,\,\chi\text{ even}}\frac{b(n)\overline{\chi}(n)}{n^{1-\beta}}\chi(f)\tau(\chi)^{m^2}F_2(nY/fq^{m^2}),$$

which by Deligne's bound is

$$\ll\sum_{q\sim Q}(fq^{m^2})^{-\Re(\beta)}\sum_{(n,q)=1}\frac{b(n)}{n^{1-\Re(\beta)}}q^{(m^2+1)/2}F_2(nY/fq^{m^2}).$$

This is easily estimated by partial summation as

$$\ll \sum_{q\sim Q} (fq^{m^2})^{-\Re(\beta)} q^{(m^2+1)/2} \int_1^\infty F_2(Yt/fq^{m^2}) \frac{dt}{t^{1-\Re(\beta)}}$$

upon using the fact that

$$\sum_{n\leq x} b(n) \ll x.$$

Now using the bound for $F_2(x)$ provided by the lemma leads to a final estimate of

$$\ll Q^{1+(m^2+1)/2} Y^{-\Re(\beta)}$$

because F_2 is rapidly decreasing. With our choice of Y, we see that the main term is bigger than the error term if

$$\beta > 1 - \frac{2}{m^2+1}.$$

This leads to:

Theorem 2. *Let π be a cuspidal automorphic representation of $GL_m(\mathbb{A}_\mathbb{Q})$ with π_∞ spherical. Then,*

$$|\Re(\mu_{j,\infty})| \leq \frac{1}{2} - \frac{1}{m^2+1}.$$

In a similar way, by following the Duke-Iwaniec method [1] one gets the estimate

$$|\log_p |\alpha_{j,p}|| \leq \frac{1}{2} - \frac{1}{m^2+1}.$$

In [2] the method described is actually applied to

$$f(s) = L(s, \pi, \mathrm{Sym}^2),$$

which was shown by Kim [3] to be holomorphic if π is not self-contragredient. The functional equation was established by Shahidi [7]. If χ is a Dirichlet character of conductor q which we take to be prime and large, we have

$$L(s, \pi \times \chi, \mathrm{Sym}^2) = L(s, \pi, \mathrm{Sym}^2 \times \chi^2)$$

so that as long as χ not one of at most two characters mod q, $\pi \times \chi$ is not self-contragredient. It will be noted that the positivity of the $b(n)$ was not used in a vital way and only the weaker estimate

$$\sum_{n\leq x} b(n) \ll x$$

was used. Thus, we may apply the method to $f(s)$ above and deduce as in [2] the following:

Theorem 3. *Let π be a cuspidal automorphic representation of $GL_n(\mathbb{A}_\mathbb{Q})$. For π_∞ unramified,*

$$|\Re(\mu_{j,\infty})| \leq \frac{1}{2} - \frac{1}{\frac{n(n+1)}{2}+1},$$

and for $p < \infty$ at which π_p is unramified,

$$|\log_p |\alpha_{j,p}|| \leq \frac{1}{2} - \frac{1}{\frac{n(n+1)}{2}+1}.$$

Applying this to GL_2 over the rational number field gives

$$|\Re(\mu_{j,\infty})| \leq \frac{7}{64}, \quad j = 1, 2$$

when π_∞ is unramified. If $p < \infty$ and π_p is unramified, we have

$$|\log_p |\alpha_{j,p}|| \leq \frac{7}{64}, \quad j = 1, 2.$$

For the Selberg eigenvalue conjecture, this translates as

$$\lambda_1 \geq \frac{975}{4096} = .238...$$

For the general number field, one has the weaker bound of $1/2 - 1/(n^2 + 1)$ (see [5]).

References

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