7. REFINED ESTIMATES FOR FOURIER COEFFICIENTS OF CUSP FORMS

1. Sieve Theory and Kloosterman Sums

Last time, we indicated how Kloosterman sums are connected with Selberg's eigenvalue conjecture. This connection has profound implications to questions of classical analytic number theory. Much of the work of Iwaniec reflects this theme.

Kloosterman sums first arose in connection with the circle method. That they are more ubiquitous than first thought is brought out by a fundamental paper of Atkinson on the fourth power moment of the Riemann zeta function. Below, I want to illustrate how Kloosterman sums enter sieve theory and relate the study to the Brun-Titchmarsh theorem. Our discussion will be brief.

The basic set-up of the sieve is as follows. We are given a set $\mathcal{A}$ together with a set conditions indexed by prime numbers $p \in \mathcal{P}$. For each $p \in \mathcal{P}$, we denote by $\mathcal{A}_p$ to be the set of elements of $\mathcal{A}$ which satisfies the conditions indicated by p. The sieve problem is to estimate the size of

$$\mathcal{S}(\mathcal{A}, \mathcal{P}) := \mathcal{A} \backslash \cup_{p \in \mathcal{P}} \mathcal{A}_p.$$

If for every squarefree number d composed of primes $p \in \mathcal{P}$, we define

$$A_d = \cap_{p|d} A_p,$$

then, the usual inclusion-exclusion process gives

$$\mathcal{S}(\mathcal{A}, \mathcal{P}) = \sum_d \mu(d) |\mathcal{A}_d|$$

when the set $\mathcal{A}$ is finite. This is sometimes referred to as the Sieve of Eratosthenes.

One of the significant applications of sieve theory is to the estimation of the number of primes in a given arithmetic progression. The best result in this direction is given by Montgomery and Vaughan [4] where they show that the number of primes $p \leq x$ and $p \equiv a (\operatorname{mod} c)$ is

$$\leq \frac{2x}{\phi(c) \log(x/c)}, \qquad c < x.$$

It is an observation due to Chowla that any improvement in the constant 2 above will imply that there are no "Siegel" zeros. As indicated above, this entails precise estimation of

$$\#\{n \leq x : n \equiv a (\operatorname{mod} c),\ d|n\}.$$

Writing $n = dt$, this means $t \equiv \overline{d} a (\operatorname{mod} c)$. Now the number of natural numbers $t \leq x$ in a fixed residue class $v (\operatorname{mod} c)$ is easily seen to be

$$\left[\frac{x - v}{c}\right] - \left[\frac{-v}{c}\right].$$

Usually, one writes this as

$$\frac{x}{c} + O(1)$$

This is the text of Lecture 7 by Ram Murty given on March 20, 2003 at the Fields Institute, Toronto, Canada.

and the error term is too crude for many applications. Hooley [3] had the idea that one can Fourier analyse the error term in the following way. Write

$$\psi(x) = x - [x] - 1/2$$

and observe that it has a Fourier series (for x non-integral):

$$\psi(x) = \sum_{h=1}^{\infty} \frac{\sin 2\pi hx}{\pi h}$$

which can be rewritten as a finite sum

$$\sum_{1 \le h \le N} \frac{\sin 2\pi ht}{\pi h} + O\left(\min\left(1, \frac{1}{N||x||}\right)\right).$$

Here $||x||$ denotes the distance from x to the integer nearest to x.

2. Gauss Sums and Hyper-Kloosterman Sums

The Gauss sum is defined by

$$g(\chi) = \sum_{a \bmod q} \chi(a) e^{2\pi i a/q}.$$

Notice that

$$\begin{aligned}
\sum_{\chi} \overline{\chi}(b)\chi^2(c)g(\chi)^r &= \sum_{\chi} \overline{\chi}(b)\chi^2(c) \sum_{a_1,\ldots,a_r} \chi(a_1)\cdots\chi(a_r) e^{\frac{2\pi i}{q}(a_1+\cdots+a_r)} \\
&= \sum_{a_1,\ldots,a_r} e^{\frac{2\pi i}{q}(a_1+\cdots+a_r)} \sum_{\chi} \overline{\chi}(b)\chi(c^2 a_1\cdots a_r) \\
&= \phi(q) \sum_{\substack{a_1,\ldots,a_r \\ a_1\ldots a_r \equiv c^{-2} b (\bmod q)}} e^{\frac{2\pi i}{q}(a_1+\cdots+a_r)}
\end{aligned}$$

which is a hyper-Kloosterman sum. Deligne [1], as a consequence of his work on the Weil conjectures has estimated this sum to be $O(q^{(r-1)/2})$. If we normalize our Gauss sums to have absolute value 1, we see that the quantity above is $O(q^{1/2})$. In the method to be discussed in the next section, we will see that hyper-Kloosterman sums enter in a natural way.

3. The Duke-Iwaniec Method

Duke and Iwaniec [2] introduced a general method of obtaining estimates for coefficients of Dirichlet series that satisfy appropriate functional equations. We now outline this method.

Let $A = \{a_n\}$ be a sequence of complex numbers. Suppose

$$A(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

converges absolutely for $\Re(s) > 1$. For any Dirichlet character χ mod q, let us set

$$A(s,\chi) = \sum_{n=1}^{\infty} a_n \chi(n) n^{-s}.$$

For technical reasons, we suppose q is prime, so that any non-trivial character mod q is primitive. We assume $A(s,\chi)$ can be analytically continued to an entire function and that it satisfies a functional equation

$$A(1-s,\chi) = \epsilon_\chi \Phi(s) A(s,\overline{\chi})$$

where

$$|\epsilon_\chi| = 1, \quad \Phi(s) \text{ is holomorphic for } \Re(s) > 1.$$

In practice,

$$\Phi(s) = \frac{\gamma(s)}{\gamma(1-s)}$$

where

$$\gamma(s) = (q/\pi)^{ds/2} \prod_{j=1}^{d} \Gamma(s/2+\mu_j)$$

with

$$\Re(\mu_j) \geq -1/2, \quad \epsilon_\chi = (g(\chi)/\sqrt{q})^d.$$

We will assume that there is a constant $c \geq 1$ such that

$$\Phi(s) \ll (q|s|)^{(2\sigma-1)c}$$

on $\Re(s) = \sigma > 1$ (the implied constant depending on σ). In practice, $2c = d$, as can be seen easily by Stirling's formula when $\gamma(s)$ is given as above. The sign in the functional equation is assumed to randomly distributed on the unit circle in the following sense. Namely, we suppose that

$$K_q(a) := \sum_\chi \overline{\chi}(a)\epsilon_\chi \ll q^{1/2}$$

for all characters χ mod q. (It is possible to weaken this condition and allow for some exceptional characters which are not included in the sum.)

Theorem 1. *If these conditons hold for a set of primes q of positive density, then*

$$a_n \ll n^{\frac{2c-1}{2c+1}+\epsilon}$$

for any $\epsilon > 0$.

Proof. Let f be a smooth, compactly supported function in $\mathbb{R}^+$. We shall study

$$A_f(q,\ell) = \sum_{n \equiv \ell (\bmod q)} a_n f(n)$$

and

$$A_f(\chi) = \sum_n a_n \chi(n) f(n)$$

so that

$$A_f(q,\ell) = \frac{1}{\phi(q)} \sum_{\chi (\bmod q)} \overline{\chi}(\ell) A_f(\chi).$$

If we define

$$F(s) = \int_0^\infty f(x) x^{-s} dx$$

then by Mellin inversion formula, we have

$$f(x) = \frac{1}{2\pi i} \int_{(2)} F(s) x^{-s} ds.$$

Thus,

$$A_f(\chi) = \sum_n a_n \chi(n) f(n) = \frac{1}{2\pi i} \int_{(2)} A(s,\chi) F(s) ds.$$

We move the line of integration to $\Re(s) = -1$ and apply the functional equation to get

$$A_f(\chi) = \epsilon_\chi A_g(\overline{\chi})$$

where

$$g(y) = \frac{1}{2\pi i} \int_{(2)} F(s)\Phi(s) y^{-s} ds.$$

Note that g depends on the parity of χ but not on χ otherwise. We will reflect this dependence by writing g_+ and g_- in place of g with self-evident notation. Accordingly, we will split the sum $K_q(a)$ into even and odd characters, getting

$$S_\pm(a) = \frac{1}{2}\left(K_q(a) \pm K_q(-a)\right).$$

We remove the contribution from the trivial character so that

$$A_f(q,\ell) - \frac{1}{\phi(q)} A_f(\chi_0) = \frac{1}{\phi(q)} \sum_{m,\pm} a_m g_\pm(m) S_\pm(\ell m).$$

Thus, this quantity is

$$\ll q^{-1/2} \sum_m |a_m g(m)|.$$

It remains to estimate $g(m)$. To this end, let us assume that the Mellin integral is bounded by

$$|F(s)| \ll |s|^{-r}$$

for some $r > c+1$. Then,

$$g(m) \ll m^{-1-\epsilon} q^{c(1+2\epsilon)}.$$

This gives a final estimate of $O(q^{c-1/2+\epsilon})$.

The above condition on $F(s)$ is easily satisfied with any $r \geq 0$ for the function of type $f(x) = \omega(x/\ell)$ where $\omega(t)$ is a smooth function supported in the interval $[1/2, 2]$ and $q \geq 1$. To see this, we need only integrate by parts the equation defining $F(s)$. We apply the calculation to this test function and obtain

$$\sum_{n \equiv \ell (\operatorname{mod} q)} a_n \omega(n/\ell) \ll \frac{1}{q} \sum_{n < 2\ell} |a_n| + p^{c-1/2+\epsilon}.$$

This holds for primes q in a set of positive density. So we sum it over such primes in an interval of the form $[P, 2P]$. (If q belongs to such an interval, we sometimes denote it by $q \sim P$.) On the left hand side the term $a_\ell \omega(1)$ occurs with high multiplicity. For $n \neq \ell$, there are at most $O(\log \ell)$ prime divisors of $n - \ell$. Thus, summing over $q \sim P$, we get

$$P|a_\ell| \ll \sum_{n<2\ell} |a_n| (\log \ell)^2 + P^{c+1/2+\epsilon}.$$

As the Dirichlet series $A(s)$ converges absolutely for $\Re(s) > 1$, we have

$$\sum_{n \leq x} |a_n| \ll \sum_{n=1}^\infty |a_n| (x/n)^{1+\epsilon} \ll x^{1+\epsilon}.$$

Hence,

$$\sum_{n<2\ell} a_n \ll \ell^{1+\epsilon}$$

and we get

$$|a_\ell| \ll \left(\frac{\ell}{P} + P^{c-1/2}\right)\ell^{\epsilon}.$$

Now choose $P = \ell^{2/(2c+1)}$ to deduce the result.

□

References

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