5. POINCARÉ SERIES

1. Poincaré Series for $SL_2(\mathbb{Z})$

The Poincaré series for $SL_2(\mathbb{Z})$ are defined by

$$G_r(z) = \frac{1}{2} \sum_{(c,d)=1} (cz+d)^{-k} e^{2\pi i r \frac{az+b}{cz+d}}$$

where a, b are any integers such that $ad - bc = 1$. Observe that if $r = 0$, this reduces to the classical Eisenstein series $E_k(z)$ (upto a constant). Thus, the Poincaré series are to be viewed as generalisations of Eisenstein series. It is easy to see that the inner summand does not depend on the choice of a solution. Indeed, by the Euclidean algorithm, any other solution for (a, b) has the form $(a + tc, b + td)$ and

$$\frac{(a+tc)z + (b+td)}{cz+d} = \frac{az+b}{cz+d} + t, \quad t \in \mathbb{Z}$$

so that

$$e^{2\pi i r\left(\frac{az+b}{cz+d}+t\right)} = e^{2\pi i r\left(\frac{az+b}{cz+d}\right)}.$$

We can rewrite the series in a more invariant form by setting

$$j(\gamma, z) = cz + d, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and then

$$G_r(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} j(\gamma, z)^{-k} e^{2\pi i r(\gamma z)}.$$

The important thing to note is that $G_r(z)$ is a modular form of weight k. To see this, let $\delta \in \Gamma = SL_2(\mathbb{Z})$. Then,

$$G_r(\delta z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} j(\gamma, \delta z)^{-k} e^{2\pi r(\gamma \delta z)}.$$

Now, we have the so-called cocycle relation:

$$j(\gamma\delta, z) = j(\gamma, \delta z) j(\delta, z)$$

as is easily verified, so that

$$j(\gamma, \delta z) = \frac{j(\gamma\delta, z)}{j(\delta, z)}$$

and

$$\begin{aligned} G_r(\delta z) &= j(\delta, z)^k \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} j(\gamma\delta, z)^{-k} e^{2\pi i r(\gamma\delta z)} \\ &= j(\delta, z)^k G_r(z). \end{aligned}$$

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Holomorphy is easy to verify using standard tests of complex analysis. In addition, $G_r(i\infty) = 0$ if $r \geq 1$. We conclude that for every $r \geq 1$, $G_r(z)$ is a cusp form of weight k. Thus, Poincaré series give explicit constructions of cusp forms. For a detailed treatment of this theory, we refer the reader to Rankin's book [4] (especially Chapter 5).

Now let f be any cusp form of weight k. We would like to compute the inner product (f, G_r).

First observe that $e^{2\pi iz} = e^{2\pi ix} \cdot e^{-2\pi y}$ so that

$$\overline{e^{2\pi iz}} = e^{-2\pi ix} \cdot e^{-2\pi y} = e^{-2\pi i(\overline{z})}.$$

Thus,

$$\begin{aligned}
(f, G_r) &= \int_{\Gamma\backslash H} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} f(z) e^{-2\pi ir\overline{\gamma z}} (\overline{cz+d})^{-k} y^k \frac{dxdy}{y^2} \\
&= \int_{\Gamma\backslash H} \sum_{\Gamma_\infty\backslash\Gamma} (cz+d)^k f(z) e^{-2\pi ir(\overline{\gamma z})} \frac{y^k}{|cz+d|^{2k}} \frac{dxdy}{y^2} \\
&= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \int_{\Gamma\backslash H} f(\gamma z) \Im(\gamma z)^k e^{-2\pi ir(\overline{\gamma z})} \frac{dxdy}{y^2} \\
&= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \int_{\gamma(\Gamma\backslash H)} f(z) \Im(z)^k e^{-2\pi ir\overline{z}} \frac{dxdy}{y^2} \\
&= \int_0^\infty \int_0^1 f(x+iy) y^k e^{-2\pi irx} e^{-2\pi ry} \frac{dxdy}{y^2}.
\end{aligned}$$

Now from the Fourier expansion of $f(z)$:

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi inx} \cdot e^{-2\pi ny}$$

we see that the x-integral picks up the r-th Fourier coefficient. Thus,

$$(f, G_r) = a_r \int_0^\infty e^{-4\pi ry} y^{k-2} dy.$$

By setting $4\pi ry = t$ in the integrand and simplifying, we deduce

Theorem 1. *Let f be any cusp form of weight k for Γ. Then,*

$$(f, G_r) = \frac{\Gamma(k-1)a_r}{(4\pi r)^{k-1}}.$$

An important corollary is:

Corollary 2. *Every cusp form is a finite linear combination of Poincaré series $G_r(z)$.*

Proof. The set of Poincaré series spans a closed subspace in the space of cusp forms. If f is a cusp form not in this space, all of its Fourier coefficients must vanish by the previous theorem. Thus, the orthogonal complement is zero. □

As an example, consider the case $k = 12$. Each of the $G_r(z)$ is a cusp form of weight 12. But any cusp form of weight 12 must be a constant multiple of Δ, Ramanujan's cusp form. Thus,

$$G_r(z) = c_r \Delta(z).$$

What is c_r? By the above theorem,

$$(\Delta, G_r) = \frac{\Gamma(11)\tau(r)}{(4\pi r)^{11}} = c_r(\Delta, \Delta).$$

Hence,

$$\tau(r) = \frac{(4\pi r)^{11}}{10!} \int_{\Gamma\backslash H} y^{12}\Delta(z)\overline{G_r(z)}\frac{dxdy}{y^2}.$$

2. Fourier coefficients and Kloosterman sums

Emanating largely from the work of Petersson [3]in the 1930's and Selberg [5], an explicit formula can be given for the Fourier coefficients of $G_r(z)$. This striking formula involves the Kloosterman sums and their appearance has opened new connections to the Selberg eigenvalue conjecture as well as applications to classical questions of analytic number theory. We now derive this remarkable formula.

We begin by writing

$$G_r(z) = \sum_{n=1}^{\infty} g_{rn} e^{2\pi i n z}.$$

Then,

$$g_{rn} = \int_0^1 G_r(x) e^{-2\pi i n x}.$$

More precisely, for reasons of convergence, we should consider

$$\int_{i\delta}^{1+i\delta} G_r(z) e^{-2\pi i n z}$$

with $\delta > 0$, but we leave this technical modification to the reader. We have

$$g_{rn} = \frac{1}{2}\sum_{(c,d)=1} \int_0^1 (cx+d)^{-k} e^{2\pi i r\left(\frac{ax+b}{cx+d}\right) - 2\pi i n x} dx.$$

Put $cx + d = t$. The argument in the exponential becomes

$$\frac{r}{t}\left(\frac{a}{c}(t-d) + b\right) - \frac{n}{c}(t-d) = \frac{nd+ar}{c} - \frac{nt}{c} - \frac{r}{tc}$$

since $ad - bc = 1$. Thus,

$$g_{rn} = \frac{1}{2}\sum_{c\neq 0}\frac{1}{c} \sum_{\substack{d \pmod c \\ ad\equiv 1 \pmod c}} e^{\frac{2\pi i}{c}(nd+ar)} \int_{-\infty}^{\infty} t^{-k} e^{-\frac{2\pi i}{c}\left(\frac{r}{t}+nt\right)} dt$$

because

$$\sum_{(c,d)=1} \int_0^1 t^{-k} e^{\frac{2\pi i}{c}(nd+ar-nt)} e^{-\frac{2\pi i r}{t}} dt$$

depends only on $d \pmod c$. Writing d as $d_0 + (m+1)c$ with varying m, we transform the integral from 0 to 1 into an integral from $-\infty$ to ∞. This integral turns out to be a Bessel function:

$$\int_{-\infty+ci}^{\infty+ci} t^{-k} \exp\left(-\frac{2\pi i}{c}\left(\frac{r}{t}+nt\right)\right) dt = 2\pi(n/r)^{(k-1)/2} J_{k-1}(4\pi\sqrt{rn}/c)$$

where

$$J_k(z) = \frac{1}{2\pi i}\int_C t^{-k-1} e^{\frac{z}{2}(t-1/t)} dt$$

where C is the unit circle. The sum

$$S(r,n,c) := \sum_{\substack{d \pmod c \\ ad\equiv 1 \pmod c}} e^{\frac{2\pi i}{c}(nd+ar)}$$

is called a Kloosterman sum. Using this notation, we obtain the beautiful formula due to Petersson:

$$g_{rn} = (n/r)^{(k-1)/2}\left\{\delta_{rn} + \pi\sum_{c=1}^{\infty}\frac{S(r,n,c)}{c}J_{k-1}\left(\frac{4\pi\sqrt{rn}}{c}\right)\right\}$$

where δ_{rn} denotes the Kronecker delta function.

We have already noted that the Poincaré series span the space of cusp forms. Thus, to prove the Ramanujan conjecture, it suffices to show that

$$g_{rn} = O(n^{\frac{k-1}{2}+\epsilon})$$

for every r. This is tantamount to showing that the expression in parentheses in the above sum is $O(n^{\epsilon})$.

Selberg, using this expression and Weil's estimate for Kloosterman sums:

$$|S(r,n,c)| \le d(c)c^{1/2}(r,n,c)^{1/2}$$

as well as the bound

$$J_{k-1}(x) \le A\min(x^{k-1}, x^{-1/2})$$

obtained that

$$g_{rn} = O(n^{k/2-1/4+\epsilon}).$$

Note that this is better than what we obtained earlier by the Rankin-Selberg method. Since the estimate was obtained crudely, Selberg felt that there must be cancellation among the Kloosterman sums. This led him to formulate the following conjecture (which was also arrived at independently by Linnik):

Conjecture. *(Selberg-Linnik)*

$$G(x) := \sum_{c\le x}\frac{S(r,n,c)}{c} = O(x^{\epsilon})$$

for $x \ge \gcd(r,n)^{1/2+\epsilon}$ *for any* $\epsilon > 0$.

In his 1965 paper, Selberg stated that this would lead to a proof of the Ramanujan conjecture (for Maass forms as well) but did not indicate a proof. We will indicate below how such a proof can be obtained for the full modular group. The argument is adapted from the author's [2].

Let us first observe that Weil's estimate for Kloosterman sums leads to the estimate

$$G(x) = O(x^{1/2}\log x)$$

for $(r,n) = 1$. Kuznetsov[1] proved that $G(x) = O(x^{1/6+\epsilon})$, but the O-constant depends on r, n so it is not applicable to the estimation of the Fourier coefficients. Let

$$H(x) := \sum_{c\le x} S(r,n,c).$$

By partial summation, the Selberg-Linnik conjecture is equivalent to

$$H(x) = O(x^{1+\epsilon}).$$

We begin by considering

$$\sum_{c>\sqrt{n}} \frac{S(r,n,c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{rn}}{c}\right) = \sum_{c>\sqrt{n}} G(c)\left\{J_{k-1}\left(\frac{4\pi\sqrt{rn}}{c+1}\right) - J_{k-1}\left(\frac{4\pi\sqrt{rn}}{c}\right)\right\}$$

by partial summation. By the mean value theorem, the expression in parentheses is

$$\frac{4\pi\sqrt{rn}}{c(c+1)} J'_{k-1}(\xi_c)$$

for some $\xi_c \in (4\pi\sqrt{rn}/(c+1), 4\pi\sqrt{rn}/c)$. Using the estimate

$$J'_{k-1}(x) \ll x^{-1/2}$$

we get

$$\sum_{c>\sqrt{rn}} \frac{S(r,n,c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{rn}}{c}\right) \ll n^{1/4} \sum_{c>\sqrt{n}} \frac{|G(c)|}{c^{3/2}} \ll n^{\epsilon},$$

by the Selberg-Linnik conjecture. Thus, we need only consider

$$\sum_{c\le\sqrt{n}} \frac{S(r,n,c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{rn}}{c}\right).$$

To estimate this, we apply an inductive argument. As there are no cusp forms of weight 10, we have

$$\sum_{c\le\sqrt{n}} \frac{S(r,n,c)}{c} J_9\left(\frac{4\pi\sqrt{rn}}{c}\right) = O(n^{\epsilon}).$$

So, if for example, we were trying to establish the conjecture for $k = 12$, then it suffices to estimate for $k = 10$ the quantity

$$\sum_{c\le\sqrt{n}} \frac{S(r,n,c)}{c} \left\{J_{k+1}\left(\frac{4\pi\sqrt{rn}}{c}\right) + J_{k-1}\left(\frac{4\pi\sqrt{rn}}{c}\right)\right\}.$$

By the familiar identity

$$\frac{2kJ_k(x)}{x} = J_{k+1}(x) + J_{k-1}(x)$$

it suffices to estimate

$$\frac{1}{\sqrt{n}} \sum_{c\le\sqrt{n}} S(r,n,c) J_k\left(\frac{4\pi\sqrt{rn}}{c}\right).$$

Again, by partial summation, we may write this as

$$\frac{1}{\sqrt{n}} \sum_{c\le\sqrt{n}} H(c)\left\{J_k\left(\frac{4\pi\sqrt{rn}}{c+1}\right) - J_k\left(\frac{4\pi\sqrt{rn}}{c}\right)\right\}.$$

Again, the expression in the brackets is

$$\frac{4\pi\sqrt{rn}}{c(c+1)} J'_k(\xi_c).$$

Using the estimate

$$J'_k(x) \ll x^{-1/2}$$

as before, and the fact that $H(c) = O(c^{1+\epsilon})$, we deduce a final estimate of $O(n^\epsilon)$ as desired. This completes the proof of the fact that the Selberg-Linnik conjecture implies the Ramanujan conjecture (for the full modular group). A similar argument can be applied to higher levels. However, the non-existence of cusp forms of small weight is not guaranteed. In this case, we exploit the fact that we know the Ramanujan conjecture in the weight two case (a result due to Eichler and Shimura).

3. The Kloosterman-Selberg Zeta Function

In order to gain more insight into the Selberg-Linnik conjecture, we will consider (with Selberg[6]) the series

$$Z(r,n,s) = \sum_{c \neq 0} \frac{S(r,n,c)}{|c|^{2s}}.$$

To study this series, Selberg [6] considers the cognate Poincaré series

$$U_n(z,s) = \sum_{\Gamma_\infty \backslash \Gamma} \Im(\gamma z)^s e^{2\pi i n \gamma z}.$$

Clearly, U_n is Γ-invariant. Moreover, if

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

then

$$\Delta U_n(z,s) = s(1-s)U_n(z,s) + 4\pi n U_n(z,s+1)s.$$

As we will show next time, the Fourier expansion of $U_n(z,s)$ contains $Z(r,n,s)$ in it. This allows us to relate the eigenvalues of Δ with the abscissa of convergence of $Z(r,n,s)$. More precisely,

$$U_n(z,s) = \sum_{m=-\infty}^{\infty} B_n(m,y,s) e^{2\pi i m x}$$

where

$$B_n(m,y,s) = \delta_{nm} y^s e^{-2\pi n y} + \frac{1}{2} \sum_{c \neq 0} \frac{S(m,n,c)}{|c|^{2s}} y^{1-s} \int_{-\infty}^{\infty} \exp\left(-2\pi i m y v - \frac{2\pi n}{c^2 y (1 - iv)} \right) \frac{dv}{(1+v^2)^s}.$$

It then turns out that

$$(2\pi\sqrt{nm})^{2s-1} \sum_{c=1}^{\infty} \frac{S(n,m,c)}{|c|^{2s}} = \frac{\sin \pi s}{2} \sum_{j=1}^{\infty} \frac{a_j(n)\overline{a_j(m)}}{\cosh \pi r_j} \Gamma(s - \frac{1}{2} + i r_j) \Gamma(s - \frac{1}{2} - i r_j)$$

$$-\frac{\delta_{nm}}{2\pi} \frac{\Gamma(s)}{\Gamma(1-s)} + \frac{1}{\pi} \int_{-\infty}^{\infty} (n/m)^{ir} \sigma_{2ir}(m) \sigma_{-2ir}(m) \frac{h(r,s)}{|\zeta(1+2ir)|^2} dr$$

where

$$h(r,s) = \frac{\sin \pi s}{2} \Gamma(s - \frac{1}{2} + ir) \Gamma(s - \frac{1}{2} - ir)$$

and the $a_j(n)$'s are the Fourier coefficients of the Maass form corresponding to the eigenvalue $\lambda_j = 1/4 + r_j^2$. This remarkable formula establishes a striking relationship between the eigenvalues λ_j and the Kloosterman-Selberg zeta function.

References

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