4. OSCILLATIONS OF FOURIER COEFFICIENTS OF CUSP FORMS

1. Preliminaries

Last time, we discussed the Rankin-Selberg L-function on $GL_n \times GL_m$ over a number field. This is one of the most powerful methods in the theory that enables us to deduce the meromorphic continuation of the symmetric power L-functions. The general strategy has been first to derive a meromorphic continuation, then to establish holomorphy everywhere and finally by some form of converse theory (again involving some application of the Rankin-Selberg method or the Langlands-Shahidi method) to establish the automorphy of the desired L-function. In this way, one hopes to inductively deduce the holomorphy of the symmetric power L-functions. This strategy has worked so far for only small dimensions and is perhaps illustrated as follows.

Following conventional notation, we shall now denote by $L(s, \pi, r_m)$ the symmetric power L-function attached to a cuspidal automorphic representation π of $GL_2(\mathbb{A}_F)$ which we had previously denoted by $L_m(s)$. As mentioned earlier, $L(s, \pi \times \pi)$ decomposes as

$$\zeta_F(s) L(s, \pi, r_2)$$

where $\zeta_F(s)$ is the Dedekind zeta function of F. This already gives a meromorphic continuation of $L(s, \pi, r_2)$. When $F = \mathbb{Q}$ and π corresponds to a holomorphic modular form, Shimura [11] had established the holomorphy of $L(s, \pi, r_2)$ by extending the Rankin-Selberg method and making ingenious use of the classical theta function. Gelbart and Jacquet [1] extended this work to all cuspidal automorphic representations of $GL_2(\mathbb{A}_F)$ and in addition proved the existence of a cuspidal automorphic representation Π of $GL_3(\mathbb{A}_F)$ such that

$$L(s, \Pi) = L(s, \pi, r_2).$$

This Π is often called the Gelbart-Jacquet lift of π.

But now, we can apply the Rankin-Selberg method to Π. We find,

$$L(s, \Pi \times \Pi) = \zeta_F(s) L(s, \pi, r_2) L(s, \pi, r_4)$$

and thus, we immediately deduce the meromorphic continuation of the 4-th symmetric power L-function.

We could also consider

$$L(s, \pi \times \Pi) = L(s, \pi, r_1) L(s, \pi, r_3)$$

and this gives us the meromorphy of $L(s, \pi, r_3)$. One expects that for each $L(s, \pi, r_m)$ there exists a cuspidal automorphic representation Π_m on $GL_{m+1}(\mathbb{A}_F)$ such that

$$L(s, \Pi_m) = L(s, \pi, r_m).$$

Now by the Clebsch-Gordon formulas for $SU(2, \mathbb{C})$, we have

$$r_m \otimes r_n = r_{m+n} \oplus r_{m+n-2} \oplus \cdots \oplus r_{|m-n|}$$

which is essentially the trigonometric identity

$$\left(\frac{\sin(m+1)\theta}{\sin\theta}\right)\left(\frac{\sin(n+1)\theta}{\sin\theta}\right) = \sum_{j=0}^{n} \frac{\sin(m+n-2j)\theta}{\sin\theta}$$

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for $n \leq m$. We leave this as an exercise for the reader.

The essential point is that we may use the Rankin-Selberg method to inductively deduce the meromorphic continuation of the symmetric power L-functions once we have shown the automorphy property. Since we know that each of the symmetric cube and fourth power L-functions are automorphic by the work of Kim-Shahidi [4] and Kim [3], we can inductively obtain the meromorphic continuation of the $L_m(s)$ for $m \leq 8$. Finer analysis of the location of the poles leads to the holomorphy of the $L_m(s)$ for $m \leq 8$ (which uses the Langlands-Shahidi method of Eisenstein series).

2. Rankin's Theorem

The discussion below applies equally well to Maass forms. However, for the sake of clarity, we will specialise to the case of classical Hecke eigenforms.

Given a normalized Hecke eigenform of weight k, we let $a(n)$ be the n-th Fourier coefficient and write

$$a(p) = 2p^{(k-1)/2} \cos \theta_p.$$

In general, we have

$$a(p^a) = (p^a)^{(k-1)/2} \frac{\sin(a+1)\theta_p}{\sin \theta_p}$$

as can be easily checked from the recursion for the Hecke operators. Thus, for example, if $a = 1$, we retrieve our formula for $a(p)$ above. Now

$$\frac{\sin(a+1)\theta}{\sin \theta} = \frac{e^{i(a+1)\theta} - e^{-i(a+1)\theta}}{e^{i\theta} - e^{-i\theta}} = \frac{x^{a+1} - y^{a+1}}{x - y}$$

with $x = e^{i\theta}, y = e^{-i\theta}$, so we see from

$$\frac{x^{a+1} - y^{a+1}}{x - y} = x^a + x^{a-1}y + \cdots + y^a$$

that

$$\left| \frac{\sin(a+1)\theta}{\sin \theta} \right| \leq a + 1.$$

Therefore,

$$|a(n)| \leq n^{(k-1)/2} d(n)$$

where $d(n)$ is the number of divisors of n. The maximal order of $d(n)$ is easily determined (see for example, [2]) and we have

$$a(n) = O\left(n^{(k-1)/2} \exp\left(\frac{c \log n}{\log \log n} \right) \right)$$

for some suitable constant $c > 0$.

In 1973, Rankin [7] investigated if this is the optimal error term. He proved that

$$\limsup_{n \to \infty} \frac{|a(n)|}{n^{(k-1)/2}} = +\infty.$$

In the same paper, Rankin indicated that if the Sato-Tate conjecture is true, then

$$a(n) = \Omega_{\pm} \left(n^{(k-1)/2} \exp\left(\frac{c \log n}{\log \log n} \right) \right)$$

for some $c > 0$. By the Sato-Tate conjecture, we have

$$\#\{p \leq x : \theta_p \in [-\pi/6, \pi/6]\} \geq c\pi(x).$$

Put

$$N = \prod_{\substack{p \le x \\ \theta_p \in [-\pi/6, \pi/6]}} p.$$

Then

$$a(N) = \prod_{\substack{p \le x \\ \theta_p \in [-\pi/6, \pi/6]}} a(p).$$

Because

$$|a(p)| = 2p^{(k-1)/2} |\cos\theta_p| \ge \sqrt{3} p^{(k-1)/2}$$

we obtain

$$|a(N)| \ge N^{(k-1)/2} (\sqrt{3})^{c\pi(x)}.$$

Now, by partial summation

$$\log N = \sum_{\substack{p \le x \\ \theta_p \in [-\pi/6, \pi/6]}} \log p \ge c_1 x.$$

Also, by Chebycheff's estimate

$$\log N \le c_2 x.$$

In any case

$$|a(N)| \ge N^{(k-1)/2} (\sqrt{3})^{c_0 \log N / \log\log N}$$

and the omega theorem is deduced from this.

But since we don't have the Sato-Tate conjecture in its entirety until all the symmetric power L-functions are shown to be entire, (no pun intended) it makes sense to ask how much of the Sato-Tate conjecture can be proved if we only had analyticity of $L_m(s)$ for $m \le R$ (say). For instance, can we aim for Chebycheff type estimates for the Sato-Tate problem based on only partial information. The goal of this lecture is to indicate how we may deduce the following.

Theorem 1. *Suppose that $L_r(s)$ extends to an analytic function for $\Re(s) \ge 1/2$ for all $r \le 2m+2$. Then, each of the statements*

(1) *for any $\delta > 0$, $-\delta < 2\cos\theta_p < \frac{2}{\delta(m+2)}$;*
(2) *for any $\epsilon > 0$,*
$$|2\cos\theta_p| > \sqrt{\frac{4m+2}{m+2}} - \epsilon;$$
(3) *for any $\epsilon > 0$, $2\cos\theta_p > \beta_m - \epsilon$ where*
$$\beta_m = \left\{ \frac{1}{4(m+2)} \binom{2m+2}{m+1} \right\}^{\frac{1}{2m+1}};$$

holds for a positive density of primes.

Corollary 2. *Setting $\delta = \sqrt{2/(m+2)}$ in (1), we deduce that there is a positive density of primes p satisfying*

$$-\sqrt{\frac{2}{m+2}} < 2\cos\theta_p < \sqrt{\frac{2}{m+2}}.$$

Putting $m = 1$ in (2), we deduce

Corollary 3. *For a positive density of primes p, we have*

$$|a(p)| \ge (\sqrt{2} - \epsilon) p^{(k-1)/2}.$$

This last result is what we need to obtain Rankin's oscillation theorem without the Sato-Tate conjecture. For this, we need the analyticity for $\Re(s) \geq 1/2$ for each of $L_r(s)$, $r \leq 4$.

In 1981, Shahidi [9] proved that $L_3(s)$ and $L_4(s)$ are analytic in this region. Recently Kim and Shahidi [4] established that these are in fact automorphic L-functions and hence entire. This is more information than we need and it is quite possible that this can be used to refine our results.

3. A review of Symmetric power L-functions

Let us look at

$$L_r(s) = \prod_p \prod_{j=0}^{r} \left(1 - \frac{\alpha_p^{r-j}\beta_p^j}{p^s}\right)^{-1}.$$

Now $\alpha_p = e^{i\theta_p}, \beta_p = e^{-i\theta_p}$ so that the Euler factor is

$$\prod_{j=0}^{r} \left(1 - \frac{e^{i(r-2j)\theta_p}}{p^s}\right)^{-1}.$$

If $L_r(s)$ is analytic for $\Re(s) \geq 1$, and non-vanishing there, we may apply the Tauberian theorem to deduce

$$\sum_{p \leq x} \left(\sum_{j=0}^{p} e^{i(r-2j)\theta_p}\right) = o(x/\log x).$$

By an easy exercise,

$$\frac{\sin(r+1)\theta}{\sin\theta} = \sum_{j=0}^{r} e^{i(r-2j)\theta}$$

so that for $r \geq 1$,

$$\sum_{p \leq x} \frac{\sin(r+1)\theta_p}{\sin\theta_p} = o(x/\log x).$$

Let

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta},$$

and $T_n(\cos\theta) = \cos n\theta$ be the Chebycheff polynomials of the first and second kind respectively.

We have the identity

$$2T_n(x) = U_n(x) - U_{n-2}(x).$$

Thus, for $n = 2$,

$$2\sum_{p \leq x} T_2(\cos\theta_p) = \sum_{p \leq x} U_2(\cos\theta_p) - \pi(x)$$

which implies

$$\sum_{p \leq x} T_2(\cos\theta_p) = (-\frac{1}{2} + o(1))\pi(x).$$

We also have

$$\sum_{p \leq x} T_1(\cos\theta_p) = o(x/\log x).$$

Now we can write powers of the cosine function using Chebycheff polynomials of the second kind:

$$(2\cos\theta)^r = 2\sum_{k=0}^{r'} \binom{r}{k} T_{r-2k}(\cos\theta), \quad r' = [r/2].$$

From this identity, we deduce

$$\sum_{p\le x}(2\cos\theta_p)^r = 2\sum_{k=0}^{r'} \binom{r}{k} \sum_{p\le x} T_{r-2k}(\cos\theta_p).$$

Each of the inner sums is $o(x/\log x)$ unless $r-2k=0$ or 2 in which case it is $\pi(x)$ or $-\pi(x)/2$ respectively. Hence, if r is not even, the sum is $o(x/\log x)$. The final result is

$$\sum_{p\le x}(2\cos\theta_p)^{2r} = \left(-\binom{2r}{r-1} + \binom{2r}{r}\right)(1+o(1))\frac{x}{\log x}.$$

The term inside the brackets is

$$\frac{1}{r+1}\binom{2r}{r}.$$

We can state the final result as:

Theorem 4.

$$\sum_{p\le x}(2\cos\theta_p)^{2r} = \frac{1}{r+1}\binom{2r}{r}(1+o(1))\frac{x}{\log x}$$

as x tends to infinity.

For example, when $r=1$, we get

$$\sum_{p\le x}(2\cos\theta_p)^2 = (1+o(1))\frac{x}{\log x}$$

and for $r=2$,

$$\sum_{p\le x}(2\cos\theta_p)^4 = (2+o(1))\frac{x}{\log x}.$$

This last result immediately gives that for a positive proportion of primes, we have

$$|2\cos\theta_p| \ge 2^{1/4} - \epsilon$$

which is greater than 1. From this result, we may deduce the Ω-result stated earlier. See [6] for further details.

4. Proof of Theorem 1

To prove Theorem 1, we need to obtain finer information about sign changes and so forth, and a slightly subtler analysis is needed.

We will need the following combinatorial identities.

Lemma 5.

$$\sum_{j=0}^{r}(-1)^j\binom{r}{j}\binom{2j}{j}\frac{2^{-2j}}{j+1} = 2^{-2r-1}\binom{2r+2}{r+1} \tag{1}$$

$$\sum_{j=0}^{r}(-1)^j\binom{r}{j}\binom{2j+2}{j+1}\frac{2^{-2j}}{j+2} = \frac{2^{-2r}}{r+2}\binom{2r+2}{r+1}. \tag{2}$$

Now consider the polynomial

$$P_m(x) = (x^2 - 4)^m (x - a)(x - b)$$

where a and b will be chosen later. Now

$$P_m(x) = (x^2 - (a+b)x + ab) \sum_{j=0}^{m} \binom{m}{j} x^{2j} (-1)^{m-j} 4^{m-j}$$

so that (after some calculation) we find

$$\frac{\log x}{x} \sum_{p \le x} P_m(2\cos\theta_p) \sim \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} 4^{m-j} \left(\frac{1}{j+2} \binom{2j+2}{j+1} + \frac{ab}{j+1} \binom{2j}{j} \right)$$

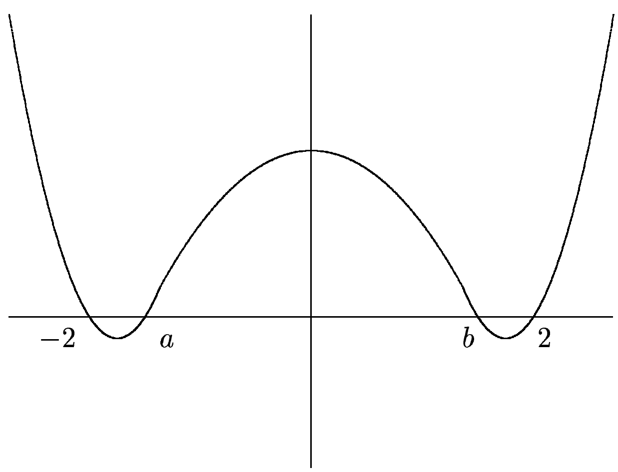
which by the lemma is

$$\sim (-1)^m \binom{2m+2}{m+1} \left(\frac{ab}{2} + \frac{1}{m+2} \right).$$

We conclude that

$$\sum_{p \le x} P_m(2\cos\theta_p) \sim (-1)^m \binom{2m+2}{m+1} \left(\frac{ab}{2} + \frac{1}{m+2} \right) \frac{x}{\log x}.$$

Now, we examine the graph of $P_m(x)$.



The graph of $P_m(x)$

Choosing $a = -\delta$, b so that $ab > -2/(m+2)$ if m is even and $ab < -2/(m+2)$ if m is odd, we deduce that

$$\sum_{p \le x} P_m(2\cos\theta_p) \gg \frac{x}{\log x}.$$

This means that a positive proportion of primes will have

$$a < 2\cos\theta_p < b$$

so we get

$$-\delta < 2\cos\theta_p < \frac{2}{\delta(m+2)}$$

as stated in Theorem 1.

The remaining part (2) of the Theorem are obtained by using the polynomial

$$Q_m(x) = x^{2m}(x^2 - \gamma)$$

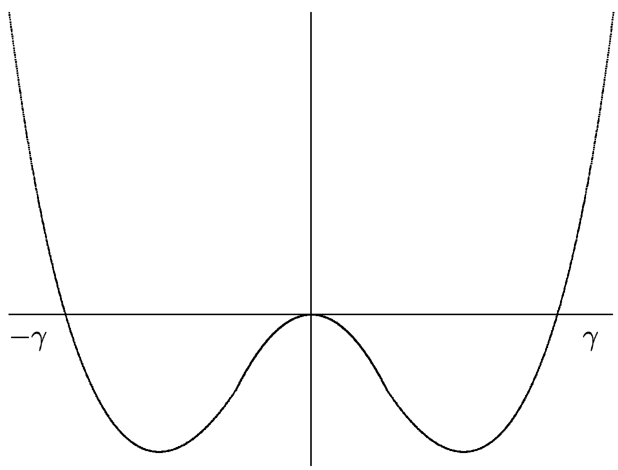
where γ is to be chosen. As before, we deduce

$$\sum_{p\le x} Q_m(2\cos\theta_p) \sim \left(\frac{1}{m+2}\binom{2m+2}{m+1} - \frac{\gamma}{m+1}\binom{2m}{m}\right)\frac{x}{\log x}$$

as x tends to infinity. Again, examining the graph of $Q_m(x)$, we see that if

$$\gamma = \frac{m+1}{m+2}\binom{2m+2}{m+1}\binom{2m}{m}^{-1} - \epsilon = \frac{2(2m+1)}{m+2} - \epsilon$$

we get (2).



The graph of $Q_m(x)$

Finally,

$$\sum_{p\le x} |2\cos\theta_p|^{2m+1} \ge \frac{1}{2}\sum_{p\le x}(2\cos\theta_p)^{2m+2}$$

which implies that

$$\sum_{p\le x,\, a_p>0} (2\cos\theta_p)^{2m+1} \gtrsim \frac{1}{4(m+2)}\binom{2m+2}{m+1}\frac{x}{\log x}.$$

Thus, for a positive proportion of primes

$$2\cos\theta_p > \left\{\frac{1}{4(m+2)}\binom{2m+2}{m+1}\right\}^{\frac{1}{2m+1}} - \epsilon.$$

By choosing better polynomials, Rankin [8], Serre and Shahidi [10] have obtained refined results. Most notable is Rankin's result[8] that for some $\delta > 0$, we have

$$\sum_{n \le x} |a_n/n^{(k-1)/2}| \ll \frac{x}{(\log x)^\delta}.$$

Here is a sketch of Rankin's argument. Let

$$b_n = a_n/n^{(k-1)/2}.$$

For each r, define the series

$$\psi_r(s) = \prod_p (1 - 2(\cos r\theta_p)p^{-s} + p^{-2s})^{-1}.$$

Then,

$$\zeta(s)\psi_2(s) = L_2(s)$$

and

$$\zeta(s)\psi_2(s)\psi_4(s) = L_4(s),$$

as can be easily checked by comparing the Euler factors of both sides.

By Gelbart-Jacquet [1] (for $L_2(s)$) and Shahidi [9] (for $L_4(s)$) we see that

$$\zeta(s)\psi_2(s)$$

and

$$\zeta(s)\psi_2(s)\psi_4(s)$$

are holomorphic and non-vanishing for $\Re(s) \ge 1$.

Rankin [8] shows that there are functions K, L, M in β satisfying

$$K - L = F(\beta) + 1$$

such that if

$$u^+(\theta) = K + 2L\cos 2\theta + 2M\cos 4\theta$$

then

$$|2\cos\theta|^{2\beta} \le u^+(\theta)$$

and

$$F(\beta) = \frac{2^{\beta-1}}{5}(2^\beta + 3^{2-\beta}) - 1.$$

We consider the Dirichlet series

$$\sum_{n=1}^\infty \frac{a^+(n)}{n^s} = \prod_p A_p^+(s)$$

where

$$A_p^+(s) = 1 + u^+(\theta_p)p^{-s} + \sum_{v=2}^\infty (v+1)^{2\beta} p^{-vs}$$

so that for all real values of s,

$$\sum_{n=1}^\infty \frac{|b_n|^{2\beta}}{n^s} \le \sum_{n=1}^\infty \frac{a^+(n)}{n^s} = \zeta(s)^K \psi_2(s)^L \psi_4(s)^M H_3(s)$$

where $H_3(s)$ is holomorphic and on-zero for $\Re(s) > 1/2$. By an extended version of the Tauberian theorem (due to Delange), we obtain

$$\sum_{n\leq x} a^+(n) \sim cx(\log x)^{K-L-1}$$

with $c \neq 0$, and if $K - L \leq 1$ (note that there is a typo in [10]).

We now use the fact that $K - L = F(\beta) + 1$ and for $\beta = 1/2$, $F(1/2) = \frac{\sqrt{2}}{5}(\sqrt{2} + 3\sqrt{3}) - 1 < 0$ as is easily checked. This completes the proof (sketch) of Rankin's theorem.

Rankin's theorem was used by Murty-Murty[5] in proving a crucial non-vanishing theorem which was an essential ingredient for Kolyvagin's theorem about finiteness of Tate-Shafarevich groups of modular elliptic curves with Mordell-Weil rank ≤ 1.

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