3. THE RANKIN-SELBERG METHOD

1. Eisenstein series and non-vanishing of $\zeta(s)$ on $\Re(s) = 1$

I want to indicate a proof of the non-vanishing of $\zeta(s)$ on $\Re(s) = 1$ which uses the theory of Eisenstein series and as a consequence does not use the Euler product of $\zeta(s)$ as most conventional proofs do. The idea was used by Jacquet and Shalika [4] in their general result about the non-vanishing on $\Re(s) = 1$ of automorphic L-functions associated with GL_n.

Recall that

$$E(z,s) = \pi^{-s}\Gamma(s)\frac{1}{2}\sum_{(m,n)\neq(0,0)}\frac{y^s}{|mz+n|^{2s}}.$$

Notice that we may also write this as

$$E(z,s) = \pi^{-s}\Gamma(s)\frac{1}{2}\zeta(2s)\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\Im(\gamma z)^s$$

where Γ_∞ is the stabilizer of the cusp at infinity.

We showed last time that

$$E(z,s) = \pi^{-s}\Gamma(s)\zeta(2s)y^s + \pi^{s-1}\Gamma(1-s)\zeta(2-2s)y^{1-s} + \sum_{r\neq 0}|r|^{s-1/2}\sigma_{1-2s}(|r|)\sqrt{y}K_{s-1/2}(2\pi|r|y)e^{2\pi i r x}$$

where $\sigma_v(n) = \sum_{d|n} d^v$ and

$$K_s(y) = \frac{1}{2}\int_0^\infty e^{-y(t+t^{-1})/2}t^s\frac{dt}{t}.$$

One can prove directly that $K_s(y) = K_{-s}(y)$ and $r^s\sigma_{-2s}(r) = r^s\sigma_{2s}(r)$ which allows us to deduce the functional equation of $E(z,s)$ from its Fourier expansion.

This result lies at the heart of the Langlands-Shahidi method of analytic continuation of Eisenstein series. It is also at the core of the Rankin-Selberg method of analytic continuation which we outline below.

Now suppose that $\zeta(1+it_0) = 0$ for some t_0 real. Then, $\zeta(1-it_0) = 0$ also. We put $s = (1+t_0)/2$ in $E(z,s)$. Then, the constant term vanishes and we get a Maass cusp form:

$$E(z,(1+it_0)/2) = 4\sqrt{y}\sum_{r=1}^\infty r^{it_0/2}\sigma_{-it_0}(r)\cos(2\pi r x)\int_0^\infty e^{-\pi r y(t+t^{-1})}\frac{dt}{t^{1-it_0/2}}.$$

Using standard estimates for the integral, one can show that the sum is $O(e^{-cy})$ for some $c > 0$. Hence the constant term of $E(z,(1+it_0)/2)$ is zero and we have a genuine Maass cusp form on our hands.

In particular,

$$\int_0^1 E(x+iy,(1+it_0)/2)dx = 0.$$

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Mulitplying this equation by y^{s-2} and integrating from 0 to ∞, we get

$$\int_0^\infty \int_0^1 E(x+iy,(1+it_0)/2)y^{s-2}dxdy = 0.$$

Now we use the fundamental idea that

$$\cup_{\gamma\in\Gamma_\infty\backslash\Gamma}\gamma(\Gamma\backslash H) = [0,1]\times[0,\infty],$$

usually referred to as the "unfolding" of the domain of integration. Thus,

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\int_{\gamma(\Gamma\backslash H)} E(z,(1+it_0)/2)\Im(z)^s\frac{dxdy}{y^2} = 0.$$

As $E(\gamma z,s) = E(z,s)$, we may change variables and get:

$$0 = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\int_{\Gamma\backslash H} E(z,(1+it_0)/2)\Im(\gamma z)^s\frac{dxdy}{y^2} = \int_{\Gamma\backslash H} E(z,(1+it_0)/2)E(z,s)\frac{dxdy}{y^2},$$

valid for all $s\in\mathbb{C}$.

From the definition of $E(z,s)$ (or its Fourier expansion) we see that

$$E(z,\bar{s}) = \overline{E(z,s)}.$$

Therefore, putting $s = (1-it_0)/2$, we get from the penultimate equation,

$$0 = \int_{\Gamma\backslash H}|E(z,(1+it_0)/2)|^2\frac{dxdy}{y^2}.$$

Thus, the integrand is identically zero. That is, we have proved that $\zeta(1+it_0) = 0$ implies that

$$E(z,(1+it_0)/2)\equiv 0.$$

We now show that this is a contradiction. We do this by showing that some Fourier coefficient of $E(z,(1+it_0)/2)$ is non-zero. That is, we need to check

$$\int_0^\infty e^{-\pi ry(u+u^{-1})}\frac{du}{u^{1+it_0}} \neq 0.$$

If we set $u = e^\theta$, we have to show that

$$\int_{-\infty}^\infty e^{-\pi ry(e^\theta+e^{-\theta})-it_0\theta}d\theta \neq 0.$$

In other words, we must show that

$$\int_0^\infty e^{-\pi ry(e^\theta+e^{-\theta})}\cos t\theta d\theta \neq 0.$$

This integral is of the form

$$\int_0^\infty e^{-y(a^\theta+a^{-\theta})}\cos\theta d\theta, \quad a>1.$$

We would like to determine its behaviour as y tends to infinity. To do this, we can apply Laplace's saddle point method: if f has two continuous derivatives, with $f(0) = f'(0) = 0$ and $f''(0) > 0$, and f is increasing in $[0,A]$, then

$$I(x) := \int_0^A e^{-xf(t)}dt \sim \sqrt{\frac{\pi}{2xf''(0)}}$$

as x tends to infinity and provided $I(x_0)$ exists for some x_0. A slightly generalized version of this says that if g is continuous on $[0, A]$, then

$$\int_0^A g(t)e^{-xf(t)}dt \sim g(0)\sqrt{\frac{\pi}{2xf''(0)}}.$$

Now choose $f(t) = a^t + a^{-t} - 2$, $g(t) = \cos t$ so that

$$e^{-2x}\int_0^\infty e^{-x(a^\theta+a^{-\theta}-2)}\cos\theta d\theta \sim e^{-2x}2\log a\sqrt{\frac{\pi}{x}}$$

from which we see that $E(z, (1+it_0)/2) \not\equiv 0$, as required. This gives the desired contradiction.

It is possible to deduce the non-vanishing of the above integrals directly without appealing to Laplace's saddle point method. With some work, it may also be possible to derive a zero-free region for $\zeta(s)$.

2. Explicit construction of Maass cusp forms

The first examples of Maass cusp forms were constructed by Maass [6] in 1949. Alternate treatments of this subject can also be found in [2] and [8].

Let F be a quadratic field over $\mathbb{Q}$ with narrow class number one. (This means that the order of the narrow ideal class group is one, where the equivalence relation for narrow ideals is modulo principal ideals with a totally positive generator.) Let ψ be a Hecke character. Such a character has the form $\psi = \psi_\infty\psi_f$ for some some finite order character ψ_f with conductor f. We will consider only characters with $f = O_F$ so that $\psi(\mathfrak{a}) = \psi_\infty(\alpha)$ where α is a totally positive generator of $\mathfrak{a}$. Let v and e be as follows. v is purely imaginary, and e, equals 0 or 1. Then

$$\psi_\infty(x) = \mathrm{sgn}(x_1)^e\mathrm{sgn}(x_2)^e|x_1/x_2|^v$$

where x_1 and x_2 are the Galois conjugates of x. It is necessary to have that $\psi_\infty(\eta) = 1$ for $\eta \in O_F^\times$. The fact that F has narrow class number one implies there is a fundamental unit $\epsilon > 1$ whose norm is -1. This forces $v = mi\pi/2\log\epsilon$ with m an ordinary integer. If $m \neq 0$, we get a family of Maass cusp forms:

$$\theta_\psi(z) = \sum_{\mathfrak{a}} \psi(\mathfrak{a})\sqrt{y}K_v(2\pi N(\mathfrak{a})y)\cos 2\pi N(\mathfrak{a})x$$

if $e = 0$.

If $e = 1$, we may take

$$\theta_\psi(z) = \sum_{\mathfrak{a}} \psi(\mathfrak{a})\sqrt{y}K_v(2\pi N(\mathfrak{a}y))\sin 2\pi N(\mathfrak{a})x.$$

Maass [6] (see [2] also) shows that each of these is a cusp form for $\Gamma_0(D)$ where D is the quadratic field of F. The corresponding eigenvalues is

$$\frac{1}{4} + \frac{m^2\pi^2}{4(\log\epsilon)^2}.$$

This construction is really a special case of Langlands functoriality, namely automorphic induction.

The fact that θ_ψ is a Maass form is proved using converse theory in [2]. In general, one expects a map

$$A(K) \to A(k)$$

from the space of automorphic representations of $GL_n(\mathbb{A}_K)$ to the space of automorphic representations of $GL_{nd}(\mathbb{A}_k)$ where $d = [K : k]$ where the map is given as follows. Let Π be a cuspidal automorphic representation of K and suppose

$$L(s, \Pi) = \prod_w L(s, \Pi_w),$$

where the product is over all places w of K. One expects that there is a π which is a cuspidal automorphic representation of k so that

$$L_(s, \pi_v) = \prod_{w|v} L(s, \Pi_w).$$

This special case of functoriality has been established by Arthur and Clozel [1] when K/k is cyclic.

3. The Rankin-Selberg L-function

The unfolding technique of section 1 has wider ramifications. It can be used to establish the analytic continuation and functional equation for a large class of L-functions which fall under the umbrage of Rankin-Selberg theory.

Let $F : H \to \mathbb{C}$ be a Γ-invariant function which is of rapid decay (that is, $F(x+iy) = O(y^N)$ for all $N \geq 1$.) Let

$$C(F, y) = \int_0^1 F(x+iy)dx, \quad y > 0$$

be the constant term of the Fourier expansion. Let

$$L(F, s) = \int_0^\infty C(F, y) y^s \frac{dy}{y^2}$$

be the Mellin transform of $C(F, y)$.

Theorem 1. *Let $L^*(F, s) = \pi^{-s}\Gamma(s)\zeta(2s)L(F, s)$. Then, $L(F, s)$ has analytic continuation to the whole complex plane, regular everywhere except for a simple pole at $s = 1$ with residue equal to*

$$\frac{3}{\pi} \int_{\Gamma\backslash H} F(z)dz.$$

The function $L^*(F, s)$ is regular for all $s \neq 0, 1$ and satisfies a functional equation

$$L^*(F, s) = L^*(F, 1-s).$$

Proof. The key idea is to use the decomposition described earlier. We have

$$L(F, s) = \int_0^\infty \int_0^1 F(x+iy) y^{s-2} dxdy.$$

Decomposing the domain of integration as in the "unfolding" technique, this becomes

$$= \sum_{\gamma \in \Gamma_\infty\backslash\Gamma} \int_{\gamma(\Gamma\backslash H)} F(z) y^s \frac{dxdy}{y^2}$$

This can be rewritten as

$$\sum_{\gamma \in \Gamma_\infty\backslash\Gamma} \int_{\Gamma\backslash H} F(\gamma z)(\Im(\gamma z))^s \frac{dxdy}{y^2} = \sum_{\gamma \in \Gamma_\infty\backslash\Gamma} \int_{\Gamma\backslash H} F(z)(\Im(\gamma z))^s \frac{dxdy}{y^2}$$

because F is Γ-invariant. Moving the summation inside the integral shows that this is equal to

$$\int_{\Gamma\backslash H} F(z)E(z,s)\frac{dxdy}{y^2}.$$

As $E(z,s)$ has analytic continuation and functional equation, we get the same for $L(F,s)$.

□

we now give a few examples on how to apply this theorem.

In the special case that f is a cusp form of weight k, we may apply the above result to $F(z) = y^k|f(z)|^2$ which is easily checked to be Γ-invariant.

A straightforward computation shows that the constant term is

$$y^k \sum_{n=1}^{\infty} |a_n|^2 e^{-4\pi n y}.$$

The Mellin transform of the constatnt term is

$$\int_0^{\infty} y^{k+s} \sum_{n=1}^{\infty} |a_n|^2 e^{-4\pi n y} \frac{dy}{y^2} = (4\pi)^{-s-k+1}\Gamma(s+k-1)\sum_{n=1}^{\infty}\frac{|a_n|^2}{n^{s+k-1}}.$$

This proves:

Theorem 2. *Let f be a cusp form of weight k for $SL_2(\mathbb{Z})$. If*

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

is its Fourier expansion at infinity, then the Dirichlet series

$$\sum_{n=1}^{\infty}\frac{|a_n|^2}{n^s}$$

has a meromorphic continuation to the whole complex plane. In fact, if

$$\psi(s) = \pi^{-2s-k+1}2^{-2s}\Gamma(s)\Gamma(s+k-1)\zeta(2s)\sum_{n=1}^{\infty}\frac{|a_n|^2}{n^s}$$

then $\psi(s)$ extends to function which is regular for all $s \in \mathbb{C}$ except at $s=1$ where it has a simple pole and residue equal to

$$\frac{3}{\pi}\int_{\Gamma\backslash H} y^k|f(z)|^2\frac{dxdy}{y^2} = \frac{3}{\pi}(f,f).$$

Moreover, $\psi(s)$ satisfies the functional equation $\psi(s) = \psi(1-s)$.

If we apply the theorem of Chandrasekharan and Narasimhan [3] mentioned in the previous lectures, we deduce that

$$\sum_{n\leq x}|a_n|^2 = \frac{3}{\pi}(f,f)x^k + O(x^{k-2/5})$$

because twice the sum of the coefficients in the Gamma factors (or equivalently the degree in the sense of Selberg) is equal to 4. By taking a single summand in the sum on the left, we deduce that $a_n = O(n^{k/2-1/5})$. The same technique applied to Maass forms gives us $a_n = O(n^{3/10})$.

If we take f and g to be cusp forms (or even with one of them a cusp form), we consider

$$y^k f(z)\overline{g(z)}$$

which is Γ-invariant. If

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

and

$$g(z) = \sum_{n=1}^{\infty} b_n e^{2\pi i n z}$$

are the respective Fourier expansions at infinity, then the constant term is easily computed to be equal to

$$y^k \sum_{n=1}^{\infty} a_n \overline{b_n} e^{-4\pi n y}.$$

One could also take forms of different weights k_1 and k_2 and consider

$$y^{(k_1+k_2)/2} f(z)\overline{g(z)}.$$

In the end, applying Theorem 1 we deduce that

$$\sum_{n=1}^{\infty} \frac{a_n \overline{b_n}}{n^s}.$$

A suitably normalized version of this series (with appropriate Γ-factors, $\zeta(2s)$ and so forth) extends to a function which is regular everywhere except possibly at $s = 1$ where it may have a simple pole with residue equal to

$$\frac{3}{\pi}(f, g).$$

Thus, if f and g are orthogonal to each other, then the normalized series extends to an entire function.

Kronecker's limit formula states that

$$\lim_{s\to 1} \left[E(z,s) - \frac{1}{s-1} \right] = \log(e^{\gamma}/4\pi) - 2\log(\sqrt{y}|\eta(z)|^2)$$

where $\eta(z) = q^{1/24} \prod_{n=1}^{\infty}(1-q^n)$, with $q = e^{2\pi i z}$. If f and g are Hecke eigenforms with π_f, π_g being the associated automorphic representations, the Kronecker limit formula allows us to write down an exact formula for the special value $L(1, \pi_f \otimes \pi_g)$.

4. Rankin-Selberg L-functions for GL_n

The general theory for GL_n was initated and developed by Jacquet, Piatetski-Shapiro and Shalika [5], Shahidi [9] and finally completed by Moeglin-Waldspurger [7]. If π_1 and π_2 are cuspidal automorphic representations of GL_m and GL_n of the adele ring over the rationals (say), then the Rankin-Selberg L-function is defined by the Euler product

$$L(s, \pi_1 \otimes \pi_2) = \prod_p L(s, \pi_{1,p} \otimes \pi_{2,p})$$

where for all but finitely many primes p, the Euler factors are given by the formula

$$L(s, \pi_{1,p} \otimes \pi_{2,p}) = \prod_{i,j} \left(1 - \frac{\alpha_{i,p}^{(1)} \alpha_{j,p}^{(2)}}{p^s}\right)^{-1}$$

and

$$L(s,\pi_{r,p}) = \prod_i \left(1 - \frac{\alpha_{i,p}^{(r)}}{p^s}\right)^{-1}$$

for $r = 1, 2$. It is possible to define the Euler factors at all the places so that the final product converges for $\Re(s) > 1$. The completed L-function turns out to be entire unless

$$\pi_2 \simeq \pi_1 \otimes |\det|^{it}$$

for some real number t in which case the function is regular everywhere except at $s = 1 - it$ where it has a simple pole.

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