## 2. MAASS WAVE FORMS

### 1. Maass forms of weight zero

If we consider modular forms without the holomorphy condition but insist that our function is an eigenfunction of the non-Euclidean Laplacian:

$$\Delta = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

we arrive at the notion of real analytic forms. We may write such a function, as a function of the variables $x, y$ and since $f(z+1) = f(z)$, we have

$$f(x, y) = \sum_n a_n(f, y) e^{2\pi i n x}.$$

Suppose that $\Delta f = \lambda f$. This gives us a condition on the coefficients $a_n(f, y)$, namely that they satisfy

$$-y^2 \frac{d^2}{dy^2} a_n(f, y) = (\lambda - 4\pi^2 n^2 y^2) a_n(f, y).$$

One can renormalize and show that

$$f(x, y) = a_0(f) y^s + a_0' y^{1-s} + \sum_{n \neq 0} a_n(f) \sqrt{y} K_{ir}(2\pi |n| y) e^{2\pi i n x}$$

where

$$K_{ir} = \frac{1}{2} \int_{-\infty}^{\infty} e^{-y \cosh t - irt} dt$$

with $\lambda = 1/4 + r^2$.

Maass proved that the series

$$\sum_{n=1}^{\infty} \frac{a_n(f)}{n^s}$$

extends to a meromorphic function for all $s \in \mathbb{C}$ analytic everywhere except possibly at $s = 0$ and $s = 1$, and satisfies a functional equation.

We have the celebrated Ramanujan conjecture that for any $\epsilon > 0$, $a_n(f) = O(n^\epsilon)$. The Selberg conjecture is that $\lambda \geq 1/4$, or equivalently, $r$ is real and not purely imaginary.

In his 1970 paper, Langlands [5] interprets the Selberg conjecture as a Ramanujan conjecture "at infinity" and thus puts both conjectures on an equal conceptual footing.

By the work of Kim and Shahidi [4], we know that $a_n = O(n^{7/64})$ and that $\lambda \geq .238$ by the recent work of Kim and Sarnak [3]

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## 2. Maass forms with weight

Let us fix a discrete subgroup $\Gamma$ of $SL_2(\mathbb{R})$. Here we consider functions on the extended upper half plane which satisfy the following: (i) $f(\gamma z) = ((c\overline{z}+d)/|cz+d|)^k f(z)$ for all

$$\gamma \in \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma;$$

(ii) $f$ is an eigenfunction of

$$\Delta_k = -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) + iky\frac{\partial}{\partial x}$$

(iii) a growth condition of the form $f(x+iy) = O(y^N)$ for some $N > 0$ as $y$ tends to infinity.

One can show the existence of "shift" operators that will reduce the study of these spaces essentially to the study of weight zero or weight one Maass forms. Thus, often in the literature, (see for example [1] ) the focus is on weight zero or the weight one case.

If $f$ is a classical modular form of weight $k$, then it is not hard to show that $y^{k/2}f(z)$ is a Maass form of weight $k$ with eigenvalue $k(2-k)/4$. Therefore, the study of Maass forms includes the study of modular forms from this perspective.

The set of Maass forms of fixed weight forms a vector space over $\mathbb{C}$. Moreover, we have an involution acting on this space given by the map

$$\iota : f(z) \to f(-\overline{z}).$$

A form is called even if $\iota \circ f = f$ and odd if $\iota \circ f = -f$. Therefore, the space of Maass forms decomposes as a direct sum of two subspaces consisting of even forms and odd forms respectively.

The $L$-series

$$\sum_{n=1}^{\infty} \frac{a_n(f)}{n^s}$$

extends to an entire function and satisfies a functional equation:

$$Q^s\Gamma\left(\frac{s+\delta+r}{2}\right)\Gamma\left(\frac{s+\delta-r}{2}\right)L(s,f) = wQ^{1-s}\Gamma\left(\frac{1-s+\delta-r}{2}\right)\Gamma\left(\frac{1-s+\delta+r}{2}\right)L(1-s,\overline{f})$$

where $\delta = 0$ or $1$ according as $f$ is even or odd.

For $\Gamma = SL_2(\mathbb{Z})$, Selberg [7] proved that $\lambda \geq 1/4$ and this was extended to congruence subgroups of sufficiently small level by Vigneras[8]. For general arithmetic groups, Selberg showed that $\lambda \geq 3/16$.

## 3. Eisenstein series

The simplest example of a Maass form is given by the Eisenstein series

$$E(z,s) = \pi^{-s}\Gamma(s)\frac{1}{2}\sum_{(m,n)\neq(0,0)} \frac{y^s}{|mz+n|^{2s}}.$$

This series converges for $\Re(s) > 1$ and we clearly have

$$E(\gamma z, s) = E(z,s)$$

for all $\gamma \in SL_2(\mathbb{Z})$. In addition, it is easily verified that

$$\Delta E(z,s) = s(1-s)E(z,s)$$

so that $E(z,s)$ is a weight zero Maass form with eigenvalue $s(1-s)$. Since $E(z,s)$ is periodic with period 1, we can derive its Fourier series:

$$E(z,s) = \sum_{r=-\infty}^{\infty} a_r(y,s)e^{2\pi irx}$$

and

$$a_r(y,s) = \int_0^1 E(x+iy,s)e^{-2\pi irx}dx.$$

We do the obvious. We insert the series expansion for $E(z,s)$ into the integral and apply Fubini's theorem. First, the contribution to $E(z,s)$ from $m=0$ is

$$\pi^{-s}\Gamma(s)y^s\zeta(2s).$$

This is part of $a_0(y,s)$ but not all of $a_0$ as we shall see below. Now suppose $m \neq 0$. Since $(m,n)$ and $(-m,-n)$ give the same summand in $E(z,s)$, we may suppose $m>0$. Thus,

$$a_r(y,s) = \pi^{-s}\Gamma(s)y^s \sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\int_0^1 [(mx+n)^2+m^2y^2]^{-s}e^{-2\pi irx}dx.$$

If we mut $n = qm+d$ with $0 \le d < m$, the sum becomes

$$\sum_{m=1}^{\infty}\sum_{d \bmod m}\int_{-\infty}^{\infty}[(mx+d)^2+m^2y^2]^{-s}e^{-2\pi irx}dx.$$

We change variables: $x = u - d/m$ to get

$$\sum_{m=1}^{\infty} m^{-2s}\int_{-\infty}^{\infty}(u^2+y^2)^{-s}e^{-2\pi iru}\left(\sum_{d \bmod m} e^{2\pi idr/m}\right)du.$$

The innermost sum is zero unless $m|r$ in which case it is $m$. Thus, the sum becomes

$$\sum_{m|r} m^{1-2s}\int_{-\infty}^{\infty}(u^2+y^2)^{-s}e^{-2\pi iru}du$$

If $r=0$, we get

$$\pi^{-s}\Gamma(s)y^s\zeta(2s-1)\int_{-\infty}^{\infty}(u^2+y^2)^{-s}du$$

which is equal to

$$\pi^{-s}\sqrt{\pi}\Gamma(s-1/2)y^{1-s}\zeta(2s-1).$$

Thus, the constant term (on applying the functional equation for $\zeta(s)$) is equal to

$$a_0(y,s) = \pi^{-s}\Gamma(s)\zeta(2s)y^s + \pi^{s-1}\Gamma(1-s)\zeta(2-2s)y^{1-s}.$$

If $r \neq 0$, then we get

$$a_r(y,s) = 2|r|^{s-1/2}\sigma_{1-2s}(|r|)\sqrt{y}K_{s-1/2}(2\pi|r|y)$$

where

$$\sigma_{1-2s}(r) = \sum_{m|r} m^{1-2s}.$$

One can show that $a_r(y,s) = a_r(y,1-s)$ and $r^s\sigma_{-2s}(r) = r^{-s}\sigma_{2s}(r)$ from which the functional equation is easily deduced.

## 4. Upper bound for Fourier coefficients and eigenvalue estimates

We begin with the elementary observation

$$e^{-1/x} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \Gamma(s)x^s ds$$

which is easily demonstrated by contour integration and Stirling's formula. Hence,

$$\sum_{n=1}^{\infty} a_n e^{-n/x} = \frac{1}{2\pi i}\int_{(2)} \Gamma(s)f(s)x^s ds$$

where

$$f(s) = \sum_{n=1}^{\infty} a_n/n^s.$$

Now suppose that $a_n \geq 0$ and $f(s)$ is absolutely convergent for $\Re(s) \geq 1+\epsilon$. Moving the line of integration to $\Re(s) = 1+\epsilon$ gives

$$\sum_{n=1}^{\infty} a_n e^{-n/x} = O(x^{1+\epsilon}).$$

Thus, for any individual term in the sum, we have

$$a_n e^{-n/x} = O(x^{1+\epsilon}).$$

Choosing $x = n$, we deduce that $a_n = O(n^{1+\epsilon})$.

It may look as if we were wasteful in the above analysis and a finer argument would give a better estimate. This, however, is not true as can be seen by considering

$$f(s) = \sum_{n=1}^{\infty} \frac{n^{k-1}}{n^{ks}} = \zeta(ks-k+1).$$

In this example, we have

$$a_n = \Omega(n^{1-\epsilon})$$

for any $\epsilon > 0$ and so, we cannot reduce the exponent in the penultimate analysis.

Now consider

$$L_m(s) := L(s,\pi,r_m) = \prod_p \prod_{j=1}^{m} \left(1 - \frac{\alpha_p^{m-j}\beta_p^j}{p^s}\right)^{-1}$$

where we are ignoring the finitely many Euler factors that need to be modified corresponding to the ramified factors.

Consider the $L$-function

$$L(s,\pi,r_m \otimes \overline{r}_m) = \prod_{k\leq 2m,\, k \text{ odd}} L(s,\pi,r_k).$$

The proof of this identity is equivalent to the trigonometric identity

$$1 + \frac{\sin 3\theta}{\sin\theta} + \frac{\sin 5\theta}{\sin\theta} + \cdots \frac{\sin(2m-1)\theta}{\sin\theta} = \left(\frac{\sin m\theta}{\sin\theta}\right)^2$$

which is easily proved by induction and left as an exercise for the reader.

Thus, the series $L(s,\pi,r_m \otimes \overline{r_m})$ is a Dirichlet series with non-negative coefficients. If we now suppose that for each $m \geq 1$, $L(s,\pi,r_m)$ is analytic for $\Re(s) \geq 1+\epsilon$, then its $p$-th coefficient (for $p$ prime) is $O(p^{1+\epsilon})$ by the argument given above. But the $p$-th coefficient is easily calculated to be

$$\left|\sum_{j=1}^{m} \alpha_p^{m-j}\beta_p^j\right|^2 .$$

Moreover, $|\alpha_p\beta_p| = 1$ so that if the Ramanujan conjecture is false, one of these has absolute value greater than 1. Without any loss of generality, suppose it is $\alpha_p$. Then, in the above summation, $\alpha_p^m$ dominates the sum so we deduce

$$|\alpha_p|^{2m} = O(p^{1+\epsilon}).$$

Taking, $2m$-th roots, we obtain

$$\alpha_p = O(p^{(1+\epsilon)/2m}),$$

and letting $m$ tend to infinity, we obtain $\alpha_p = O(1)$ which is the Ramanujan conjecture.

As we remarked above, this reasoning cannot be sharpened. However, using the fact that each of the $L$-functions $L(s,\pi,r_m)$ satisfies a functional equation, one can improve the estimate using a classical result of Chandrasekharan and Narasimhan [2]. This result says that if $a_n \geq 0$ and $f(s) = \sum_{n=1}^{\infty} a_n/n^s$ is convergent in some half-plane, has analytic continuation for all $s$ except for a pole at $s = 1$ of order $k$ and it satisfies a functional equation of the form

$$Q^s\Delta(s)f(s) = wQ^{1-s}\Delta(1-s)f(1-s)$$

where $Q > 0$ and

$$\Delta(s) = \prod_i \Gamma(\alpha_i s + \beta_i)$$

then

$$\sum_{n\leq x} a_n = xP_{k-1}(\log x) + O(x^{\frac{2A-1}{2A+1}}\log^{k-1} x)$$

where $A = \sum_i \alpha_i$. Taking differences, we deduce that

$$a_n = O(n^{\frac{2A-1}{2A+1}}\log^{k-1} n).$$

In [6], this result is stated with a typo on page 525. (On lines 3 and 7 of [6], $(2A-1)(2A+1)$ should be $(2A-1)/(2A+1)$ in both instances.)

A similar reasoning can be applied to obtain bounds in the Selberg eigenvalue conjecture. If $\pi$ corresponds to a Maass form with eigenvalue $\lambda$, then the Gamma factors in the functional equation of $L(s,\pi,r_m)$ will have the following shape:

$$\Gamma(s,\pi,r_m) = \prod_{j=0}^{m} \Gamma\left(\frac{s-\lambda_j}{2}\right), \quad \lambda_j = i(m-2j)r, \quad \lambda = \frac{1}{4} + r^2.$$

One can also study oscillations of Fourier coefficients of modular forms as well as Dirichlet series constructed out of Kloosterman sums. This we will take up in later lectures.

## References


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