10. INTRODUCTION TO ARTIN L-FUNCTIONS

1. Hecke L-functions

Dirichlet's work on primes in arithmetic progression gave birth to a family of L-functions attached to characters of the group of coprime residue classes mod q. Using the analyticity of these L-functions, and most importantly, their non-vanishing at $s = 1$, Dirichlet deduced the infinitude of primes in a given arithmetic progression $a \pmod q$ with $(a, q) = 1$.

If one wants to generalize Dirichlet's theorem, several natural questions arise. First, the ring of integers O_K of a number field K does not, in general, have the unique factorization property. Thus, we must speak of prime ideals rather than prime elements. Having decided this, the next question is to understand the notion of a residue class. The natural object to take is the ideal class group of a number field and inquire if there are infinitely many prime ideals in a given ideal class. This was the approach taken by Hecke.

Given a number field K, and an ideal $\mathfrak{q}$, we have the notion of the $\mathfrak{q}$-ideal class group defined as follows. We consider the group of fractional ideals of K which are coprime to $\mathfrak{q}$ modulo the principal ideals (α) with $\alpha \equiv 1 \bmod \mathfrak{q}$. Thus, a natural generalization of Dirichlet's theorem is to inquire if there are infinitely many prime ideals in a given $\mathfrak{q}$-ideal class.

In the case $K = \mathbb{Q}$, all ideals are principal and the q-ideal class group is easily seen to be $(\mathbb{Z}/q\mathbb{Z})^*/\pm 1$. Thus, we don't realize $(\mathbb{Z}/q\mathbb{Z})^*$ as a q-ideal class group. Clearly the problem arises with the choice of generator for the principal ideals. To rectify this, we introduce the real embeddings of K in the following way.

We introduce the notion of a generalized ideal $\mathfrak{f} = \mathfrak{f}_0\mathfrak{f}_\infty$ where $\mathfrak{f}_0$ is an ordinary ideal and $\mathfrak{f}_\infty$ is a collection of real embeddings of K. The $\mathfrak{f}$-ideal class group consists of the group of fractional ideals coprime to $\mathfrak{f}_0$ modulo the subgroup of principal fractional ideals (α) with

$$\alpha \equiv 1 \bmod {}^\times \mathfrak{f}$$

which means that $\alpha \equiv 1 \bmod \mathfrak{f}_0$ and $\sigma(\alpha) > 0$ for all $\sigma \in \mathfrak{f}_\infty$. If $K = \mathbb{Q}$, and ∞ denotes the usual embedding of $\mathbb{Q}$ into $\mathbb{R}$, then $q\infty$-ideal class group of $\mathbb{Z}$ retrieves the coprime residue classes mod q. From this perspective, Dirichlet's theorem is to be viewed as a special case of a theorem about the distribution of prime ideals in generalized ideal classes.

The special case when $\mathfrak{f} = \mathfrak{f}_\infty$ includes all the real embeddings, the ideal class group is called the narrow ideal class group and its order is called the narrow class number.

Following Hecke, we may now consider characters of $\mathfrak{f}$-ideal class groups and for each character χ, we set

$$L(s, \chi) = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s}.$$

These L-functions are referred to as Hecke L-functions of finite type and χ is called a character of finite order. Hecke showed that these L-functions extend to entire functions and satisfy a suitable functional equation. He also proved that they do not vanish on $\Re(s) = 1$. Thus, following the Tauberian theorem explained in Lecture 1, we deduce that for any generalized $\mathfrak{f}$-ideal class, there are infinitely many prime ideals in that class.

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But Hecke considered a more general question in his researches. To any idele class character χ of $\mathbb{A}_K^*/K^*$, he showed how one can associate an L-function, extend it to an entire function and establish a functional equation. These characters are called "grossencharacters" and if we view the idele class group as $GL_1(\mathbb{A}_K)/GL_1(K)$, then Hecke's work is the first level in the Langlands program. For more on Hecke L-functions, see [5].

2. Artin L-functions

Let K/k be finite Galois extension of algebraic number fields. Let $G = \mathrm{Gal}(K/k)$. For each prime ideal $\mathfrak{p}$ of K, let us consider a prime ideal $\wp$ of K dividing $\mathfrak{p}$. We define the decomposition group $D_\wp$ and inertia groups $I_\wp$ by

$$D_\wp = \{\sigma \in G : \wp^\sigma = \wp\}$$

and

$$I_\wp = \{\sigma \in G : \sigma(x) \equiv x^{N(\mathfrak{p})} \bmod \wp \quad \text{for all}\, x \in O_K\}$$

respectively. Clearly $I_\wp$ is a normal subgroup of $D_\wp$ and one can show that

$$D_\wp/I_\wp \simeq \mathrm{Gal}((O_K/\wp)/(O_k/\mathfrak{p}))$$

as the latter is the Galois group of a finite extension of a finite field. As such, the latter is a cyclic group generated by the Frobenius automorphism

$$\mathrm{Frob}_\mathfrak{p} : x \mapsto x^{N(\mathfrak{p})}.$$

The pull-back of this element to $D_\wp$ which is well-defined up to an element of $I_\wp$ is called the Frobenius element (denoted $\sigma_\wp$) attached to $\wp$. As we vary over the $\wp$ with $\wp|\mathfrak{p}$, the elements $\sigma_\wp$ determine a conjugacy class $\sigma_\mathfrak{p}$ of elements which we call the Artin symbol attached to $\mathfrak{p}$. Of course, this is only well-defined when the inertia group is trivial. In general, it is well-defined modulo inertia.

In the special case that $k = \mathbb{Q}$ and $K = \mathbb{Q}(\sqrt{D})$, the Artin symbol turns out to be the Legendre symbol (D/p).

A natural question to ask is the following. Given a conjugacy class C of G, how often do we have $\sigma_\mathfrak{p} \in C$? The Chebotarev density theorem states that

$$\#\{\mathfrak{p},\, N(\mathfrak{p}) \leq x \,:\, \sigma_\mathfrak{p} \in C\} \sim \frac{|C|}{|G|}\frac{x}{\log x}$$

as x tends to infinity. One way to prove this theorem (although historically this was not the case) is to introduce the non-abelian L-series of Artin as follows. Let V be a finite dimensional vector space over $\mathbb{C}$ and let

$$\rho : G \to GL(V)$$

be a representation. The Artin L-function is defined as

$$L(s, \rho; K/k) = \prod_\mathfrak{p} \det(1 - \rho(\sigma_\wp)N(\mathfrak{p})^{-s}|V^{I_\wp})^{-1}.$$

Sometimes, we write $L(s, \chi, K/k)$ for $L(s, \rho, K/k)$ where $\chi = \mathrm{tr}\,\rho$ is the character of ρ.

Artin's conjecture is the assertion that if ρ is irreducible and unequal to the trivial representation, then $L(s, \rho, K/k)$ extends to an entire function. In the case that ρ is one-dimensional, Artin showed that there is a Hecke character ψ of k of finite order, so that

$$L(s, \rho, K/k) = L(s, \psi).$$

This is usually referred to as Artin's reciprocity law. If K is a quadratic extension of k, then this theorem is precisely the law of quadratic reciprocity for algebraic number fields.

Langlands [4] has enunciated a more general conjecture. Namely, given ρ of degree n, he predicts that there is an automorphic representation $\pi(\rho)$ of $GL_n(\mathbb{A}_k)$ with

$$L(s, \rho, K/k) = L(s, \pi(\rho))$$

and the latter L-function, being a GL_n L-function has been shown by Godement and Jacquet [3] to extend to an entire function. This conjecture of Langlands is referred to as the Langlands reciprocity conjecture or sometimes as the strong Artin conjecture. By the work of Arthur-Clozel [1], we know that it holds for any nilpotent Galois extension K/k.

Before we state what is currently known about this conjecture, it will be useful to review some functorial properties of Artin L-functions. They are:

(1) $L(s, 1, K/k) = \zeta_k(s)$;
(2) $L(s, \chi_1 + \chi_2, K/k) = L(s, \chi_1, K/k)L(s, \chi_2, K/k)$;
(3) if η is a character of a subgroup H of G, then

$$L(s, \eta, K/K^H) = L(s, \mathrm{Ind}_H^G \eta, K/k);$$

(4) if M/k is Galois with $M \subseteq K$, and τ is a character of $\mathrm{Gal}(M/k)$ then,

$$L(s, \tau, M/k) = L(s, \tilde{\tau}, K/k).$$

Properties (1), (2) and (4) are easy to verify. Property (3) involves some group theory and algebraic number theory. The details can be found in [5].

Motivated by Artin's conjecture, Brauer [2] was led to prove the following fundamental theorem in group theory. Let G be a finite group and χ any character of G. Then, there exist nilpotent subgroups H_i of G, one-dimensional characters ψ_i of H_i and integers n_i so that

$$\chi = \sum_i n_i \mathrm{Ind}_{H_i}^G \psi_i.$$

An immediate consequence is:

Theorem 1. *(Brauer, 1947) The Artin L-function $L(s, \rho, K/k)$ can be written as a quotient of products of Hecke L-functions and consequently, it extends to a meromorphic function.*

Proof. We first use Brauer's theorem to write $\chi = \mathrm{tr}\,\rho$ as a sum of abelian characters induced from nilpotent subgroups. By property (2),

$$L(s, \chi, K/k) = \prod_i L(s, \mathrm{Ind}_{H_i}^G \psi_i, K/k)^{n_i}.$$

By property (3), each of the factors can be written as

$$L(s, \mathrm{Ind}_{H_i}^G \psi_i, K/k) = L(s, \psi_i, K/K^{H_i})^{n_i}.$$

By Artin's reciprocity law, $L(s, \psi_i, K/K^{H_i})$ is a Hecke L-function $L(s, \eta_i)$ (say), which by Hecke's theorem extends to an entire function. This completes the proof of the theorem. ∎

3. Automorphic induction and Artin's conjecture

Our goal now is to show how a special case of the automorphic induction conjecture in the Langlands program suffices to establish the Langlands reciprocity conjecture and consequently Artin's conjecture.

Conjecture 1. *(Automorphic induction of Hecke characters) Let K/k be an arbitrary finite extension of algebraic number fields. If ψ is a Hecke character of K, then there is a cuspidal automorphic representation $\pi(\psi)$ of K such that*

$$L(s,\psi) = L(s,\pi(\psi)).$$

Now we can prove:

Theorem 2. *If we have automorphic induction of Hecke characters, then the Langlands reciprocity conjecture follows.*

Proof. By the proof of Brauer's theorem, we may write

$$L(s,\chi,K/k) = \prod_i L(s,\psi_i)^{n_i}$$

where ψ_i are Hecke characters of some extension K^{H_i} of k. By the automorphic induction conjecture, we may write

$$L(s,\psi_i) = L(s,\pi_i)$$

for some automorphic representation π_i of k. After regrouping some factors if necessary, we may write

$$L(s,\chi,K/k) = \prod_i L(s,\pi_i)^{e_i}$$

where all of the π_i's are distinct cuspidal automorphic representations. (Here, we are viewing the L-function $L(s,\pi)$ as a product over the finite primes only.) Writing

$$L(s,\pi) = \sum_{\mathfrak{n}} \frac{a_\pi(\mathfrak{n})}{N(\mathfrak{n})^s}$$

and comparing coefficients of Dirichlet series for a prime ideal $\mathfrak{p}$ in the penultimate equality, we get

$$\chi(\sigma_{\mathfrak{p}}) = \sum_i e_i a_{\pi_i}(\mathfrak{p}).$$

Now we compare poles of the "Rankin-Selberg" L-function of both sides. The order of the pole is equal to the multiplicity of the trivial character in $\chi\overline{\chi}$. As χ is irreducible, the multiplicity is one. On the other hand, the right hand side contributes a pole (by Rankin-Selberg theory) of order

$$\sum_i e_i^2.$$

Thus, all the $e_i = 0$ with one exception e_1 (say) which must be ± 1. If $e_1 = -1$, the Artin L-function would have "trivial poles" at certain negative integers and by Brauer's theorem, we know that all the poles of an Artin L-function lie in the critical strip. ∎

References

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