1. THE SATO-TATE CONJECTURE

1. Introduction

There are many significant applications of the theory of symmetric power L-functions to questions arising from classical analytic number theory. In these notes, we will touch upon only a few of them. In this lecture, we will discuss the Sato-Tate conjecture and discuss the relationship between this conjecture and the analyticity of the symmetric power L-functions. In the next lecture, we will discuss the Ramanujan conjecture and the Selberg eigenvalue conjecture.

Let E be an elliptic curve over a number field F. For each prime ideal v of F where E has good reduction, the number of points of E mod v is given by

$$N(v) + 1 - a_v$$

where $N(v)$ denotes the norm of v and a_v satisfies Hasse's inequality

$$|a_v| \leq 2(N(v))^{1/2}.$$

Thus, we can write

$$a_v = 2N(v)^{1/2} \cos \theta_v$$

for a uniquely defined angle θ_v satisfying $0 \leq \theta_v < \pi$. The Sato-Tate conjecture is a statement about how the angles θ_v are distributed in the interval $[0, \pi]$ as v varies.

To study the distribution of the angles θ_v attached to an elliptic curve, we have to consider two cases. The first case is when the elliptic curve has CM (complex multiplication). This refers to the well-known fact that the ring of endomorphisms of an elliptic curve E is either isomorphic to the ring of ordinary integers or is an order in an imaginary quadratic field k. In the former case we say E has no CM (no complex multplication) and in the latter case, we say E has CM.

Let us now look at the CM case. For simplicity, let us suppose that k is contained in F, the field over which E is defined. Then, the sequence $\{\theta_v, -\theta_v\}$, as v ranges over the places of F, is uniformly distributed in $[-\pi, \pi]$. If F does not contain k, the situation is a little more complicated with a slightly different density function that has been determined (see [4]).

In the non-CM case, the distribution is unknown at present. We will show below that the angles are **not** uniformly distributed when $F = \mathbb{Q}$. Sato and Tate (independently) predicted another law of distribution for the angles θ_v. More precisely, they predict that

$$\#\{v : N(v) \leq x : \theta_v \in (\alpha, \beta)\} \sim \left(\frac{2}{\pi} \int_\alpha^\beta \sin^2 \theta d\theta\right) \pi_F(x)$$

as x tends to infinity, where $\pi_F(x)$ is the number of prime ideals of F whose norm is less than x.

2. Uniform Distribution

We will begin with a general discussion of the classical setting for uniform distribution. A sequence of real numbers $\{x_n\}$ is called **uniformly distributed** (modulo 1) if for any pair of real numbers α, β with $0 \leq \alpha < \beta < 1$, we have

$$\#\{n \leq N : \quad x_n \in (\alpha, \beta)\} \sim (\beta - \alpha)N$$

This is the text of Lecture 1 by Ram Murty given on January 23, 2003 at the Fields Institute, Toronto, Canada.

as N tends to infinity.

Theorem 1. *(Weyl's Criterion) The sequence $\{x_n\}$ is uniformly distributed mod 1 if and only if for all $m \geq 1$,*

$$\sum_{n \leq N} e^{2\pi i m x_n} = o(N)$$

as N tends to infinity.

Proof. (Sketch) First suppose that the sequence is uniformly distributed. We will show the condition is necessary. Let us observe that any continuous function f can be approximated by a linear combination of step functions so that for any given $\epsilon > 0$, we have

$$\sup_{x \in [0,1]} |f(x) - \sum_i c_i \chi_{I_i}(x)| \leq \epsilon,$$

where χ_I denotes the characteristic function of the interval I. Then,

$$\sum_{n \leq N} f(x_n) = \sum_i c_i \left(\sum_{n \leq N} \chi_{I_i}(x_n) \right) + O(\epsilon N).$$

By hypothesis,

$$\sum_{n \leq N} \chi_{I_i}(x_n) = \mu(I_i) N + o(N),$$

where $\mu(I)$ denotes the measure of the interval I. Now the sum

$$\sum_i c_i \mu(I_i)$$

is a Riemann sum and as our epsilon gets smaller, the sum converges to the integral

$$\int_0^1 f(x) dx.$$

Thus, we have proved that

$$\frac{1}{N} \sum_{n \leq N} f(x_n) \to \int_0^1 f(x) dx.$$

In particular, we can apply this to $\cos mx$ and $\sin mx$ to deduce the required result.

For the converse, we approximate $\chi_I(x)$ by trigonometric polynomials (which can be done by the Stone-Weierstrass theorem). In fact, one can be more precise. For any positive integer K, there are trigonometric polynomials $m(x)$ and $M(x)$ of degree $\leq K$ such that

$$m(x) \leq \chi_I(x) \leq M(x)$$

with

$$m(x) = \sum_{|m| \leq K} a_m e^{2\pi i m x}, \quad M(x) = \sum_{|m| \leq K} b_m e^{2\pi i m x}$$

with

$$a_0 = b_0 = \mu(I) + O(1/K).$$

Therefore,

$$\#\{n \leq N : \quad x_n \in I\} = \sum_{n \leq N} \chi_I(x_n) = \mu(I) N + o(N),$$

as required. □

Theorem 1 says that to establish uniform distribution of the angles θ_v, we need to study the exponential sums

$$\sum_{N(v)\leq x} e^{2\pi i m\theta_v}.$$

In the CM case, Hecke proved a theorem that implies that the series

$$L(s,\chi) := \prod_v \left(1 - \frac{\chi(v)}{N(v)^s}\right)^{-1}$$

with $\chi(v) = e^{2\pi i\theta_v}$, extends to an entire function for $\Re(s) \geq 1$ and does not vanish there. The same applies to $L(s,\chi^m)$ for each natural number m. Thus, we can now apply a classical Tauberian argument to deduce the uniform distribution of the θ_v. We briefly review the relevant theorem in the next section.

3. Wiener-Ikehara Tauberian Theorem

Theorem 2. *Let $f(s) = \sum_{n=1}^{\infty} a_n/n^s$, with $a_n \geq 0$, and $g(s) = \sum_{n=1}^{\infty} b_n/n^s$ be two Dirichlet series with $|b_n| \leq a_n$ for all n. Assume that $f(s)$ and $g(s)$ extend analytically to $\Re(s) \geq 1$ except possibly at $s = 1$ where they have a simple pole with residues R and r (which may be zero) respectively. Then*

$$\sum_{n\leq x} b_n \sim rx$$

as x tends to infinity.

The classical application of this theorem is the deduction of the prime number theorem. Let

$$f(s) = -\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

where $\Lambda(n) = \log p$ when $n = p^a$ for some prime p and zero otherwise. Taking $g(s) = f(s)$ in the above theorem allows us to deduce the prime number theorem

$$\sum_{n\leq x} \Lambda(n) \sim x$$

using the well-known fact that the Riemann zeta function does not vanish on $\Re(s) = 1$.

One can apply this theorem to $L(s,\chi^m)$ above and deduce the uniform distribution of the angles after a routine application of partial summation.

In a fundamental paper written in 1970, Langlands [3] outlined an approach to the Sato-Tate conjecture using the theory of automorphic forms. To simplify matters and notation, we will give only a rough outline of this approach and relegate to later lectures the more precise details.

Firstly, Langlands suggested the automorphic viewpoint. Thus, the conjecture of Sato-Tate was applicable in a larger context of modular forms, or more generally, to automorphic forms on $GL(2)$. For example, one could take the celebrated Ramanujan τ function attached to the unique newform of weight 12 and level 1, and write

$$\tau(p) = 2p^{11/2}\cos\theta_p.$$

One expects the same Sato-Tate distribution for these angles θ_p as well.

Here is a brief description of the strategy of Langlands [3]. For each natural number m, put

$$L_m(s) = \prod_v \prod_{j=0}^{m} \left(1 - \frac{\alpha_v^{m-j}\beta_v^j}{Nv^s}\right)^{-1}$$

where $\alpha_v = e^{2\pi i\theta_v}, \beta_v = e^{-2\pi i\theta_v}$. Langlands indicated that the theory of automorphic forms predicts that each $L_m(s)$ should extend to an entire function. In fact, if each $L_m(s)$ extends analytically for $\Re(s) \geq 1$, and does not vanish there, then by the Tauberian theorem, the Sato-Tate conjecture follows. Kumar Murty [4] showed that the non-vanishing hypothesis can be dispensed with because a very elegant argument extending the classical one of Hadamard and de la Vallée Poussin allows one to show non-vanishing from having analytic continuation to $\Re(s) \geq 1$.

In the case F is the rational number field, it is now a theorem due to Wiles and others that $L_1(s)$ is essentially the L-function attached by Hecke to a classical cusp form of weight 2. Thus, in this particular case, the Langlands conjecture is established. The non-vanishing of $L_1(s)$ on $\Re(s) = 1$, is a result due to Rankin. For $m = 2$, Rankin-Selberg theory allows one to deduce that $L_2(s)$ extends to an entire function for $\Re(s) \geq 1$. The continuation of $L_2(s)$ to the entire complex plane was established by Shimura [6] in the case $F = \mathbb{Q}$ and in the general case by Gelbart and Jacquet [1]. In very recent work, Kim and Shahidi [2] showed that $L_3(s)$ extends to an entire function and later, Kim, showed the same for $L_4(s)$. For the cases $5 \leq m \leq 9$, Kim and Shahidi have shown that $L_m(s)$ extends to a meromorphic function for all $s \in \mathbb{C}$ which is regular for $\Re(s) \geq 1$, except in the case of $m = 9$, $L_9(s)$ may have a pole at $s = 1$.

Let us remark that Rankin's result on $L_2(s)$ is already sufficient to show that in the non-CM case, the Sato-Tate distribution does not hold. Also, if $L_9(s)$ were to have a pole at $s = 1$, then the Sato-Tate conjecture would be false, as we will indicate below.

4. Weyl's Theorem for Compact Groups

Serre [5] gave the following reformulation of the Weyl criterion for uniform distribution in the context of a compact group. Let G be a compact group and X its space of conjugacy classes. Let μ denote its normalised Haar measure. A sequence of elements $\{x_n\}$ with $x_n \in X$ is said to be uniformly distributed in X if for every continuous function f with compact support, we have

$$\sum_{n \leq N} f(x_n) \sim N \int_X f d\mu$$

as N tends to infinity.

Theorem 3. *(Weyl's criterion for compact groups) Let G be a compact group with Haar measure μ. A sequence $\{x_n\}$ is uniformly distributed in G if and only if*

$$\sum_{n \leq N} \chi(x_n) = o(N)$$

for every irreducible character χ of G.

The classical case in Theorem 1 corresponds to $G = \mathbb{R}/\mathbb{Z}$ because in this case, the irreducible characters are given by $x \mapsto e^{2\pi i x m}$.

Serre gave an interesting reformulation of this criterion in the context of L-functions. Let F be a field and for each place v of F, let $x_v \in G$. For each irreducible representation $\rho : G \to GL_n(\mathbb{C})$, we let

$$L(s, \rho) = \prod_v \det(I - \rho(x_v)Nv^{-s})^{-1}.$$

Theorem 4. *(Serre) Suppose that for each irreducible non-trivial representation ρ of G, the L-function $L(s,\rho)$ extends to an analytic function for $\Re(s) \geq 1$. Then, the sequence $\{x_v\}$ is uniformly distributed in X if and only if $L(s,\rho)$ does not vanish on $\Re(s) = 1$.*

In the context of the Sato-Tate conjecture, one considers the group $SU(2,\mathbb{C})$ where the conjugacy classes are parametrized by

$$X_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \quad 0 \leq \theta \leq \pi.$$

The image of the Haar measure in the space of conjugacy classes of $SU(2,\mathbb{C})$ is known to be

$$\frac{2}{\pi}\sin^2\theta d\theta.$$

The irreducible representations of $SU(2,\mathbb{C})$ are the symmetric power representations ρ_m of the standard representation ρ_1 of $SU(2,\mathbb{C})$ into $GL(2,\mathbb{C})$. We find that $L(s,\rho_m)$ as defined above by Serre coincide with $L_m(s)$ defined in section 3.

Since tr $\rho_m(X_\theta) = \sin(m+1)\theta/\sin\theta$, the Sato-Tate conjecture is equivalent to the assertion

$$\sum_{N(v)\leq x} \frac{\sin(m+1)\theta_v}{\sin\theta_v} = o(\pi_F(x)),$$

for each natural number m. So far, this has been established only for $m \leq 8$ by the work of Kim and Shahidi [2].

References

[1] S. Gelbart and H. Jacquet, A relation between automorphic forms on $GL(2)$ and $GL(3)$, Proc. Nat. Acad. Sci., USA, Vol. **73** (1976), 3348-3350.

[2] H. Kim and F. Shahidi, Functorial products for $GL_2 \times GL_3$ and functorial symmetric cube for GL_2, C.R. Acad. Sci. Paris Ser. 1, Math., Vol. 331, (2000), no. 8, pp. 599-604.

[3] R.P. Langlands, Problems in the theory of automorphic forms, Lectures in Modern Analysis and Applications, Vol. 170, Springer-Verlag, 1970, pp. 18-86.

[4] V. Kumar Murty, On the Sato-Tate conjecture, in Number Theory related to Fermat's Last Theorem, edited by N. Koblitz, pp. 195-205, *Progress in Math.*, **26**, (1982), Birkhäuser, Boston, Mass.

[5] J.-P. Serre, Abelian ℓ-adic representations and Elliptic curves, Benjamin, 1970.

[6] G. Shimura, On the holomorphy of certain Dirichlet series, Proc. London Math. Soc., (3) **31** (1975), no. 1, 79-98.