A Semi-Lagrangian Double Fourier Method for the Shallow Water Equations

Anita Layton

Department of Mathematics

University of North Carolina, Chapel Hill

Bill Spotz

Sandia National Laboratories

Motivation

- Goal: To construct a stable and efficient numerical method for solving SWE (and more comprehensive models) on a sphere.
- Spectral transform method is $O(N^3)$
- To maintain stability, some methods require spherical harmonic projections (SHP), which take $O(N^3)$ time.

Double Fourier Series

- Based on Fourier series (sines and cosines) rather spherical harmonics
- FFT in both longitudinal and latitudinal direction, $O(N^2 log(N))$ time
- With Eulerian time-stepping, the double Fourier method requires SHP or damping to be stable because
  - Discontinuities at the poles
  - Nonlinear terms give rise to non-isotropoic waves

(Cheong and collaborators, *JCP*, 2000)

Semi-Lagrangian Double Fourier Method—Projection Free?

- Nonlinearity in SWE arises mostly from advection terms, which are implicit in a semi-Lagrangian formulation
- ***Question***: Can the damping introduced by spatial interpolations used in a semi-Lagrangian double Fourier method be sufficient to maintain stability?

Unfortunately...

- Weak nonlinearity still persists in
  - scalar components of motion equations
  - quadratic term in continuity equation (or logarithmic term in logarithmic formulation)
- This weak nonlinearity necessities some post-processing (e.g., SHP or damping) on prognostic variables

Temporal Discretization

- Three-time-level semi-Lagrangian semi-implicit discretization
- Follow Ritchie's approach and discretize in tangential Cartesian coordinates (Ritchie, *Mon Wea Rev*, 116, 1988)

Time-Discretized SWE

$$\sin\theta \frac{d\vec{v}}{dt} + f \sin\theta\, \hat{k} \times \vec{v} + \sin\theta \vec{\nabla}\phi = 0$$

$$\frac{d\phi}{dt} + \left(\phi + \phi^*\right)\vec{\nabla} D = 0$$

$$\Downarrow$$

$$\sin\theta\left(\frac{\vec{v}^{\,n+1} - \tilde{\vec{v}}^{\,n-1}}{2\Delta t}\right) + \sin\theta\left(\frac{\vec{\nabla}\phi^{n+1} + \vec{\nabla}\tilde{\phi}^{\,n-1}}{2}\right) = -\left(f \sin\tilde{\theta}\, \hat{k} \times \vec{v}\right)^{n}$$

$$\phi^{n+1} + \Delta t \phi^* D^{n+1} = \tilde{\phi}^{\,n-1} - \Delta t \phi^* \tilde{D}^{n-1} - 2\Delta t \tilde{\phi}^{\,n} \tilde{D}^{n}$$

Time -Discretized SWE (Cont'd)

$$\Downarrow$$

algebraic manipulations,
tangential Cartesian transformation

$$\varsigma^{n+1}\sin^2\theta = \frac{1}{a}\left(-\sin\theta\frac{\partial}{\partial\theta}Q_1+\frac{\partial}{\partial\lambda}Q_2\right)\equiv L$$

$$D^{n+1}\sin^2\theta+\Delta t\sin^2\theta\nabla^2\phi^{n+1} = \frac{1}{a}\left(\frac{\partial}{\partial\lambda}Q_1+\sin\theta\frac{\partial}{\partial\theta}Q_2\right)\equiv M$$

$$\phi^{n+1}+\Delta t\phi^* D^{n+1} = \tilde{\phi}^{n-1}-\Delta t\phi^*\tilde{D}^{n-1}-2\Delta t\tilde{\phi}^{n}\tilde{D}^{n}\equiv Q_3$$

where Q1 and Q2 are functions of prognostic varibles at mid-stream and upstream points and vectors in tangential Cartesian plane.

Spatial Discretization

$$\varsigma^{n+1} \sin^2 \theta = L$$

$$D^{n+1} \sin^2 \theta + \Delta t \sin^2 \theta \nabla^2 \phi^{n+1} = M$$

$$\phi^{n+1} + \Delta t \phi^* D^{n+1} = Q_3$$

$$\Rightarrow \quad \sin^2 \theta \left(1 + \Delta t^2 \nabla^2\right) \phi^{n+1} = \sin^2 \theta Q_3 - \Delta t M$$

where $\displaystyle \xi(\lambda, \theta) = \sum_{m=-N}^{N} \xi_m(\theta) e^{im\lambda}$

$$\xi_m(\theta) = \begin{cases} \sum_{l=0}^{N} \xi_{m,l} \cos l\theta & m \; even \\ \sum_{l=0}^{N} \xi_{m,l} \sin l\theta, & m \; odd \end{cases}$$

(Yee, *Mon Wea Rev*, 1980; Orszag, *Mon Wea Rev*, 1974)

Spectral Filtering

- Use 2/3-truncation grid to prevent aliasing from quadratic term
- Apply SHP to prognostic variables at every time step by projecting variables to spherical harmonics space and back
- 6 transforms needed for each time step

Summary of Results for Standard Test Cases

- The SLSI-DF method was put through the standard test cases by Williamson et al.
- Stable for all test cases.
- First-order convergence for test case 1, owing to discontinuous second derivative in solution.
- Third-order convergence for test cases 2, 3, and 4 when cubic Lagrange interpolation was used to estimate upstream values.

Summary of Results for Standard Test Cases (Cont'd)

- Explicit, scale- and $\Delta\lambda^4$-dependent damping used in test cases 5, 6, and 7.
- Stable for these cases, but low-order convergence because solution is not smooth.
- For test cases 3 and 4, the method was stable with CFL numbers as large as 30.

Quadratic vs. Linear Truncation

- Because of the $\sin^2\theta$ operations, linear grid retains all but the highest 2 waves.
- Logarithmic formulation of the continuity equation (reduced-Gaussian grid):

$$\frac{d}{dt}\log\left(1+\frac{\phi}{\phi^*}\right)+D=0$$

$$\Rightarrow \quad \log\left(1+\frac{\phi^{n+1}}{\phi^*}\right)+\Delta t D^{n+1}=\log\left(1+\frac{\tilde{\phi}^{n-1}}{\phi^*}\right)-\Delta t\tilde{D}^{n-1}$$

(Côté and Staniforth, *Mon Wea Rev*, 1988)

Questions

- Is the logarithmic formulation projection free?
- Does the linear truncation grid generate more accurate solutions?
- Is the linear truncation grid more efficient?

Quadratic/Linear Grid Results

- The weak nonlinearity that persists in the motion equations and logarithmic term still necessitates the use of SHP.
- Linear grid yields slightly less accurate solutions than quadratic grid (!) because interpolation errors dominate.
- Results have been confirmed:
  - by truncating half of the waves—no significant reduction in accuracy
  - by replacing cubic Lagrange with quartic Lagrange interpolation—improved accuracy

Future Projects

- Replace spherical harmonics projections with artificial viscosity?
- A two-time-level SLSI formulation

http://www.amath.unc.edu/Faculty/layton/research/swe/dfourier/