

# A Semi-Lagrangian Double Fourier Method for the Shallow Water Equations

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# Motivation

- Goal: To construct a stable and efficient numerical method for solving SWE (and more comprehensive models) on a sphere.
- Spectral transform method is  $O(N^3)$
- To maintain stability, some methods require spherical harmonic projections (SHP), which take  $O(N^3)$  time.

# Double Fourier Series

- Based on Fourier series (sines and cosines) rather than spherical harmonics
- FFT in both longitudinal and latitudinal direction,  $O(N^2 \log(N))$  time
- With Eulerian time-stepping, the double Fourier method requires SHP or damping to be stable because
  - Discontinuities at the poles
  - Nonlinear terms give rise to non-isotropic waves(Cheong and collaborators, *JCP*, 2000)

# Semi-Lagrangian Double Fourier Method—Projection Free?

- Nonlinearity in SWE arises mostly from advection terms, which are implicit in a semi-Lagrangian formulation
- *Question*: Can the damping introduced by spatial interpolations used in a semi-Lagrangian double Fourier method be sufficient to maintain stability?

# Unfortunately...

- Weak nonlinearity still persists in
  - scalar components of motion equations
  - quadratic term in continuity equation (or logarithmic term in logarithmic formulation)
- This weak nonlinearity necessitates some post-processing (e.g., SHP or damping) on prognostic variables

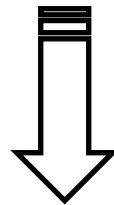
# Temporal Discretization

- Three-time-level semi-Lagrangian semi-implicit discretization
- Follow Ritchie's approach and discretize in tangential Cartesian coordinates (Ritchie, *Mon Wea Rev*, 116, 1988)

# Time-Discretized SWE

$$\sin \theta \frac{d\vec{v}}{dt} + f \sin \theta \hat{k} \times \vec{v} + \sin \theta \vec{\nabla} \phi = 0$$

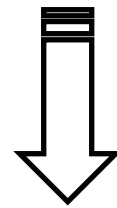
$$\frac{d\phi}{dt} + (\phi + \phi^*) \vec{\nabla} D = 0$$



$$\sin \theta \left( \frac{\vec{v}^{n+1} - \tilde{\vec{v}}^{n-1}}{2\Delta t} \right) + \sin \theta \left( \frac{\vec{\nabla} \phi^{n+1} + \vec{\nabla} \tilde{\phi}^{n-1}}{2} \right) = - \left( f \sin \tilde{\theta} \hat{k} \times \vec{v} \right)^n$$

$$\phi^{n+1} + \Delta t \phi^* D^{n+1} = \tilde{\phi}^{n-1} - \Delta t \phi^* \tilde{D}^{n-1} - 2\Delta t \tilde{\phi}^n \tilde{D}^n$$

# Time -Discretized SWE (Cont'd)



algebraic manipulations,  
tangential Cartesian transformation

$$\zeta^{n+1} \sin^2 \theta = \frac{1}{a} \left( -\sin \theta \frac{\partial}{\partial \theta} Q_1 + \frac{\partial}{\partial \lambda} Q_2 \right) \equiv L$$

$$D^{n+1} \sin^2 \theta + \Delta t \sin^2 \theta \nabla^2 \phi^{n+1} = \frac{1}{a} \left( \frac{\partial}{\partial \lambda} Q_1 + \sin \theta \frac{\partial}{\partial \theta} Q_2 \right) \equiv M$$

$$\phi^{n+1} + \Delta t \phi * D^{n+1} = \tilde{\phi}^{n-1} - \Delta t \phi * \tilde{D}^{n-1} - 2\Delta t \tilde{\phi}^n \tilde{D}^n \equiv Q_3$$

where Q1 and Q2 are functions of prognostic variables at mid-stream and upstream points and vectors in tangential Cartesian plane.



# Spatial Discretization

$$\zeta^{n+1} \sin^2 \theta = L$$

$$D^{n+1} \sin^2 \theta + \Delta t \sin^2 \theta \nabla^2 \phi^{n+1} = M$$

$$\phi^{n+1} + \Delta t \phi * D^{n+1} = Q_3$$

$$\Rightarrow \sin^2 \theta (1 + \Delta t^2 \nabla^2) \phi^{n+1} = \sin^2 \theta Q_3 - \Delta t M$$

where  $\xi(\lambda, \theta) = \sum_{m=-N}^N \xi_m(\theta) e^{im\lambda}$

$$\xi_m(\theta) = \begin{cases} \sum_{l=0}^N \xi_{m,l} \cos l\theta & m \text{ even} \\ \sum_{l=0}^N \xi_{m,l} \sin l\theta, & m \text{ odd} \end{cases}$$

(Yee, *Mon Wea Rev*, 1980; Orszag, *Mon Wea Rev*, 1974)

# Spectral Filtering

- Use 2/3-truncation grid to prevent aliasing from quadratic term
- Apply SHP to prognostic variables at every time step by projecting variables to spherical harmonics space and back
- 6 transforms needed for each time step

# Summary of Results for Standard Test Cases

- The SLSI-DF method was put through the standard test cases by Williamson et al.
- Stable for all test cases.
- First-order convergence for test case 1, owing to discontinuous second derivative in solution.
- Third-order convergence for test cases 2, 3, and 4 when cubic Lagrange interpolation was used to estimate upstream values.

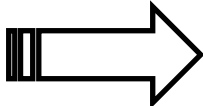
# Summary of Results for Standard Test Cases (Cont'd)

- Explicit, scale- and  $\Delta\lambda^4$ -dependent damping used in test cases 5, 6, and 7.
- Stable for these cases, but low-order convergence because solution is not smooth.
- For test cases 3 and 4, the method was stable with CFL numbers as large as 30.

# Quadratic vs. Linear Truncation

- Because of the  $\sin^2\theta$  operations, linear grid retains all but the highest 2 waves.
- Logarithmic formulation of the continuity equation (reduced-Gaussian grid):

$$\frac{d}{dt} \log \left( 1 + \frac{\phi}{\phi^*} \right) + D = 0$$


$$\log \left( 1 + \frac{\phi^{n+1}}{\phi^*} \right) + \Delta t D^{n+1} = \log \left( 1 + \frac{\tilde{\phi}^{n-1}}{\phi^*} \right) - \Delta t \tilde{D}^{n-1}$$

(Côté and Staniforth, *Mon Wea Rev*, 1988)

# Questions

- Is the logarithmic formulation projection free?
- Does the linear truncation grid generate more accurate solutions?
- Is the linear truncation grid more efficient?

# Quadratic/Linear Grid Results

- The weak nonlinearity that persists in the motion equations and logarithmic term still necessitates the use of SHP.
- Linear grid yields slightly less accurate solutions than quadratic grid (!) because interpolation errors dominate.
- Results have been confirmed:
  - by truncating half of the waves—no significant reduction in accuracy
  - by replacing cubic Lagrange with quartic Lagrange interpolation—improved accuracy

# Future Projects

- Replace spherical harmonics projections with artificial viscosity?
- A two-time-level SLSI formulation

<http://www.amath.unc.edu/Faculty/layton/research/swe/dfourier/>