Local Models and Displays

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The problem:

Express the completed local ring of a point on a PEL moduli space so that strata defined in terms of Frobenius, or in terms of endomorphisms, can be calculated.

Rephrased:

How to write the universal display in the case of a PEL Shimura variety?

We shall present a method to do that, and also be able to reconcile the crystalline and the display approach to deformation theory.

Throughout we deal with positive char. p and equi-characteristic deformations to local artinian k-algebras with residue field k

Well Known Examples

Let A/k be a g-dimensional abelian variety in char. p.

Siegel case: A is principally polarized.

Hilbert case: $\mathcal{O}_L \hookrightarrow \mathrm{End}(A)$, where L is totally real, $[L:\mathbb{Q}] = g$ (+ pol'n datum...).

Quaternion case: $g = 2$, $\mathcal{O} \hookrightarrow \mathrm{End}(A)$, $\mathcal{O}$ is a maximal order of an indefinite quaternion algebra $B/\mathbb{Q}$.

The Siegel case

The deformation space as PPAV is

$$\mathrm{Spf}\ k[[t_{ij} : 1 \le i, j \le g]]/(\{t_{ij} - t_{ji}\}).$$

Take sympl. basis $\{x_1, \ldots, x_g, y_1, \ldots, y_g\}$ to $H^1_{crys}(A/W(k))$, such that

$$x_i \in \mathrm{Ker}(V),\ y_i \in \mathrm{Ker}(F) \pmod{p}.$$

If Frobenius on $H^1_{crys}(A/W(k))$ is given by $\left(\begin{smallmatrix} A & pB \\ C & pD \end{smallmatrix}\right)$ then the universal Frobenius is

$$\begin{pmatrix} A + TC & pB + pTD \\ C & pD \end{pmatrix}, \quad T = (T_{ij}),$$

where T_{ij} is the Teichmuller lift of t_{ij}.

($A + TC \pmod{p}$ is the Hasse-Witt matrix.)

Note that in this perspective the kernel of Frobenius mod p is *constant*.

The inert Hilbert case

As in the Siegel case, only that one chooses the x_i and the y_i to be eigenvectors for the $\mathcal{O}_L \otimes k \cong k^g$ action. The defor. space is

$$\text{Spf } k[[t_{ij} : 1 \leq i, j \leq g]]/(\{t_{ij} : i \neq j\}) \\ = \text{Spf } k[[t_1, \ldots, t_g]].$$

Frobenius is given as above (A, B, C, D are sub-diagonal), but T is diagonal.

The maximally ramified Hilbert case

Write $\mathcal{O}_L \otimes k = k[T]/(T^g)$, and let $i \geq j$ be such that $H^0(A, \Omega^1_{A/k}) = (T^i)\alpha \oplus (T^j)\beta$, where $\alpha \wedge \beta$ is a unit. Let

$a = \sum_{s=0}^{i-1} a_s T^s, c = \sum_{s=0}^{i-1} c_s T^s,$

$b = \sum_{s=0}^{j-1} b_s T^s, d = \sum_{s=0}^{j-1} d_s T^s.$

Then the defor. ring is

$$\mathrm{Spf}\, k[[a,b,c,d]]/((T^i\alpha + a\alpha + b\beta) \wedge (T^j\beta + c\alpha + d\beta)).$$

The universal Frobenius is given w.r.t. the basis $\{\alpha, \beta\}$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1 + T^{-j} d^\sigma & -T^{-j} c^\sigma \\ -T^{-i} b^\sigma & 1 + T^{-i} a^\sigma \end{pmatrix},$$

where T is a lift of T to $\mathcal{O}_L \otimes W(k)$.

Note that there are $2g$ variables and g equations $\rightsquigarrow$ much more efficient than getting it from the Siegel case ($g(g+1)/2$ variables); also, evident that l.c.i..

The quaternion case

Assume p ramifies in B. Let $\pi \in \mathcal{O} \otimes \mathbb{Z}_p$ such that $\pi^2 = p$, conjugation by π induces a non-trivial automorph. on $W(\mathbb{F}_{p^2}) \subset \mathcal{O} \otimes \mathbb{Z}_p$, and

$$\mathcal{O}/p\mathcal{O} = \mathbb{F}_{p^2} \oplus \mathbb{F}_{p^2}\pi.$$

This can be related to (any) inert Hilbert case $\mathcal{O}_L \otimes \mathbb{Z}_p = W(\mathbb{F}_{p^2})$. Let $R = \mathcal{O}/p\mathcal{O}$.

Case I. If $H^0(A, \Omega^1_{A/k}) \cong \mathbb{F}_{p^2}\pi \otimes_{\mathbb{F}_p} k$ as an R-module then the universal deformation ring is defined by the ideal $(t_1 t_2)$ and Frobenius is given by restriction.

Case II. If $H^0(A, \Omega^1_{A/k}) \cong (\mathbb{F}_p \oplus \mathbb{F}_p\pi) \otimes_{\mathbb{F}_p} k$ then the deformation space is smooth and given by either (t_1) or (t_2) according to the representation of $\mathcal{O}_L$ on $\mathbb{F}_p$.

It can also be related to the maximally ramified Hilbert case. **Case I** is the case $j = 1$ and $a_0 = d_0 = 0, b_0c_0 = 0$, and **Case II** is $j = 0$ and $a_0 = a_1 = c_0 = 0$.

How to calculate in general the universal Frobenius and get a description for the endomorphisms?

Grothendiecks' Theorem (special case).

Let R be an artinian local k-algebra with residue field k, such that $\mathfrak{m}_R$ is equipped with divided powers structure. There is an equivalence of categories

abelian schemes over R	$\leftrightsquigarrow$	(A, Fil^1), A/k ab. var., Fil^1 a free direct factor of $\mathbb{D}^*(A)_R$ extending the Hodge filtration on A

Here $\mathbb{D}^*(A)$ is the Grothendieck-Messing crystal whose value on R can be calculated as $H^1_{dR}(A'/R)$, where A' is any deformation of A to R.

Local Models

Grothendieck's theorem implies that at least for some level of truncation the completed local ring of a point on a PEL moduli variety, that is "p-prudent", and the completed local ring of a point on a suitable flag variety are isomorphic.

That this holds without truncation was shown for various moduli problems and for various level and endomorphisms structures (De Jong, Deligne, Pappas, Rapoport, Zink and others). However, it is important to note that the isomorphism is not canonical.

Examples

Local structure of the Grassmann variety $G(g,V)$, $G(\langle -,-\rangle; g, V)$, $G(E, \langle -,-\rangle; g, V)\dots$

Let $W \subset V$ be a point. Choose U such that

$$V = W \oplus U.$$

An affine neighborhood of W is

$$\mathrm{Hom}(W,U) = \mathrm{Hom}(W, V/W)$$

(to f associate its graph).

If we have a perfect symplectic pairing on V such that U, W are isotropic, then an affine neighborhood is

$$\mathrm{Hom}(W,U)^{symm} = \mathrm{Hom}(W, V/W)^{symm}$$

If we have endomorphisms E this may fails. It is not always true that an affine neighborhood is given by $\mathrm{Hom}_E(W, V/W)$ (as the example of the ramified Hilbert case shows). However, it is true that the tangent space is given by $\mathrm{Hom}_E(W, V/W)$, which in our applications is $\mathrm{Hom}_E(\mathfrak{t}_A^\vee, \mathfrak{t}_{A^t})$.

The Siegel case

An affine neighborhood is $\mathrm{Hom}(W, U)^{symm}$ and upon choosing coordinates we get that the completed local ring is

$$\mathrm{Spf}\ k[[t_{ij} : 1 \leq i, j \leq g]]/(\{t_{ij} - t_{ji}\}).$$

The Hilbert inert case

In this case $\mathcal{O}_L \otimes k \cong k^g$ and the conditions imposed in the moduli problem give that both $\mathfrak{t}_A^\vee, \mathfrak{t}_{A^t}$ are free modules of rank 1 over $\mathcal{O}_L \otimes k$. We find that the completed local ring is

$$\mathrm{Spf}\ k[[t_i : 1 \leq i \leq g]].$$

The Hilbert maximally ramified case

Choosing a basis α, β to $H^1_{dR}(A/k)$, we have

$$H^0(A, \Omega^1_{A/k}) = (T^i)\alpha \oplus (T^j)\beta$$

and we choose

$U = \text{Span}(\alpha, T\alpha, \ldots, T^{i-1}\alpha, \beta, T\beta, \ldots, T^{j-1}\beta)$.

Note that U is usually not $\mathcal{O}_L$ invariant.

A map f is described by its effect on the generators:

$T^i\alpha \mapsto T^i\alpha + a\alpha + b\beta,\ T^j\beta \mapsto T^j\beta + c\alpha + d\beta$.

The condition of the pairing is that

$(T^i\alpha + a\alpha + b\beta) \wedge (T^j\beta + c\alpha + d\beta) = 0$

and it turns out that this implies the image is also $\mathcal{O}_L$ invariant.

About universal objects

The local model tells us over which ring the universal display lives. Once we construct a display how do we know it's universal?

Suppose a functor $\mathbf{F}$ is representable by an object $\mathscr{M}$. Then exists

$$h : \mathrm{Mor}(-, \mathscr{M}) \xrightarrow{\cong} \mathbf{F}(-).$$

Given h, we get a universal object

$$h(\mathscr{M} \xrightarrow{\mathrm{id}} \mathscr{M}).$$

The set of (universal) objects is a p.h.s. under End($\mathscr{M}$) (resp. Aut($\mathscr{M}$)).

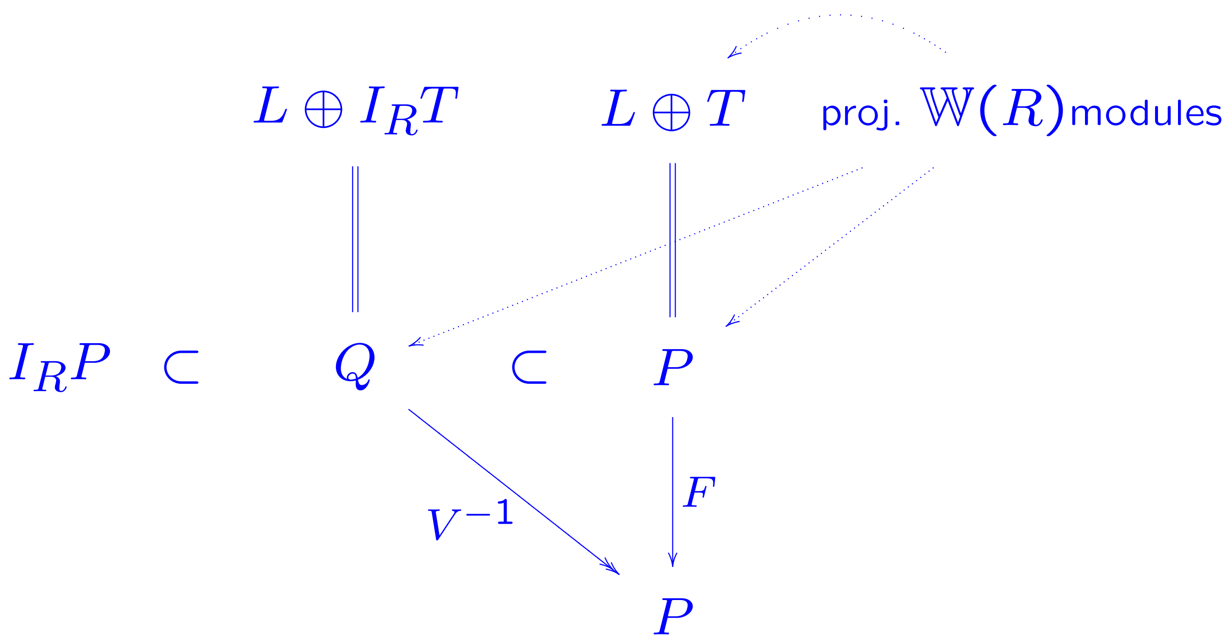
In the situation at hand

$$\mathscr{M} = \mathrm{Spf}\widehat{\mathcal{O}_x},$$

where $\widehat{\mathcal{O}_x}$ is a complete noetherian local ring. We shall use that for such a ring R, $f : R \longrightarrow R$, a local homomorphism, is an isomorphism if $f(\mathfrak{m}_R/\mathfrak{m}_R^2) = \mathfrak{m}_R/\mathfrak{m}_R^2$.

Displays

Let R be a ring and I_R the augmentation ideal of $\mathbb{W}(R)$. A display over R:



For $x \in P, w \in \mathbb{W}(R), V^{-1}({}^V wx) = wFx$.

One also imposes a certain nilpotency condition and other technical conditions.

If A/k is an abelian variety over a perfect field and M is its Dieudonné module then (M, VM, F, V^{-1}) is a display.

One can recover the Grothendieck crystal $\mathbb{D}^*$ from the theory of displays.

One can recover the Cartier-Dieudonné module of formal curves from the display.

Yet, it is amenable to computations.

Zink's Theorem:

Let R be an excellent local ring (e.g. a complete local ring), or a ring such that R/pR is an algebra of finite type over a field k. Assume that p is nilpotent in R. Then the category of displays $\mathscr{P}$ over R is equivalent to the category of formal p-divisible groups $\mathscr{G}$ over R.

Calculating the universal display

♢ It lives over the completed local ring coming from the local model which is $G(E, \langle -, - \rangle, g, V)$.

♢ Every display $D = (P, Q, F, V^{-1})$ has a Hodge filtration

$$0 \subset Q/I_R P \subset P/I_R P.$$

♢ Let $D_0 = (P_0, Q_0, F_0, V_0^{-1})$ be a display over k. There is a universal display $D_0^{\text{univ}} = (P_0^{\text{univ}}, Q_0^{\text{univ}}, F_0^{\text{univ}}, V_0^{-1\text{univ}})$, whose Hodge filtration agrees, at least on the tangent space, with the universal flag of the Grassmannian. Meaningful because of crystalline nature of displays that gives over the crystalline site, so for $k[\epsilon]$ deformations, an identification $P_0^{\text{univ}} = P_0^{\text{triv}}$.

$\diamondsuit$ The specialization map of the universal display to any other display induces a specialization of Hodge filtrations.

Some clues

In the Siegel/Hilbert unramified case we have

$$\begin{aligned}\begin{pmatrix} A+TC & B+TD \\ C & D \end{pmatrix} &= \begin{pmatrix} I & T \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \\ &= \begin{pmatrix} I & T \\ 0 & I \end{pmatrix} \left[\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I & T^\sigma \\ 0 & I \end{pmatrix} \right] \begin{pmatrix} I & -T^\sigma \\ 0 & I \end{pmatrix}\end{aligned}$$

That is, the universal Frobenius can be thought of as $F \circ \psi^{-1}$, where F is the original Frobenius, ψ is given by $\begin{pmatrix} I & -T \\ 0 & I \end{pmatrix}$, and we perform a change of basis. Note: the kernel of $F \circ \psi^{-1}$ modulo p is the universal flag and the change of basis trivializes it.

In the Hilbert maximally ramified case

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1 + T^{-j} d^\sigma & -T^{-j} c^\sigma \\ -T^{-i} b^\sigma & 1 + T^{-i} a^\sigma \end{pmatrix}.$$

A technical lemma

Let $D = (P, Q, F, V^{-1})$ be a display over the completed local ring of the local model (for subspaces of P_0/pP_0), deforming D_0, and such that its Hodge filtration is the universal flag on the level of the tangent space (mod $\mathfrak{m}_R^2$), under the canonical identification between P and P_0^{triv} over this thickening. Then D is a universal display.

(We denote by $D_0^{\text{triv}} = (P_0^{\text{triv}}, Q_0^{\text{triv}}, F_0^{\text{triv}}, V_0^{-1\text{triv}})$ the trivial deformation of the display D_0).

Example of a general theorem (we hope...)

In the Hilbert maximally ramified case indeed

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1+T^{-j}d^\sigma & -T^{-j}c^\sigma \\ -T^{-i}b^\sigma & 1+T^{-i}a^\sigma \end{pmatrix}$$

is a universal display.

The twist here is that the map

$$\begin{pmatrix} 1+T^{-j}d^\sigma & -T^{-j}c^\sigma \\ -T^{-i}b^\sigma & 1+T^{-i}a^\sigma \end{pmatrix}$$

is only defined on $P_0^{\text{triv}} \otimes \mathbb{Q}$. Yet it works. One check that it takes Q_0^{triv} to a submodule that reduces mod I_R to precisely the universal flag and one can prove that all the conditions required of display are satisfied.

Note that the original approach to displays, in which the Hodge filtration is constant, turns out to be a Herring, because, for example, in the case at hand it is impossible to achieve such a display on which the $\mathcal{O}_L$ action is extended linearly from the special fiber.

A easy sample application

The structure of the supersingular locus for $g = 2$ around a superspecial point in the maximally ramified Hilbert case.

One can prove that one can take

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix},$$

if $j = 0$, resp. $j = 1$.

The completed local ring is accordingly

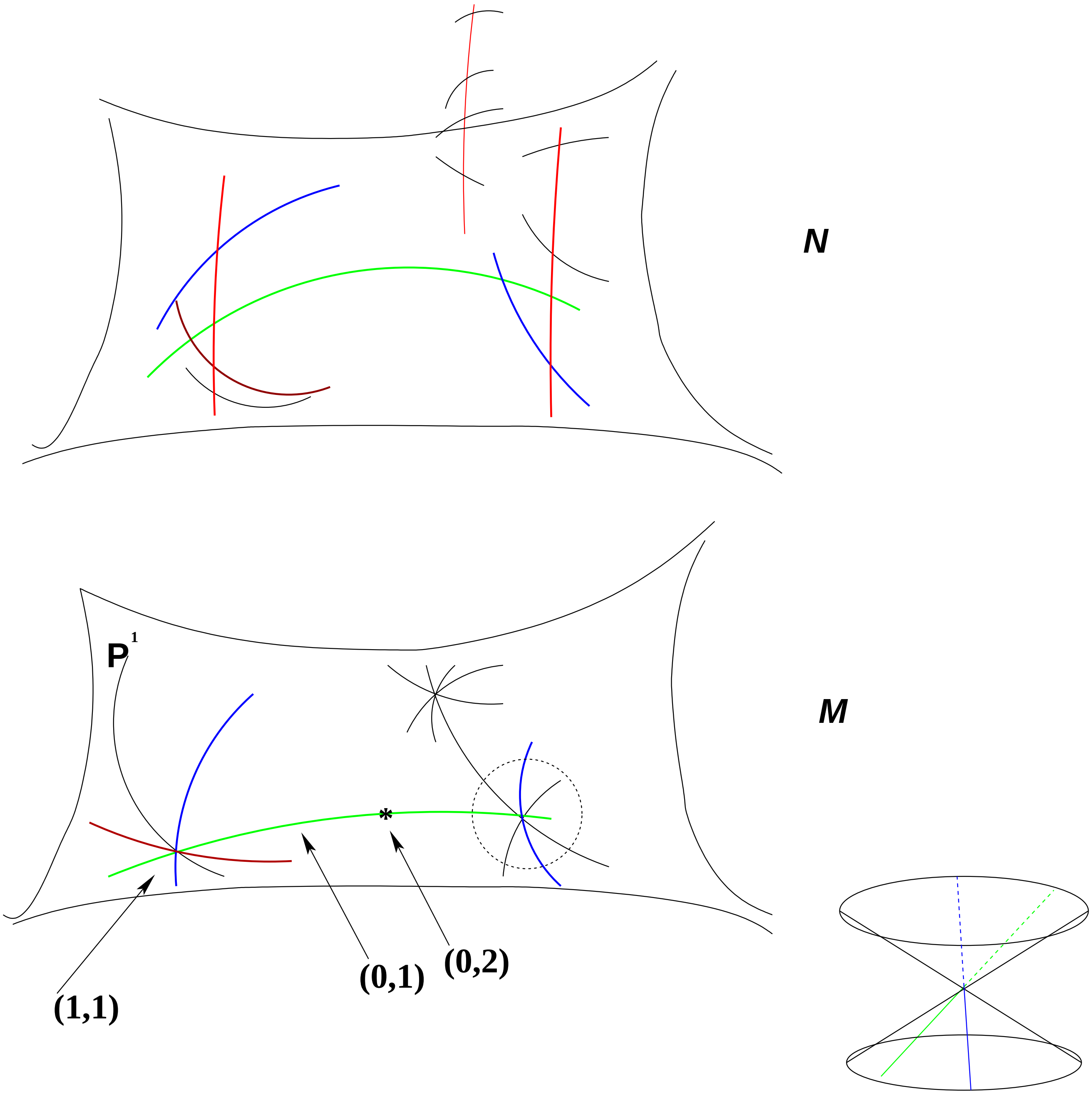
$\mathrm{Spf} k[[c_0, c_1]]$, $\mathrm{Spf} k[[a_0, b_0, c_0, d_0]]/(a_0 + d_0, a_0 d_0 - b_0 c_0)$

and the universal Frobenius mod I_R is accordingly

$$\begin{pmatrix} 0 & 0 \\ 1 & -c_0 - c_1 T \end{pmatrix}, \quad \begin{pmatrix} -b_0^p & a_0^p + T \\ d_0^p + T & -c_0^p \end{pmatrix}.$$

One checks from first principles that a deformation is supersingular iff TF^2 is zero mod p.

This implies in the first case that $c_0 = 0$ and in the second case a certain set of equations defining $p+1$ lines through the origin on the cone $a_0 + d_0, a_0 b_0 - c_0 d_0$.


N
M
P1
*
(1,1)
(0,1)
(0,2)