Debt allocation:

To fix or float?

Svein-Arne Persson
Norwegian School of Economics

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Motivation:

Stylized facts (Norway):

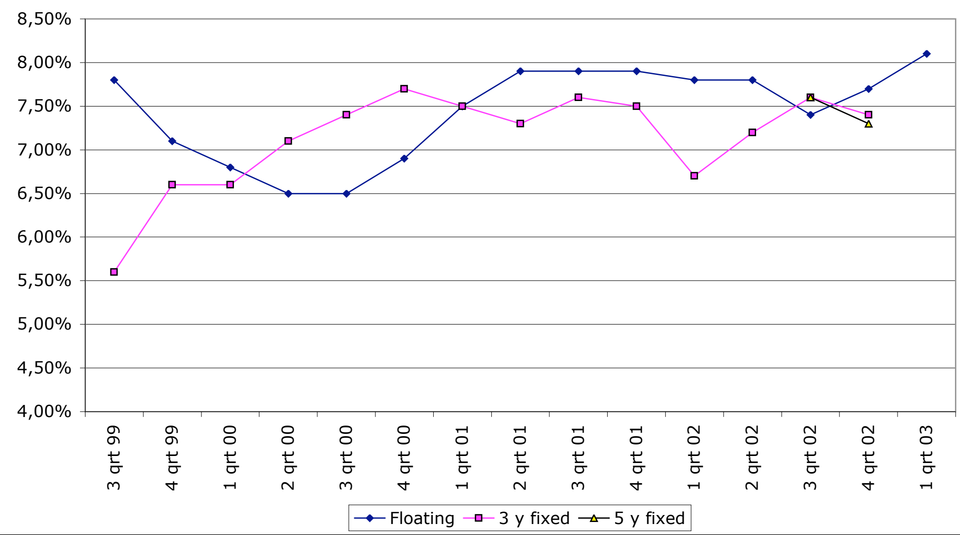
- Most private homes financed by *floating* interest rates the last 20 years.
- Last 5 years major banks introduce *fixed rate loans* alternatives.
- Floating rate loans still dominate the market.
- Advice from experts flourish in media.

Example 1: State Education Loan Fund - foundation of student financial aid in Norway, government run organization under the Ministry of Education. (`www.lanekassen.no`)

- interest rates are determined by the financial market

- customer has to choose *either*
  - 3 year fixed (Oct 1, 2002: 7.4%)
  - 5 year fixed (Oct 1, 2002: 7.3%) *or*
  - floating interest rates (Jan 1, 2003: 8.1%) (may be changed quarterly)

Interest rate development 1999-2002



Example 2: Postbanken

`www.postbanken.no`

- conditions depend on whether amount of loan is within 60% or 80% of market value.

- floating rates depend on whether loan amount is above or below NOK 500 000.

- fixed rate loans of 3, 5 or 10 years maturity

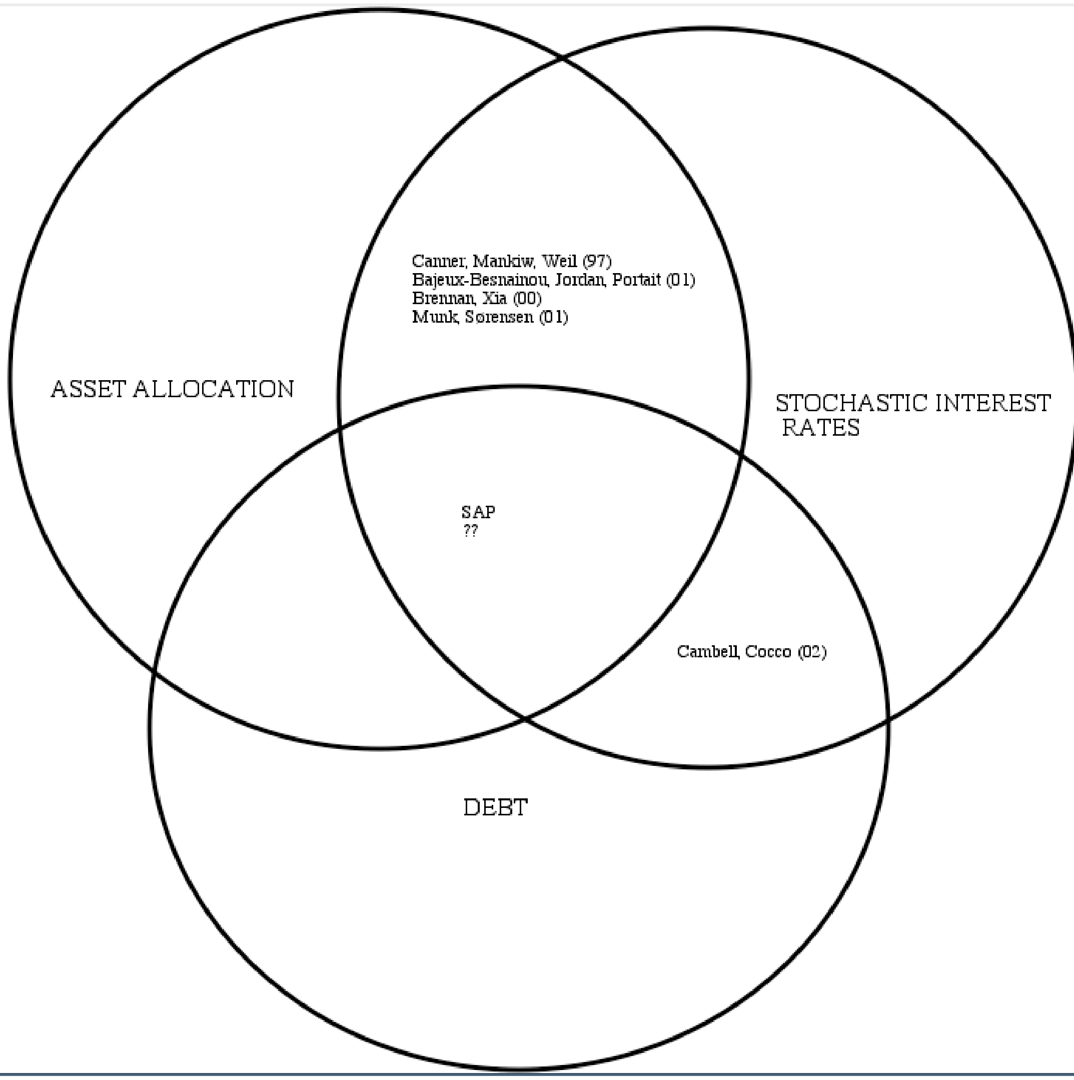
Conditions as of Oct 30, 2002.

	floating		fixed		
	$>.5$m	$<.5$m	3 y	5 y	10 y
$<60\%$	8.1%	8.45%	7.45%	7.40%	7.40%
$<80\%$	8.85%	9.15%	7.9%	7.85%	7.85%

Agenda:

- Introduction (done!)
- Literature
- Term structure model
- The agent's problem
- Static problem ("buy-and-hold")
- Dynamic problem (continuous rebalancing)
- Numerical comparison

Literature


Canner, Mankiw, Weil (97)
Bajeux-Besnainou, Jordan, Portait (01)
Brennan, Xia (00)
Munk, Sorensen (01)
ASSET ALLOCATION
STOCHASTIC INTEREST RATES
SAP
??
Cambell, Cocco (02)
DEBT

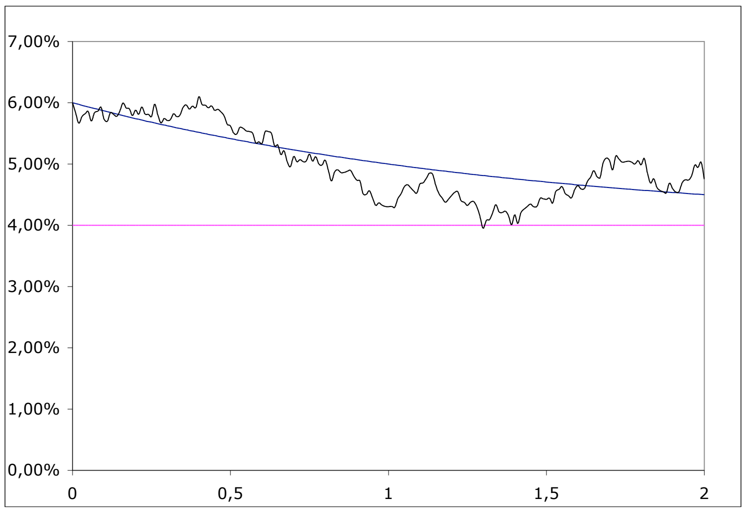
Choice of term structure model

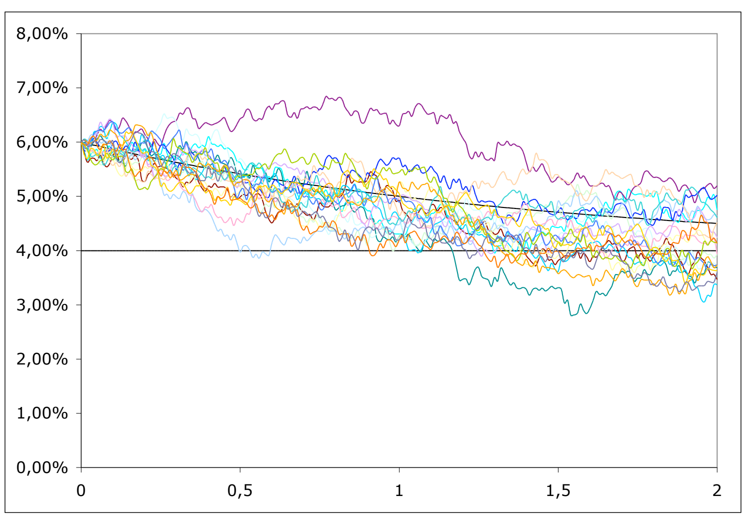
- We model stochastically fluctuating spot interest rates by an Ornstein-Uhlenbeck process

$$dr_t = q(m - r_t)dt + vdB_t,$$

where v, q, and m are constants and the initial value $r_s = r$ (constant). First used in financial economics by Vasicek (1977).

OU-processes: $q = \ln(2)$, $r = 6\%$, $m = 4\%$, $v = 0.01$.





The Hull-White model

- $P_{s,\tau}$- time s market price of a default free unit discount bond expiring at time τ.

- Vasicek (77):
  $P_{s,\tau} = P(\lambda(\cdot))$,
  $\lambda(\cdot)$ the market price of interest rate risk.

- Hull-White (90):
  $P(\lambda(\cdot)) = e^{-\int_s^\tau f_s(t)dt}$,
  $f_s(t)$ time s forward rate for time t.

- $\lambda(\cdot)$ depends on
  – v, q, m (interest rate process)
  – $f_s(t)$, $\frac{\partial}{\partial t} f_s(t)$ (initial term structure)
  $\lambda_s(t) = \frac{qm}{v} - \frac{1}{v}\left[q f_s(t) + \frac{\partial}{\partial t} f_s(t)\right] - \frac{v}{2q}(1 - e^{-2q(t-s)})$.

Set-up
(seminal Merton 1969, 1971 model)

Utility over terminal wealth (time T) only given by

$$u(x) = \frac{1}{1-\rho} x^{1-\rho}.$$

- relative risk aversion $\rho = \frac{-u''(x)}{u'(x)} x$.
- $\rho > 0$
- $\rho = 1$ corresponds to $u(x) = \ln(x)$.

Set-up

- floating rate interest equal to spot rate r_t.
- fixed rate until time T follows from forward rates $f_t(s)$ as
$$r_s^x = \frac{1}{T-s}\int_s^T f_s(t)dt.$$
- initial (time s) amount of debt $D_s = D$.
- deterministic time T wealth $\bar{W}$ (collateral!).
- all interest payments take place at the horizon T.

Static problem
No intermediate rebalancing of debt
("buy-and -hold")

- Let $R = \int_s^T r_t dt$

- $R \sim N(\mu, \sigma^2)$ (Gaussian)

- α is the fraction of floating rate debt (amount of floating rate debt $= D^L = \alpha D$)

- Let L denote the *wealth-to-debt ratio*

$$L = \frac{\bar{W} e^{-r_s^x (T-s)} - D}{D}.$$

- Terminal wealth

$$\begin{aligned} W_T &= \bar{W} - \alpha D e^R - (1-\alpha) D e^{r_s^x (T-s)} \\ &= D L e^{r_s^x (T-s)} (1 + \frac{\alpha}{L} (1 - e^{R - r_s^x (T-s)})). \end{aligned}$$

Investor's problem:

$$\max_{\alpha} E\left[u(W_T)\right].$$

The first order condition of this problem is

$$E\left[u'(W_T)(e^{r_s^x(T-s)} - e^R)\right] = 0. \qquad \text{(A)}$$

or using assumed CRRA utility,

$$E\left[(1 + \frac{\alpha}{L}(1 - e^{R - r_s^x(T-s)})^{-\rho}(1 - e^{R - r_s^x(T-s)})\right] = 0. \qquad \text{(B)}$$

Lower *fixed* rate bound for floating rate *debt*.

Study first order condition (A) for the case $\alpha = 0$ (Huang and Litzenberger(1988)):

$$r_s^x > r_L = \frac{1}{T-s}(\mu + \frac{1}{2}\sigma^2).$$

- If $r_s^x > r_L$ it is optimal to borrow to the FLOATING rate (include floating rate debt).
- The agent *lends* instead of *borrows* to the FLOATING rate if $r_s^x \leq r_L$.
- The condition is independent of specific utility function $u(x)$.

Upper fixed rate bound for fixed rate debt.

Study first order condition (B) for the case $\alpha = 1$. Define $Z = \frac{\bar{W}}{D} - e^{R}$.

$$r_s^x < r_U = \frac{1}{T-s}\left[\ln\left(\frac{\bar{W}}{D} - \frac{E[Z^{1-\rho}]}{E[Z^{-\rho}]}\right)\right]$$

- If $r_s^x < r_U$ it is optimal to borrow at the fixed rate (inlude fixed rate debt).

- If $r_s^x \geq r_U$ it is optimal to *lend* instead of *borrow* to the fixed rate (*buy* bonds instead of *issue* bonds).

- Here r_U depends on the chosen CRRA utility function.

- Condition also depends on agent characteristics such as $\frac{\bar{W}}{D}$ and ρ.

For a risk neutral agent $r_L = r_U$.

Proof:
Insert $\rho = 0$ in the previous expression for r_U.

A risk neutral agent chooses either fixed rate loan or floating rate loan, never a combination of both.

First order condition (B)

$$E\left[(1+\frac{\alpha}{L}(1-e^{R-r_s^x(T-s)})^{-\rho}(1-e^{R-r_s^x(T-s)})\right]=0. \quad \text{(B)}$$

- optimal α (α^*) proportional to wealth-to-debt ratio L
- constant relative risk aversion?

A reformulation.

- α^* depends on L in previous formulation

- express floating rate debt as a fraction β of market value of wealth
  $(D^L = \beta W = \beta(\bar{W} e^{-r_s^x(T-s)} - D))$

- $W_T = (\bar{W} - D e^{r_s^x(T-s)})(1 + \beta(1 - e^{R - r_s^x(T-s)}))$

- First order condition (C)

$$E\left[(1 + \beta(1 - e^{R - r_s^x(T-s)})^{-\rho}(1 - e^{R - r_s^x(T-s)})\right] = 0.$$

- NOT dependent on wealth-to-debt ratio L!

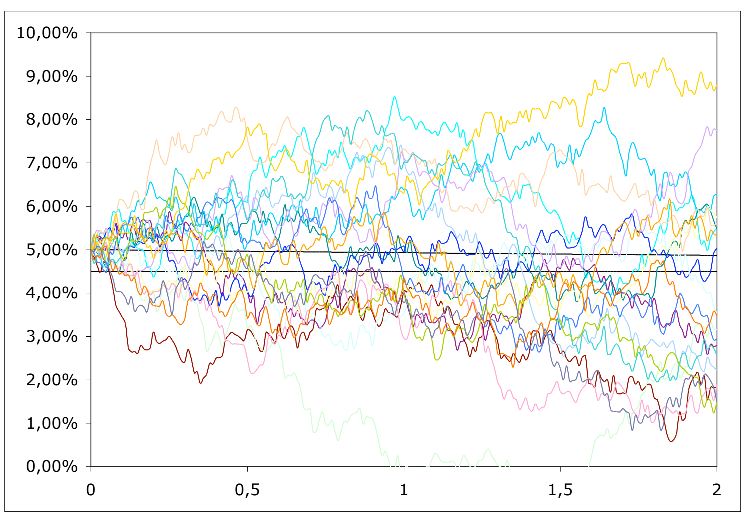
- connection between α^* and β^*:

$$\alpha^* = \beta^* L$$

The assumed parameter values of the spot interest rate with P dynamics

$$dr_t = q(m - r_t)dt + vdB_t$$

$$r = 5\%,\ q = 15\%,\ m = 4.5\%,\ v = 2\%$$



Optimal β as a function of ρ

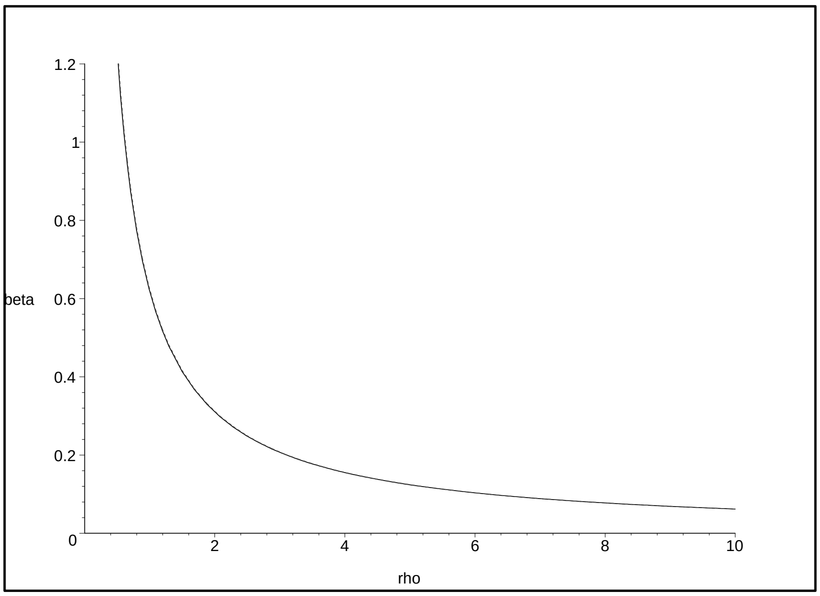


Table 1

Optimal β (β^*) and expected utility.

	$\rho = 1$	$\rho = 2$	$\rho = 4$
β^*	0.619	0.310	0.155
$E[U(W_T^*)]$	0.1505	-0.8605	-0.2125

The time horizon is $T = 3$, and the fixed rate is $r_0^x = 5\%$.

Dynamic problem
Continuous (costless) rebalancing of debt

Methodology:

- Martingale formulation
  Pliska(1986), Cox and Huang (1989) as extended Munk and Sørensen (2001)

- Stochastic control problem
  Merton (71)

Elements of the set-up

- Market value at time $t \leq T$ of time T wealth is
$$W_t = \bar{W} P_{t,T} - D_t.$$

- α_t fraction of FLOATING rate debt at time t

- Wealth process (α formulation)
$$\begin{aligned} dW_t = & [(r_t + b_{t,T})W_t + \alpha_t b_{t,T} D_t]dt \\ & + a_{t,T}[W_t + \alpha_t D_t]dB_t \end{aligned}$$

- floating rate as a fraction β_t fraction of wealth W_t at time t

- Wealth process (β formulation)
$$dW_t = [(r_t + b_{t,T}(1+\beta_t)]W_t dt + a_{t,T}(1+\beta_t)W_t dB_t$$

Solution (optimal indirect utility) of problem

$$J_s = \frac{1}{1-\rho}\left[\left(\frac{W_s}{P_{s,T}}\right)^{1-\rho} e^{\frac{1}{2}\frac{1-\rho}{\rho}V_{s,T}^2}\right] \text{ for } \rho \neq 1,$$

$$J_s = \ln\left(\frac{W_s}{P_{s,T}}\right) + \frac{1}{2}V_{s,T}^2 \text{ for } \rho = 1$$

where

$$V_{s,T}^2 = \mathrm{Var}\left(\int_s^T r_t dt + \int_s^T \lambda_s(t) dB_t | \mathcal{F}_s\right).$$

- Martingale formulation: Munk and Sørensen (2001)

- Stochastic control: Brennan and Xia (2000)

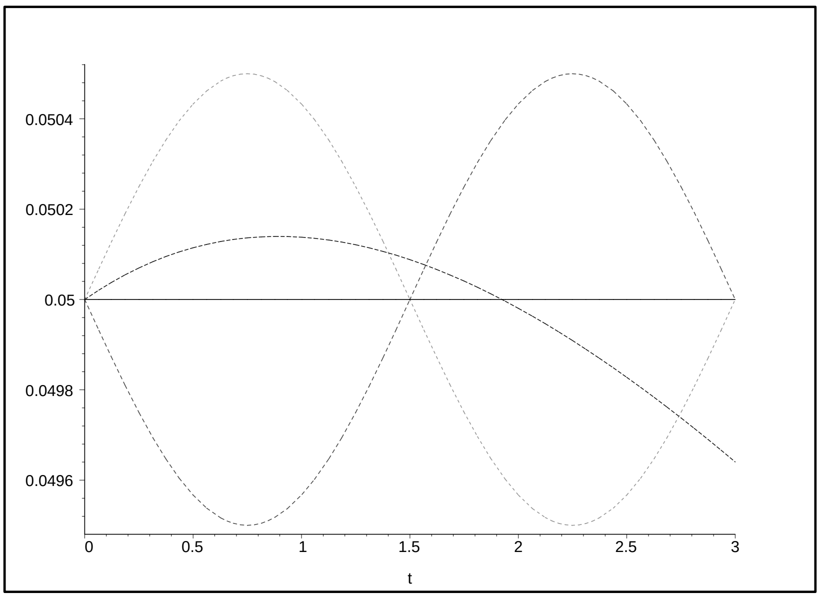
Solution (optimal fractions of floating rate debt) of problem

$$\beta_t = \frac{1}{\rho}\left(\frac{\lambda_s(t)}{a_{t,T}} - 1\right),$$

$$\alpha_t = \beta_t L_t.$$

- Both J_s (through $V_{s,T}^2$) and β_t depends on the market price of interest rate risk which again depends on the initial forward rates.

4 examples of initial term structures



- constant λ (initially increasing)
- constant
- initially incresing
- initially decreasing

Comparison of optimal expected utility

J_s	$\rho = 1$	$\rho = 2$	$\rho = 4$
static case	0.1505	-0.8605	-0.2125
constant λ	0.1515	-0.8594	-0.2121
constant $f_s(t)$	0.1527	-0.8584	-0.2117
increasing $f_s(t)$	0.1546	-0.8568	-0.2111
decreasing $f_s(t)$	0.1550	-0.8564	-0.2110

Optimal initial utility levels J_s compared with the results of the static model. The time horizon is $T = 3$, the fixed rate is $r_s^x = 5\%$.

Percentage increase in certainty equivalent wealth (ΔCE) compared with static case for the four dynamic cases.

ΔCE in %	$\rho = 1$	$\rho = 2$	$\rho = 4$
constant λ	0.10	0.13	0.06
constant $f_s(t)$	0.22	0.24	0.13
increasing $f_s(t)$	0.41	0.43	0.22
decreasing $f_s(t)$	0.45	0.48	0.24

Let $\bar{u}$ denote the optimal utility level from the previous table. The certainty equivalent wealth is then calculated as $(\bar{u}(1-\rho))^{\frac{1}{1-\rho}}$ for $\rho \neq 1$ and as $e^{\bar{u}}$ for $\rho = 1$.

Comparisons of initial fractions of floating rate debt

β_s	$\rho = 1$	$\rho = 2$	$\rho = 4$
static case	0.619	0.310	0.155
constant λ	0.116	0.058	0.029
constant $f_s(t)$	-0.224	-0.112	-0.056
increasing $f_s(t)$	0.860	0.430	0.215
decreasing $f_s(t)$	-1.31	-0.654	-0.164

Optimal initial utility fractions of floating rate debt β_s compared with the results of the previous static model. The time horizon is $T = 3$ and the fixed rate is $r_s^x = 5\%$.

Preliminary numerical examples indicate

- Debt allocation is an issue for the chosen parameters in our model.

- Surpricingly low increase in “welfare” in dynamic situation measured by certainty equivalent wealth compared to static situation.

- Initial optimal floating rate debt depends crucially on the initial slope of the initial term structure – this fact makes it difficult to undertake a direct comparison of the static and dynamic case.

Further research/extension

- Introduce (fixed) transaction cost in the spirit of Davis and Norman (1990), Korn (1998), Øksendal and Sulem (2000), Zakamouline (2002) makes set-up close to real world situations.

- introduce stochastic wealth/collateral ($\bar{W}$)