Normalizers, Ternary Rings, and All That

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(with Ivan Todorov and Vern Paulsen)

Normalizers

Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. Recall $T \in \mathcal{B}(\mathcal{H})$ **normalizes** $\mathcal{A}$ (: $T \in \mathcal{N}(\mathcal{A})$) when $T^*\mathcal{A}T \subseteq \mathcal{A}$ and $T\mathcal{A}T^* \subseteq \mathcal{A}$. Generalize:

Given $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H}_1)$ and $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H}_2)$ *reflexive algebras* (later), say $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ **normalizes** $\mathcal{B}$ **into** $\mathcal{A}$ (: $T \in \mathcal{N}(\mathcal{B}, \mathcal{A})$) when

$$T^*\mathcal{B}T \subseteq \mathcal{A} \qquad \text{and} \qquad T\mathcal{A}T^* \subseteq \mathcal{B}.$$

When T only satisfies $T^*\mathcal{B}T \subseteq \mathcal{A}$ say T **semi-normalizes**:

$$\mathcal{SN}(\mathcal{B}, \mathcal{A}) = \{T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) : T^*\mathcal{B}T \subseteq \mathcal{A}\}.$$

Reflexivity

von Neumann algebras:

$$\mathcal{A} = \{A \in \mathcal{B}(\mathcal{H}) : AL = LA \,\forall L \in \mathcal{L}\}$$

(here $\mathcal{L} =$projections in $\mathcal{A}'$).

reflexive algebras:

$$\mathcal{A} = \{A \in \mathcal{B}(\mathcal{H}) : AL = LAL \,\forall L \in \mathcal{L}\}$$

(here $\mathcal{L} = \mathrm{Lat}\mathcal{A} = \mathcal{A}$ -invariant projections).

reflexive subspaces:

$$\mathcal{M} = \{T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) : TL = \phi(L)TL \,\forall L\}.$$

Local linear structure of $\mathcal{SN}(\mathcal{B},\mathcal{A})$

$\mathcal{A},\mathcal{B}$ reflexive algebras. For $T \in \mathcal{SN}(\mathcal{B},\mathcal{A})$ let

$$\phi_T : \mathrm{Lat}\mathcal{A} \to \mathrm{Lat}\mathcal{B} \qquad L \to [\mathcal{B}TL]$$

show $TL = \phi_T(L)T$ for all $L \in \mathrm{Lat}\mathcal{A}$.

Conversely, given $\phi : \mathrm{Lat}\mathcal{A} \to \mathrm{Lat}\mathcal{B}$ (respecting zero and sups) construct

$\mathcal{U}_\phi = \{S \in \mathcal{B}(\mathcal{H}_1,\mathcal{H}_2) : SL = \phi(L)S \;\forall\, L \in \mathrm{Lat}\mathcal{A}\}$.

Show $\mathcal{U}_\phi \subseteq \mathcal{SN}(\mathcal{B},\mathcal{A})$. But $T \in \mathcal{U}_{\phi_T}$, hence

$$\mathcal{SN}(\mathcal{B},\mathcal{A}) = \textstyle\bigcup_\phi \mathcal{U}_\phi.$$

$\mathcal{U}_\phi$ is

- A reflexive linear space
- Ternary: $T,R,S \in \mathcal{U}_\phi \Rightarrow TR^*S \in \mathcal{U}_\phi$.
- Saturated: $\mathcal{B}_d\mathcal{U}_\phi\mathcal{A}_d \subseteq \mathcal{U}_\phi$ (where $\mathcal{B}_d = \mathcal{B} \cap \mathcal{B}^*$).

More generally:

Any *linear* space $\mathcal{U} \subseteq \mathcal{SN}(\mathcal{B}, \mathcal{A})$ is contained in a ternary space $\mathcal{U}_t \subseteq \mathcal{SN}(\mathcal{B}, \mathcal{A})$ which is reflexive and a bimodule over the diagonals ($\mathcal{B}_d \mathcal{U}_t \mathcal{A}_d \subseteq \mathcal{U}_t$).

Reflexive Ternary Spaces

- A ternary linear space $\mathcal{U}$ is reflexive iff w*-closed [vN bicommutant] iff it a corner of a vN algebra.

$$\begin{bmatrix} (\mathcal{U}\mathcal{U}^*)'' & \mathcal{U} \\ \mathcal{U}^* & (\mathcal{U}^*\mathcal{U})'' \end{bmatrix}$$

(NB: $\mathcal{U} \subseteq \mathcal{N}([\mathcal{U}\mathcal{U}^*], [\mathcal{U}^*\mathcal{U}])$)

- $\mathcal{SN}(\mathcal{B}, \mathcal{A})$ is generated by its partial isometries: Each $T \in \mathcal{SN}(\mathcal{B}, \mathcal{A})$ is the norm closed linear span of partial isometries in $\mathcal{SN}(\mathcal{B}, \mathcal{A})$.
- A reflexive ternary linear space $\mathcal{U}$ generated by its rank one operators is a *sum of full corners*:

$$\mathcal{U} = \bigoplus_n \mathcal{B}(\mathcal{H}_n, \mathcal{K}_n)$$

hence is a bimodule over totally atomic masas.

Masa bimodules

(X_i, μ_i): standard Borel spaces, $\mathcal{H}_i = L^2(X_i, \mu_i)$. If $\Omega \subseteq X_1 \times X_2$, say $T : \mathcal{H}_1 \to \mathcal{H}_2$ is *supported by* Ω if

$P(\beta)TP(\alpha) = 0 \; \forall \alpha, \beta$ Borel s.t. $(\alpha \times \beta) \cap \Omega = \emptyset$.

(Arveson).

$$\mathcal{M}_{\text{max}}(\Omega) = \{T : \mathcal{H}_1 \to \mathcal{H}_2 \text{ supported by } \Omega\}$$

a reflexive bimodule over the multiplication masas. Conversely

Every reflexive masa bimodule is of this form. (Erdos, K., Shulman)

(for an Ω which -if chosen to be "ω-closed"- is unique up to marginal equivalence, i.e. up to a set of the form $X_1 \times N_2 \cup N_1 \times X_2$, $\mu_i(N_i) = 0$).

Ternary masa bimodules

When is $\mathcal{U} = \mathcal{M}_{\mathrm{max}}(\Omega)$ ternary? Precisely when

$$\Omega \underset{m}{\simeq} \{(s,t) \in X_1 \times X_2 : f_1(s) = f_2(t)\}$$

for appropriate Borel functions $f_i : X_i \to [0,1]$.

(If for example the algebra $[\mathcal{U}\mathcal{U}^*]$ is abelian, then may write

$$\Omega \underset{m}{\simeq} \{(s,t) : f(s) = t\}.)$$

Also, $\mathcal{U} = \mathcal{M}_{\mathrm{max}}(\Omega)$ is ternary iff Ω has the $3 \Rightarrow 4$ property: For $x_i \in X_1, y_i \in X_2$, $x_1 \neq x_2, y_1 \neq y_2$, if 3 pairs (x_i, y_i) are in Ω, then all 4 are.

When is $\mathcal{U}_\phi$ a masa bimodule? Precisely when ϕ is determined by its action on a *nest* of projections.

Synthesis

A (ω-closed) set $\Omega \subseteq X_1 \times X_2$ is **synthetic** if there is only one w*-closed masa bimodule $\mathcal{U}$ with Ω as its support (: $\mathcal{U} = \mathcal{M}_{\max}(\Omega)$).

Arveson: There exist non-synthetic sets (from harmonic analysis). But,

A ternary masa bimodule $\mathcal{U}$ is (not only synthetic but also) "hereditarily synthetic":

Every $\mathcal{S} \subseteq \mathcal{U}$ which is a left $[\mathcal{U}^*\mathcal{U}]$-module is synthetic.

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