

# Multiscale Methods in Turbulence

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*Dedicated to Jerry Marsden on occasion of his 60<sup>th</sup> birthday*

# Outline

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Introduction

Incompressible Navier-Stokes Equations

Large Eddy Simulation

Variational Multiscale Method

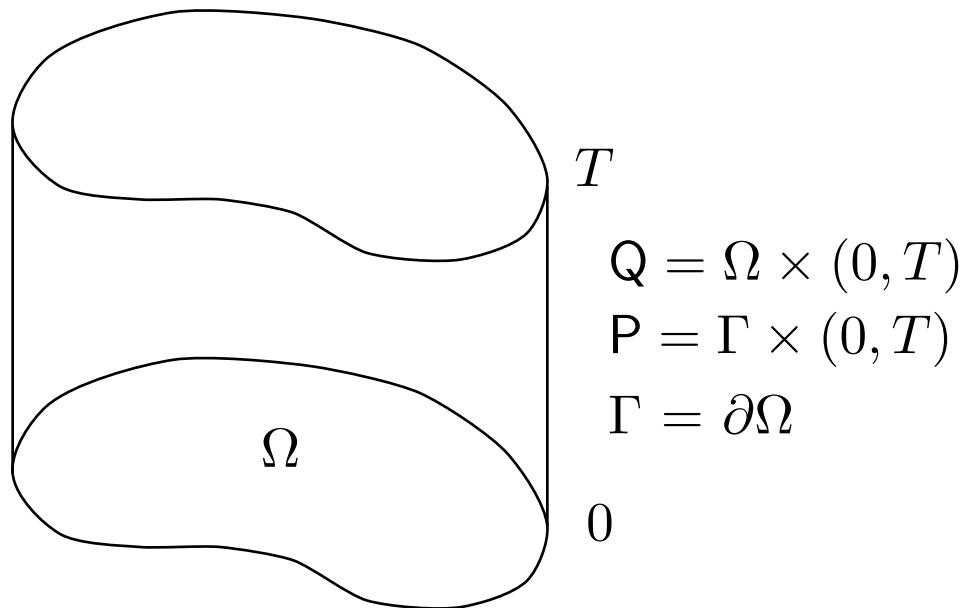
Numerical Examples

Concluding Remarks

# Incompressible Navier-Stokes Equations

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$$\begin{aligned}\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) + \nabla p &= \nu \Delta \mathbf{u} + \mathbf{f} && \text{in } Q \\ \nabla \cdot \mathbf{u} &= 0 && \text{in } Q \\ \mathbf{u} &= \mathbf{0} && \text{on } P \\ \mathbf{u}(t = 0^+) &= \mathbf{u}(t = 0^-) && \text{on } \Omega\end{aligned}$$



# Large Eddy Simulation

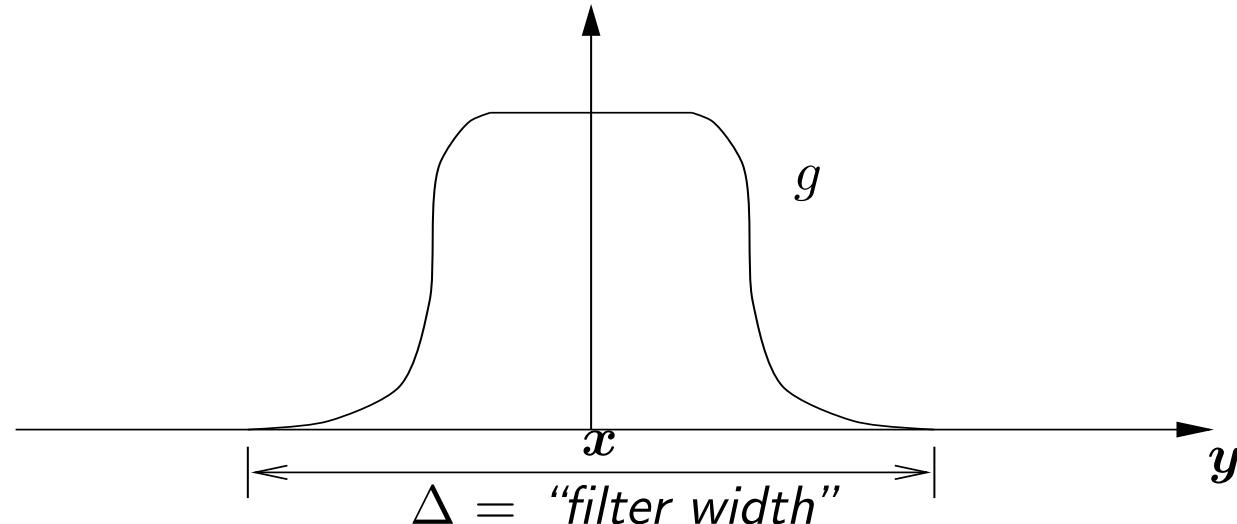
## Filter

$$\bar{u}(x, t) = \int_{D_\Delta} g(x, y) u(y, t) dy$$

where

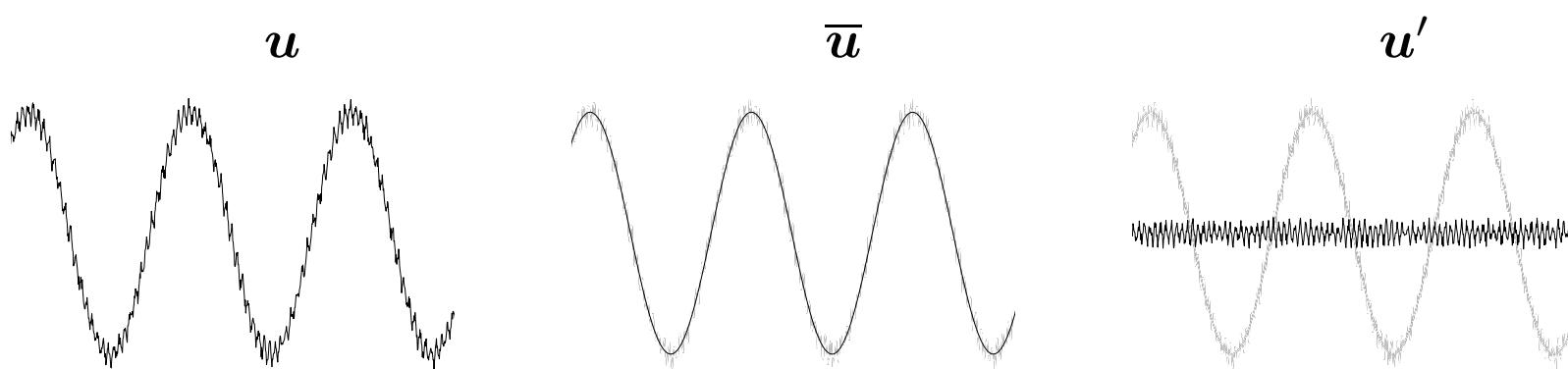
$$g(x, y) = g(x - y) \quad (\text{homogeneity}) \quad \text{and} \quad 1 = \int_{D_\Delta} g(x, y) dy$$

E.g.,



# Large Eddy Simulation

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# Large Eddy Simulation

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## Filtered equations

$$\begin{aligned}\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}} \otimes \bar{\mathbf{u}}) + \nabla \bar{p} &= \nu \Delta \bar{\mathbf{u}} + \bar{\mathbf{f}} && \text{in } Q \\ \nabla \cdot \bar{\mathbf{u}} &= 0 && \text{in } Q \\ \bar{\mathbf{u}} &= \mathbf{0} && \text{on } P \\ \bar{\mathbf{u}}(t = 0^+) &= \bar{\mathbf{u}}(t = 0^-) && \text{on } \Omega\end{aligned}$$

**Closure problem:** how to compute  $\bar{\mathbf{u}} \otimes \bar{\mathbf{u}}$ ?

# Large Eddy Simulation

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Rewrite as:

$$\begin{aligned}\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}} \otimes \bar{\mathbf{u}}) + \nabla \bar{p} &= \nu \Delta \bar{\mathbf{u}} + \nabla \cdot \mathbf{T} + \bar{\mathbf{f}} && \text{in } Q \\ \nabla \cdot \bar{\mathbf{u}} &= 0 && \text{in } Q \\ \bar{\mathbf{u}} &= \mathbf{0} && \text{on } P \\ \bar{\mathbf{u}}(t = 0^+) &= \bar{\mathbf{u}}(t = 0^-) && \text{on } \Omega\end{aligned}$$

where  $\mathbf{T}$ , the *subgrid-scale stress*, is modeled

$$\mathbf{T} = \bar{\mathbf{u}} \otimes \bar{\mathbf{u}} - \overline{\mathbf{u} \otimes \mathbf{u}} \approx \mathbf{T}_s$$

# Large Eddy Simulation

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## Smagorinsky eddy viscosity

$$\mathbf{T}_s = 2\nu_T \nabla^s \bar{\mathbf{u}} = 2(C_S \Delta)^2 |\nabla^s \bar{\mathbf{u}}| \nabla^s \bar{\mathbf{u}}$$

where

$$\begin{aligned}\nabla^s \bar{\mathbf{u}} &= \frac{1}{2}(\nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T) \\ |\nabla^s \bar{\mathbf{u}}| &= (2 \nabla^s \bar{\mathbf{u}} \cdot \nabla^s \bar{\mathbf{u}})^{1/2}\end{aligned}$$

and  $C_S$  is the **Smagorinsky constant**

# Large Eddy Simulation

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## Critique

- Too dissipative for flows with large-scale structures
- Incorrect asymptotic behavior near walls
- Poor performance on non-equilibrium flows

**Modification:** The dynamic model (Germano et al. 1991)

$$C_S = C_S(\boldsymbol{x}, t)$$

# Numerical Analysis Interpretation

**Estimation of  $\nu_T$ :** Due to Lilly

$$\Delta = O(1/\bar{k})$$

$$\nu_T = O(1/\bar{k}^{4/3})$$

where  $\bar{k}$  is the cutoff wave number.

**Assumption:**  $\bar{k} = \text{const}/h$ . It follows

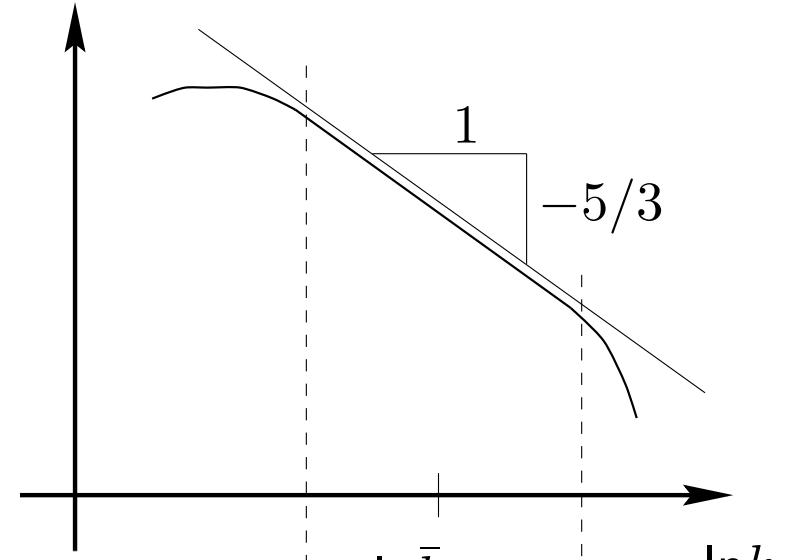
$$\Delta = O(h)$$

$$\nu_T = O(h^{4/3})$$

So

$$T_s = 2\nu_T \nabla^s \bar{u} = O(h^{4/3}) \nabla^s \bar{u}$$

acts as an **artificial viscosity** in “resolved” scales



# Spectral Eddy Viscosity

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## Navier-Stokes equations

$$\frac{\partial \hat{u}(\mathbf{k})}{\partial t} = -\nu k^2 \hat{u}(\mathbf{k}) + N(\mathbf{k} | \hat{u}(\mathbf{q}), \hat{u}(\mathbf{p}))$$

Multiply with  $\hat{u}^*(\mathbf{k})$ , sum over shells of radius  $k - \Delta k$  to  $k + \Delta k$ ,

$$\frac{\partial E(k)}{\partial t} = -2\nu k^2 E(k) + T(k | \hat{u}(\mathbf{q}), \hat{u}(\mathbf{p}))$$

Introduce a cutoff  $k_c$ . For  $k < k_c$

$$\frac{\partial E(k)}{\partial t} = -2\nu k^2 E(k) + T(k | \hat{u}(\mathbf{q}), \hat{u}(\mathbf{p})) + \textcolor{red}{T^S}(k | \hat{u}(\mathbf{q}), \hat{u}(\mathbf{p}))$$

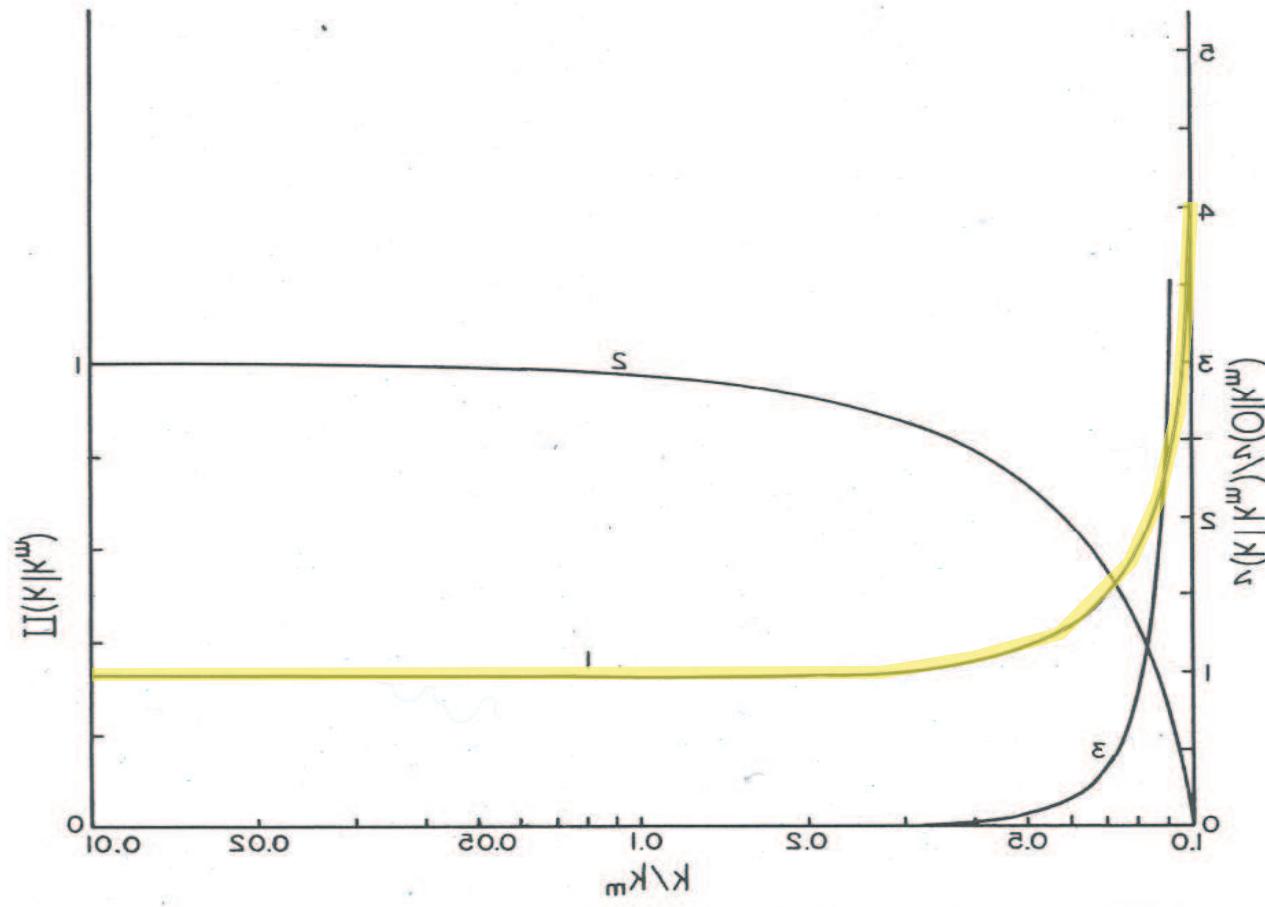
## Equivalent Eddy Viscosity

$$\nu_T(k) = -\frac{\textcolor{red}{T^S}}{2k^2 E(k)}$$

# Spectral Eddy Viscosity

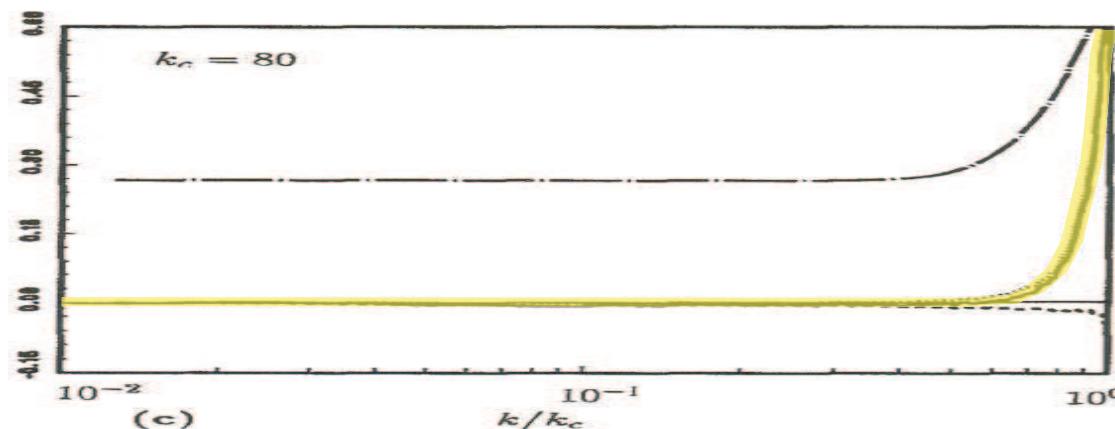
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DIA results for homogeneous turbulence from Kraichnan (1976)

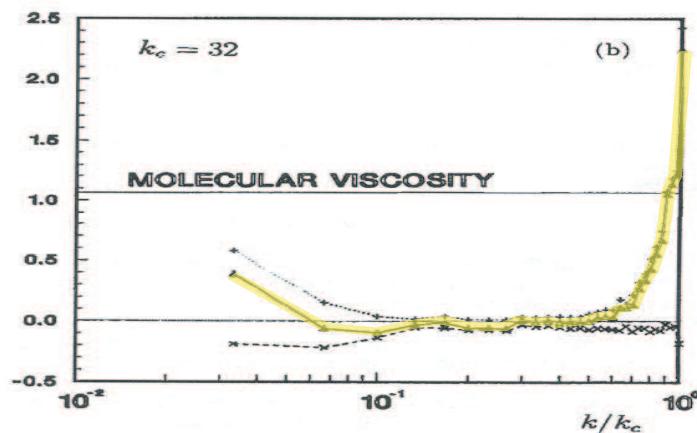


# Spectral Eddy Viscosity

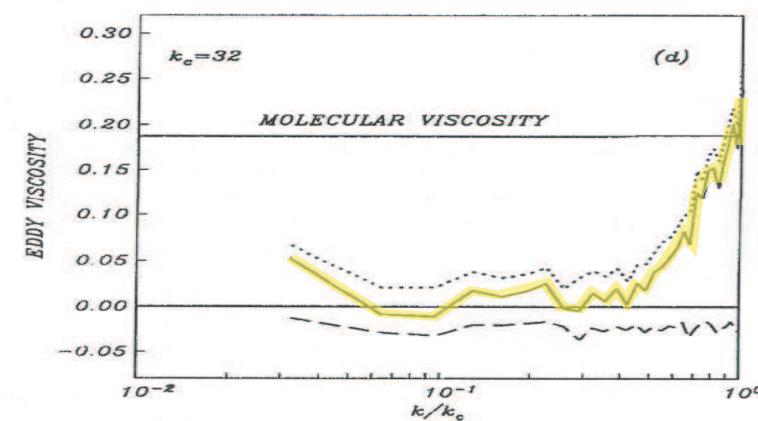
DNS Results for homogeneous turbulence, Domaradzki & Liu (1993),  
 $Re_\lambda = 70$



DNS results for wall-bounded flow, Domaradzki & Liu (1993),  $Re_\tau = 210$



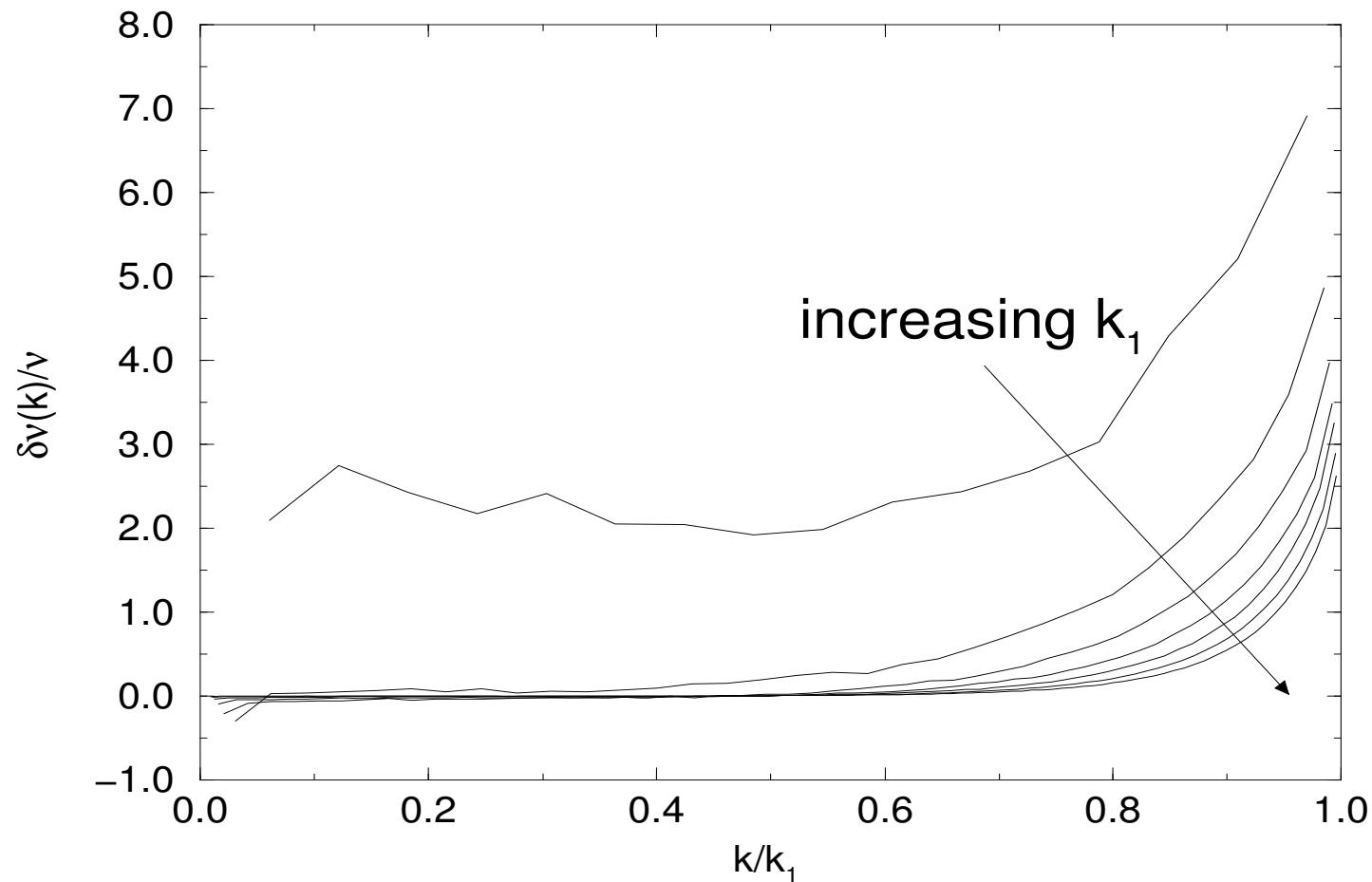
Core



Wall

# Spectral Eddy Viscosity

DNS results from Young & McComb (1998), on a  $256^3$  mesh at  $Re_\lambda = 190$ ,  $16.5 < k_1 < 112.5$



# Variational Multiscale Method

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## Space-Time Formulation

Find  $\mathbf{U} = \{\mathbf{u}, p\} \in \mathcal{V}$  such that  $\forall \mathbf{W} = \{\mathbf{w}, q\} \in \mathcal{V}$

$$B(\mathbf{W}, \mathbf{U}) = (\mathbf{W}, \mathbf{F}) \quad (\star)$$

where

$$\begin{aligned} B(\mathbf{W}, \mathbf{U}) &= (\mathbf{w}(T^-), \mathbf{u}(T^-))_\Omega - \left( \frac{\partial \mathbf{w}}{\partial t}, \mathbf{u} \right)_Q \\ &\quad - (\nabla \mathbf{w}, \mathbf{u} \otimes \mathbf{u})_Q + (q, \nabla \cdot \mathbf{u})_Q \\ &\quad - (\nabla \cdot \mathbf{w}, p)_Q + (\nabla^s \mathbf{w}, 2\nu \nabla^s \mathbf{u})_Q \end{aligned}$$

$$(\mathbf{W}, \mathbf{F}) = (\mathbf{w}, \mathbf{f})_Q + (\mathbf{w}(0^+), \mathbf{u}(0^-))_\Omega$$

# Variational Multiscale Formulation

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## Split

$$\mathcal{V} = \overline{\mathcal{V}} \oplus \mathcal{V}'$$

Identify  $\overline{\mathcal{V}} \subset \mathcal{V}$  with a *finite dimensional space*. Then  $\mathcal{V}' = \mathcal{V} \setminus \overline{\mathcal{V}}$ .

( $\star$ ) splits into two problems: Find  $\mathbf{U} = \overline{\mathbf{U}} + \mathbf{U}'$ ,  $\overline{\mathbf{U}} \in \overline{\mathcal{V}}$ ,  $\mathbf{U}' \in \mathcal{V}'$ , such that

$$\begin{array}{llll} (\overline{\mathbf{U}}) & B(\overline{\mathbf{W}}, \overline{\mathbf{U}} + \mathbf{U}') & = & (\overline{\mathbf{W}}, \mathbf{F}) & \forall \overline{\mathbf{W}} \in \overline{\mathcal{V}} \\ (\mathbf{U}') & B(\mathbf{W}', \overline{\mathbf{U}} + \mathbf{U}') & = & (\mathbf{W}', \mathbf{F}) & \forall \mathbf{W}' \in \mathcal{V}' \end{array}$$

# Variational Multiscale Method

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## Finite-dimensional approximation

$$\bar{\mathcal{V}} = \bar{\mathcal{V}}^h \quad \text{and} \quad \mathcal{V}' \approx \mathcal{V}'^h$$

$\bar{\mathcal{V}}^h$  could be a *standard FE space, spectral space, etc.* Think of  $\mathcal{V}'^h$  as hierarchical  $p$ -refinement, spectral refinement, etc.

# Variational Multiscale Method

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Introduce models to account for truncating  $\mathcal{V}'$ . So

$$(\bar{\mathbf{U}}) \quad B(\bar{\mathbf{W}}, \bar{\mathbf{U}} + \mathbf{U}') = (\bar{\mathbf{W}}, \mathbf{F})$$

$$(\mathbf{U}') \quad B(\mathbf{W}', \bar{\mathbf{U}} + \mathbf{U}') = (\mathbf{W}', \mathbf{F})$$

are approximated by :

$$(\bar{\mathbf{U}}^h) \quad B(\bar{\mathbf{W}}^h, \bar{\mathbf{U}}^h + \mathbf{U}'^h) + (\nabla^s \bar{\mathbf{w}}^h, \bar{\mathbf{R}}_s^h) = (\bar{\mathbf{W}}^h, \mathbf{F})$$

$$(\mathbf{U}'^h) \quad B(\mathbf{W}'^h, \bar{\mathbf{U}}^h + \mathbf{U}'^h) + (\nabla^s \mathbf{w}'^h, \mathbf{R}'_s^h) = (\mathbf{W}'^h, \mathbf{F})$$

where

$$\bar{\mathbf{R}}_s^h = \mathbf{0} \quad \text{and} \quad \mathbf{R}'_s^h = 2\nu'_T \nabla^s \mathbf{u}'^h$$

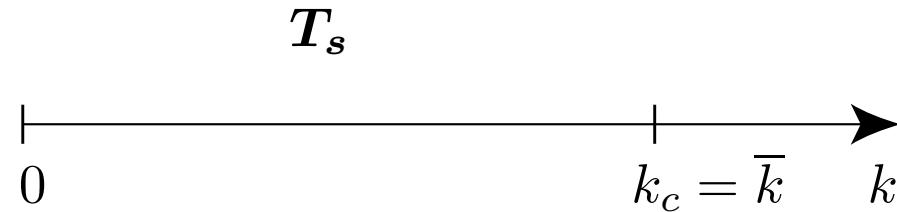
This leads to

$$(\mathbf{U}^h) \quad B(\mathbf{W}^h, \mathbf{U}^h) + (\nabla \mathbf{w}'^h, \mathbf{R}'_s^h) = (\mathbf{W}^h, \mathbf{F})$$

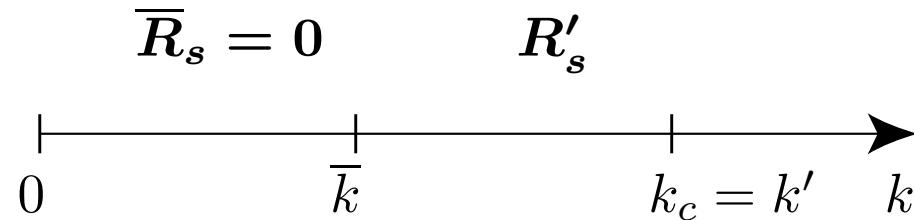
# Variational Multiscale Method

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## Classical LES



## Multiscale LES



# Variational Multiscale Formulation

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## Eddy Viscosity Models

$$\mathbf{R}'_s = 2\nu'_T \nabla^s \mathbf{u}'$$

Two *candidate* definitions of  $\nu'_T$ :

$$\nu'_T = (C'_S \Delta')^2 |\nabla^s \mathbf{u}'| \quad \text{Small-Small}$$

$$\nu'_T = (C'_S \Delta')^2 |\nabla^s \bar{\mathbf{u}}| \quad \text{Large-Small}$$

$C'_S = C_S f(\bar{k}/k')$  can be estimated by assuming an inertial subrange

# Homogeneous Isotropic Flows

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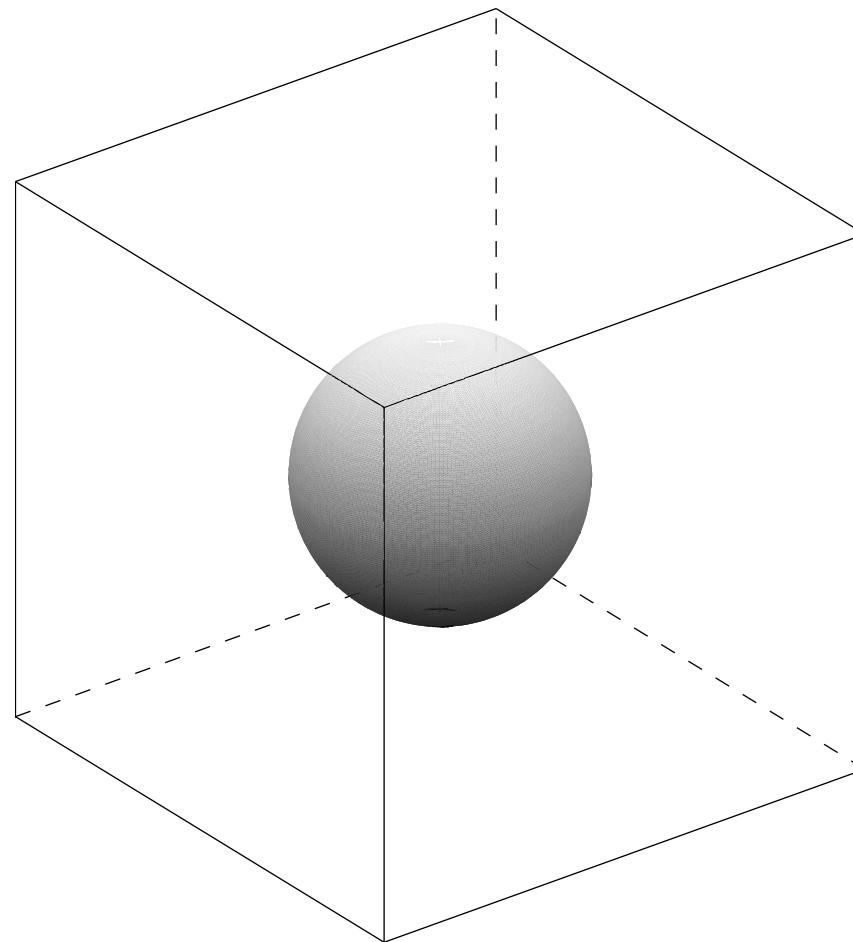
## Numerical Results

- Fourier-spectral discretization
- Third-order Runge-Kutta time advancement
- Dealiased with 3/2 rule
- Two cases:  $E(k, t = 0) = k^4 \exp(-4k/k_p)$ 
  1. Inviscid,  $k_p = 4$
  2. Viscous,  $k_p = 1$
- Comparisons:
  - DNS
  - Smagorinsky ( $C_S = 0.1$ )
  - Dynamic Smagorinsky with sharp cutoff
  - Multiscale : large-small ( $C'_S = 0.1$ )
  - Multiscale : small-small ( $C'_S = 0.1$ )

# Homogeneous Isotropic Flows

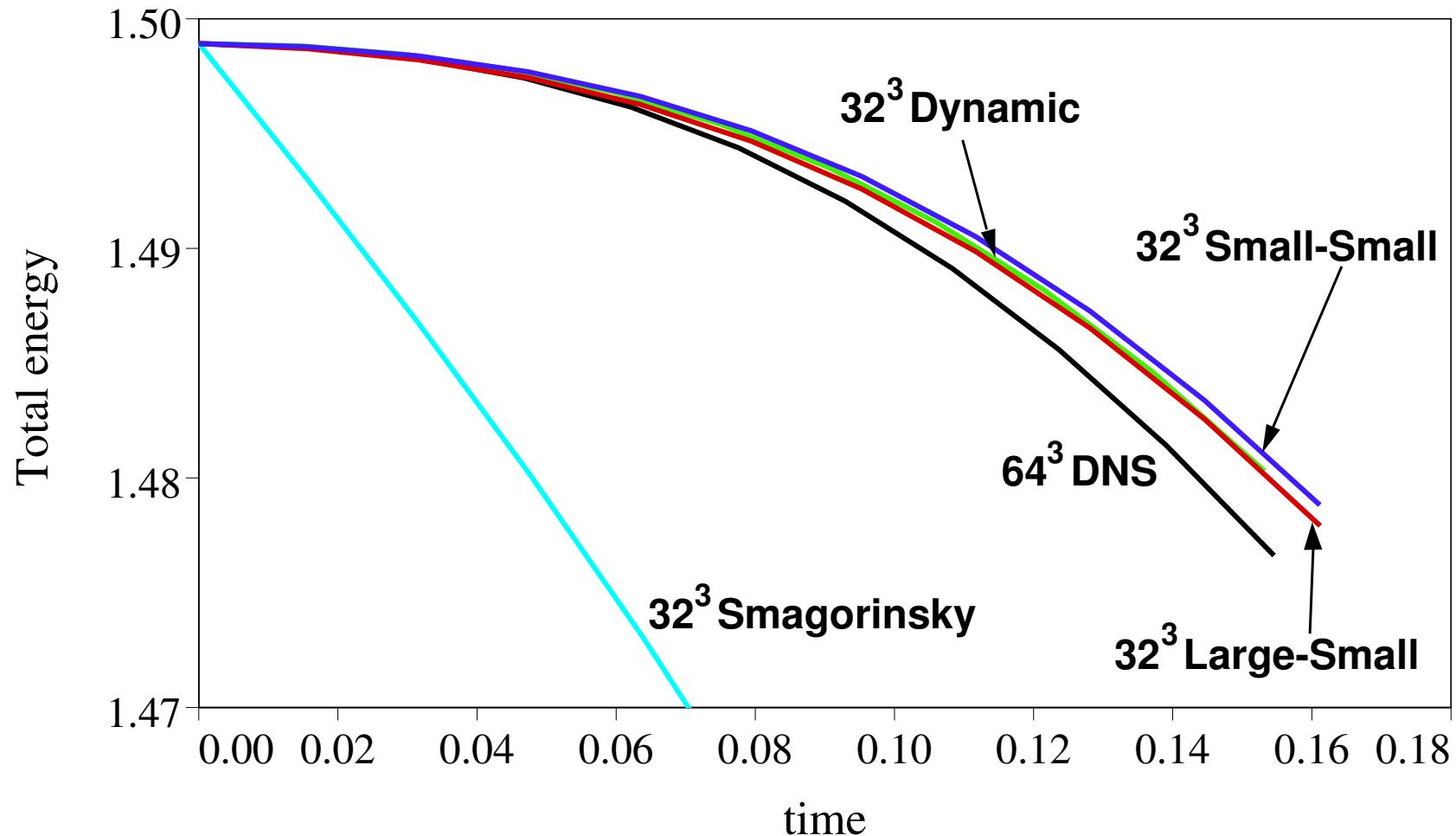
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Schematic representation  
of  $\mathcal{V}^h$ ,  $\bar{\mathcal{V}}^h$  and  $\mathcal{V}'^h$



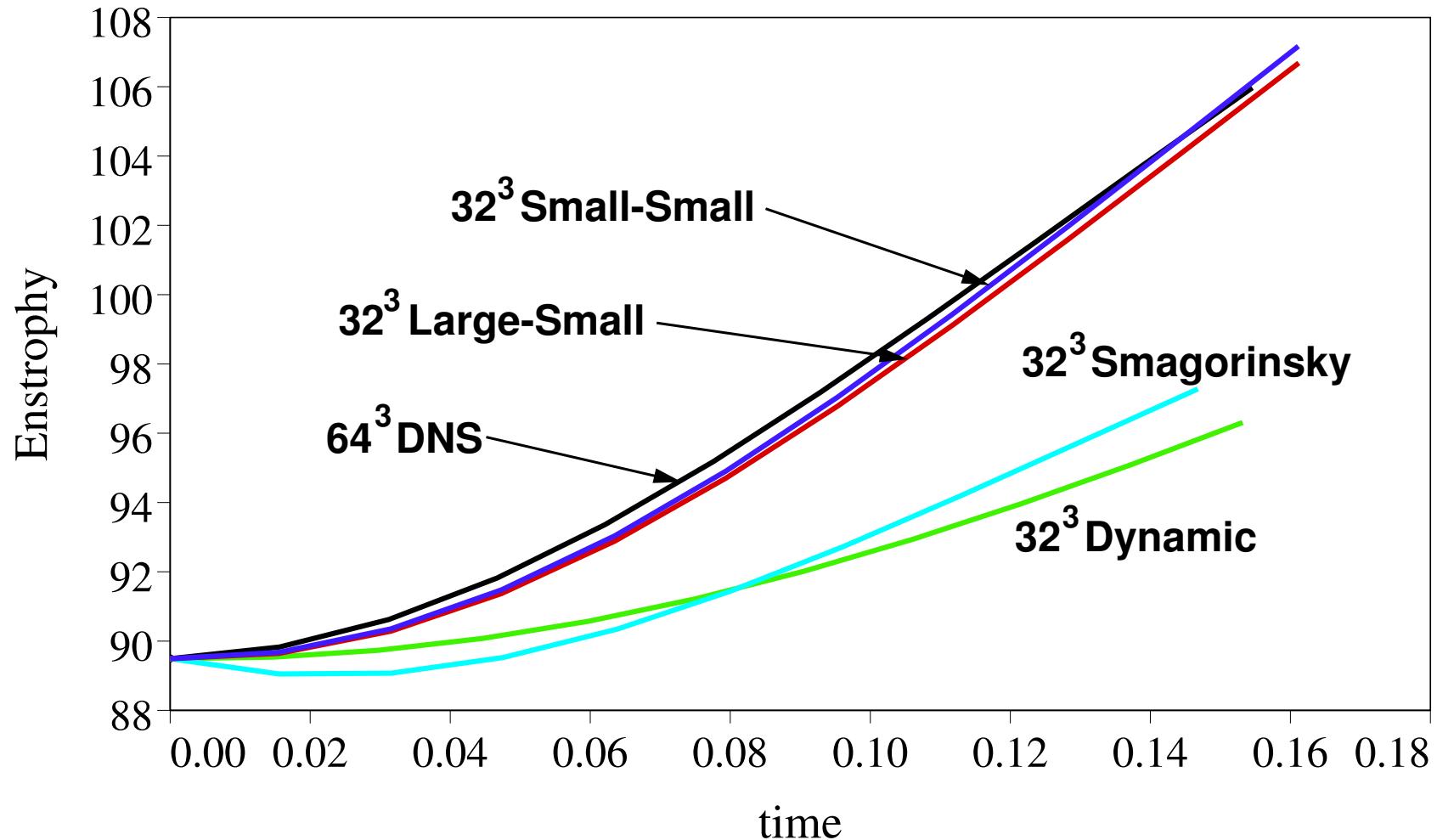
## Inviscid case

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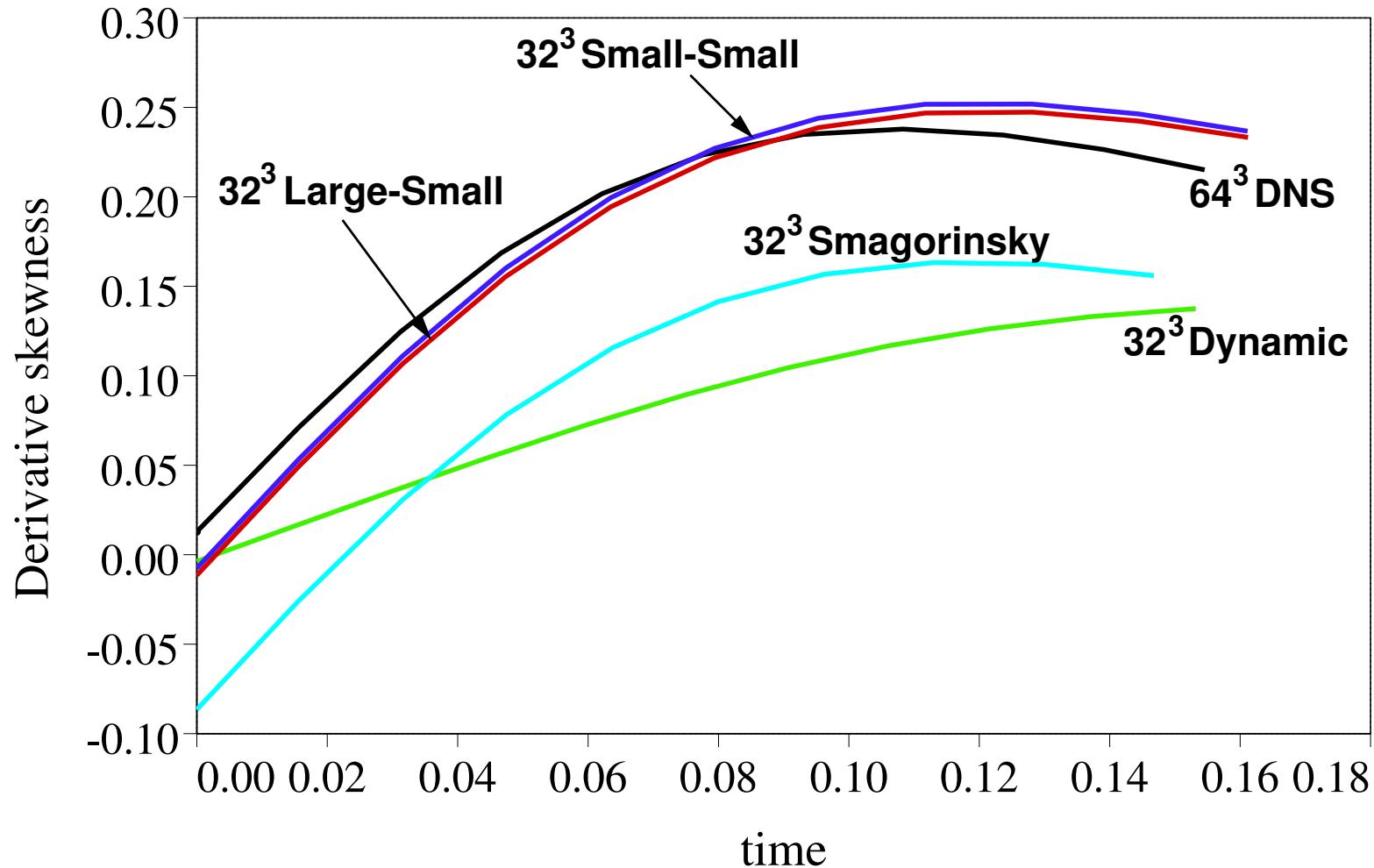
## Inviscid case

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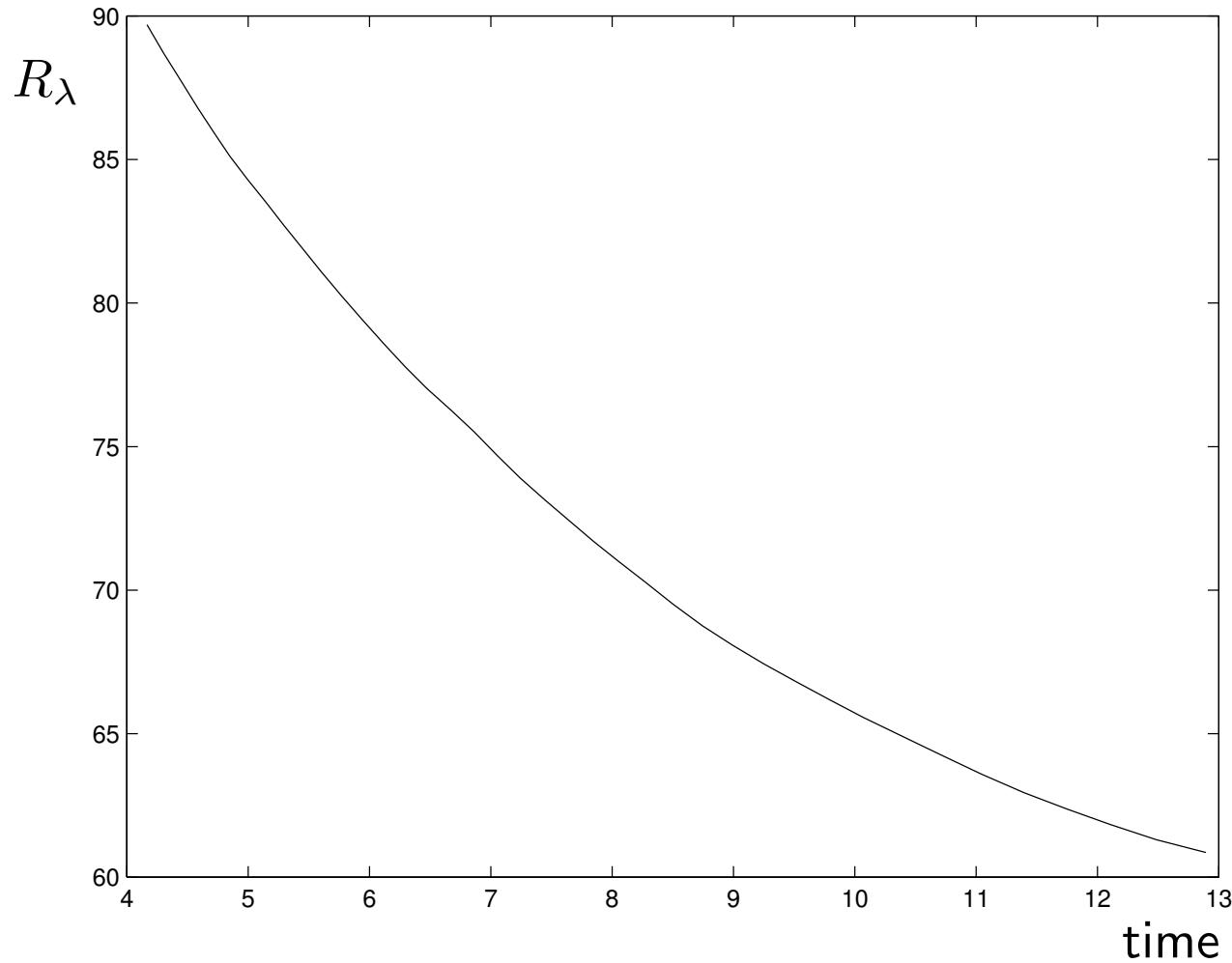
# Inviscid case

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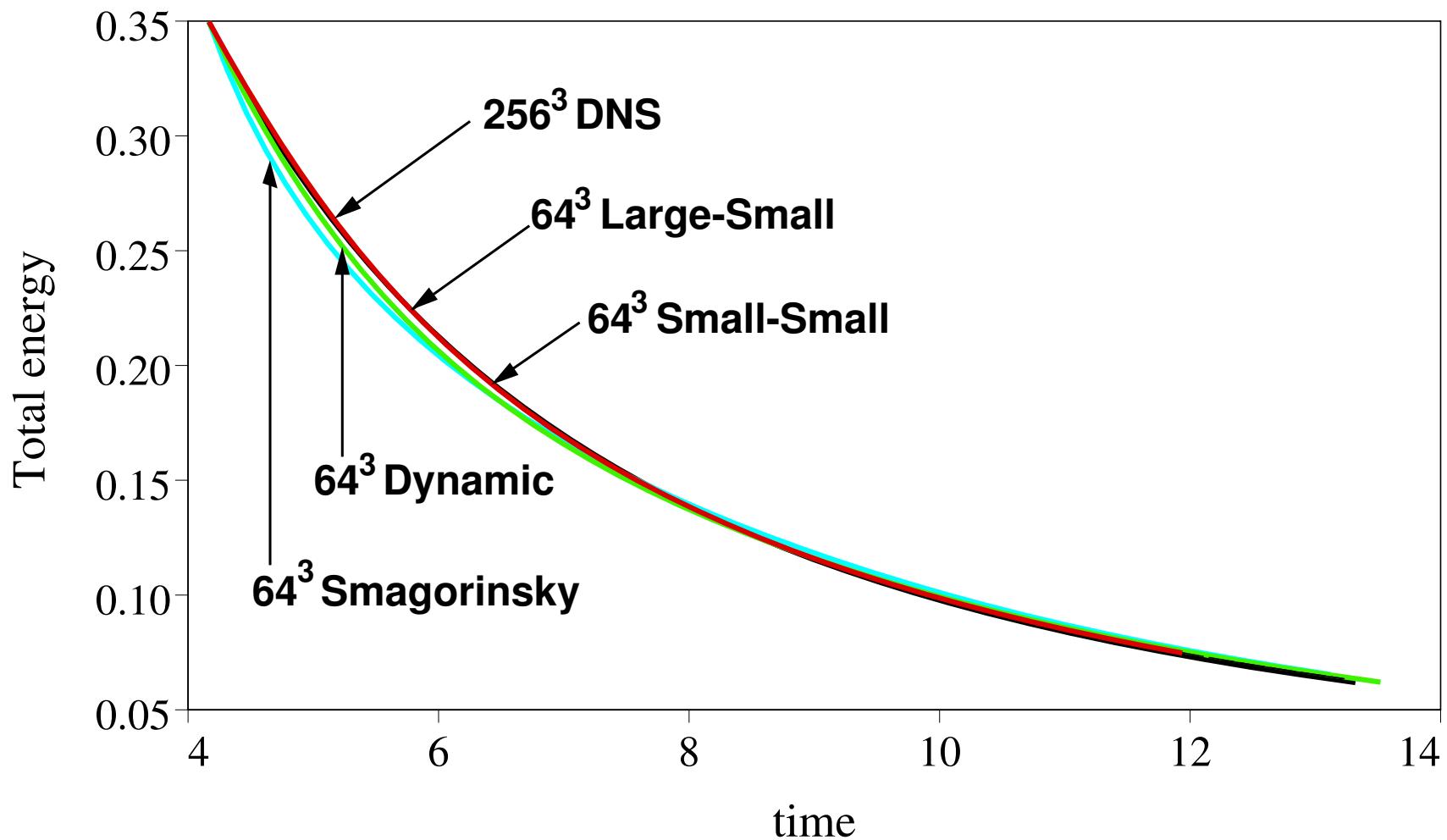
# Viscous case

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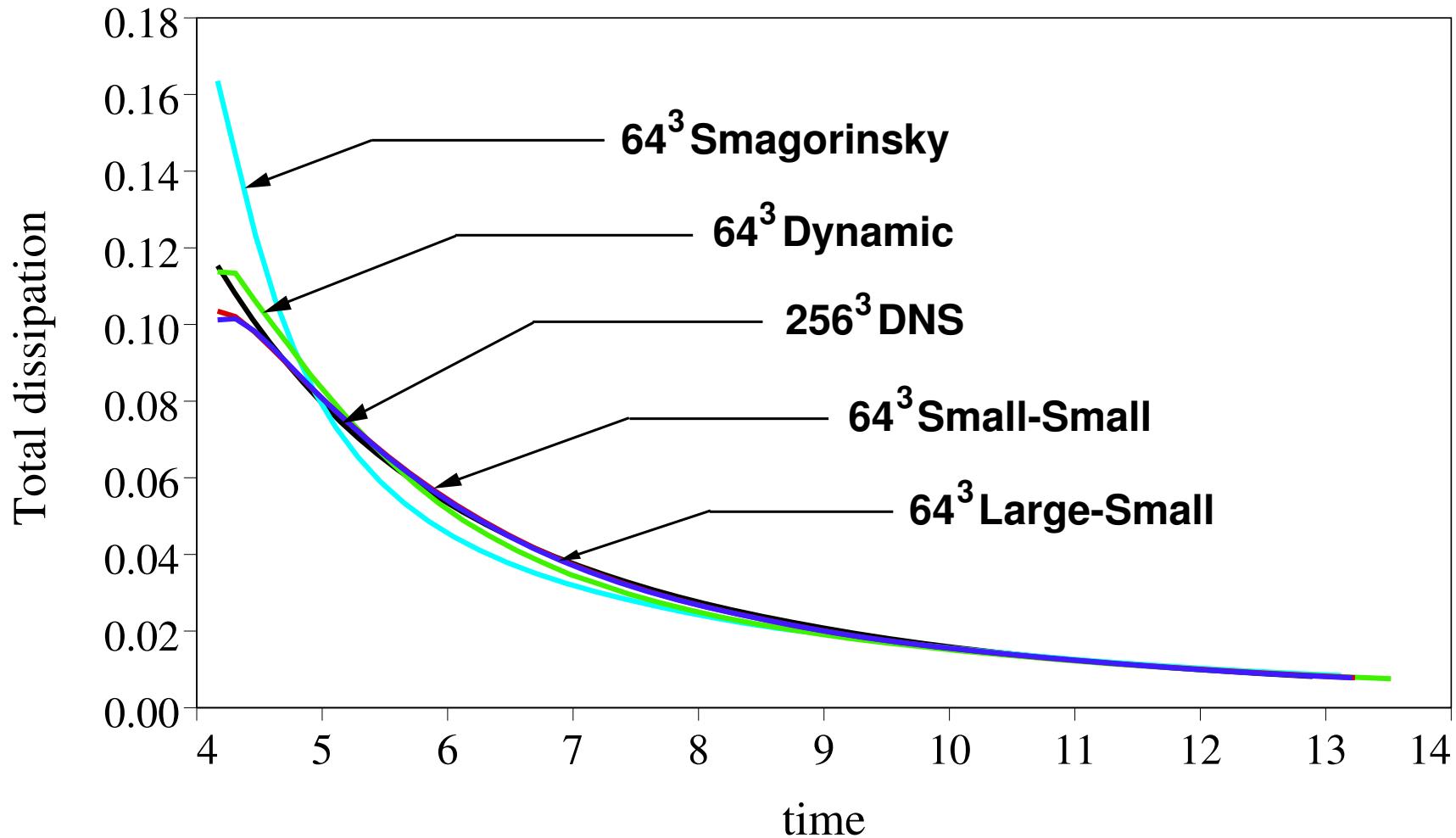
# Viscous case

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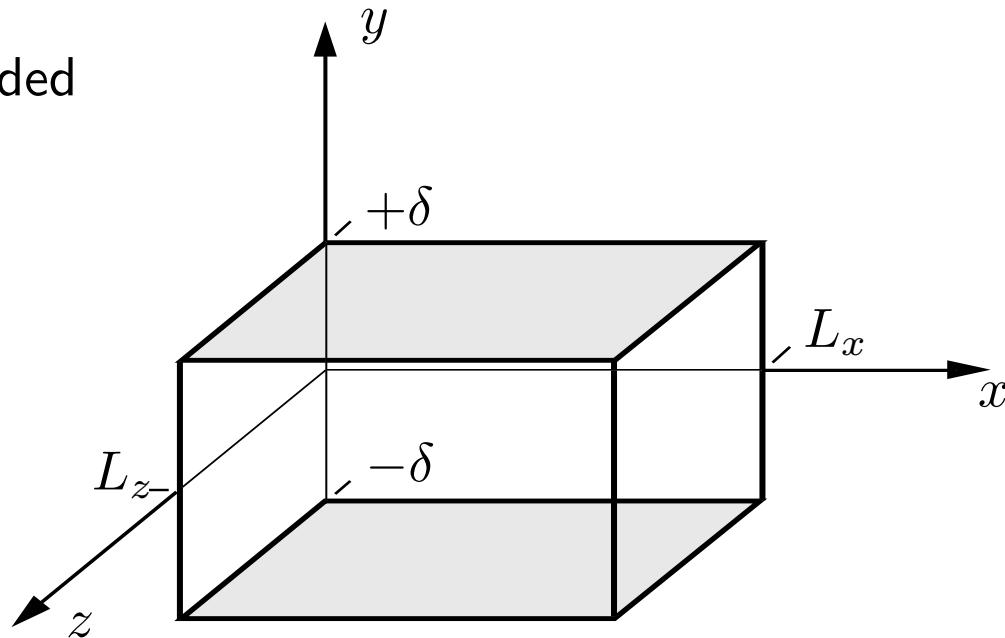
# Viscous case

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# Channel Flows

- Flow driven by a uniform pressure gradient
- Numerical method:
  - Modified version of a code written by Lopez and Moser (U.I.U.C.)
  - A reduced system of equations for  $y$ -velocity  $u_y$ , and  $y$ -vorticity  $\omega_y$
  - Basis: Fourier-spectral in  $x$  and  $z$ , modified Legendre in  $y$
  - Time-stepping scheme: Implicit in the molecular viscosity term, explicit in all others
  - De-aliasing included



# Channel Flows

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- **Notation**

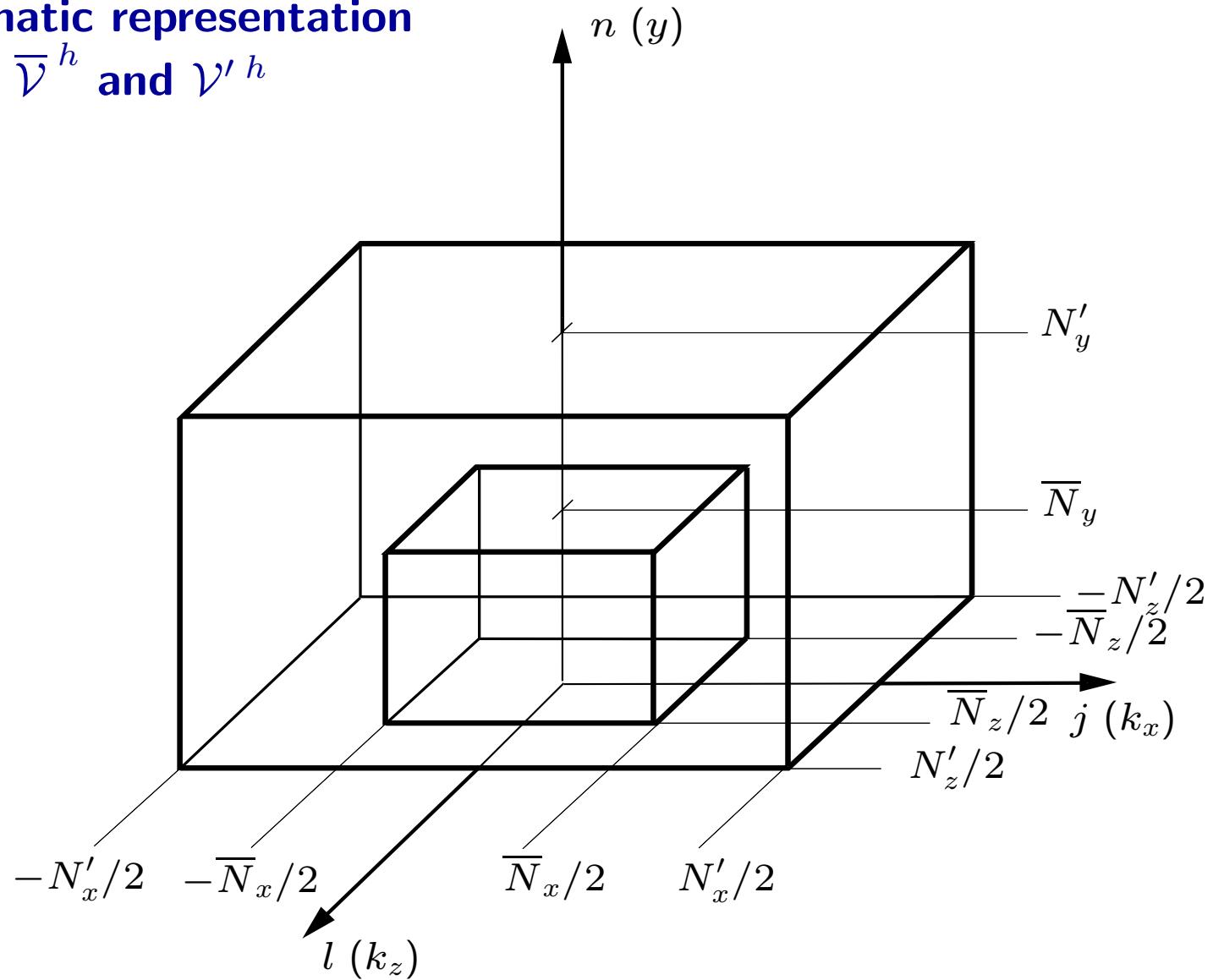
- $\langle \cdot \rangle$  denotes averages over  $x, z$  planes
- $\langle\langle \cdot \rangle\rangle$  denotes averages over  $x, z$ -planes and time
- $y^+ = (\delta - |y|)u_\tau/\nu$
- $R_{xy}$  = Reynolds shear stress (includes model)

- **Comparisons**

- DNS (Moser, Kim & Mansour, 1999)
- Smagorinsky ( $C_S = 0.1$ )
- Dynamic Smagorinsky with sharp cutoff
- Multiscale : large-small ( $C'_S = 0.1$ )
- Multiscale : small-small ( $C'_S = 0.1$ )

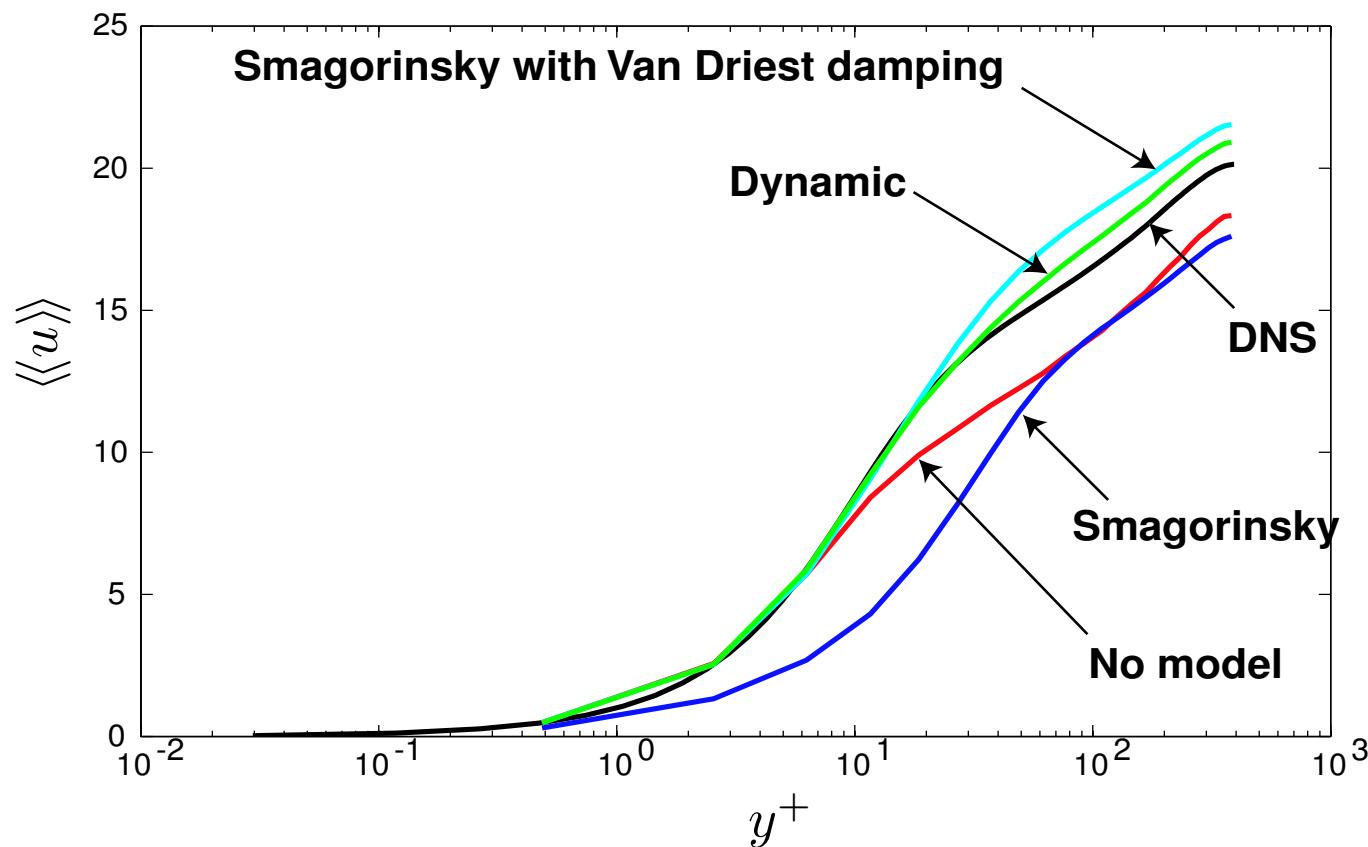
# Channel Flows

Schematic representation  
of  $\mathcal{V}^h$ ,  $\bar{\mathcal{V}}^h$  and  $\mathcal{V}'^h$



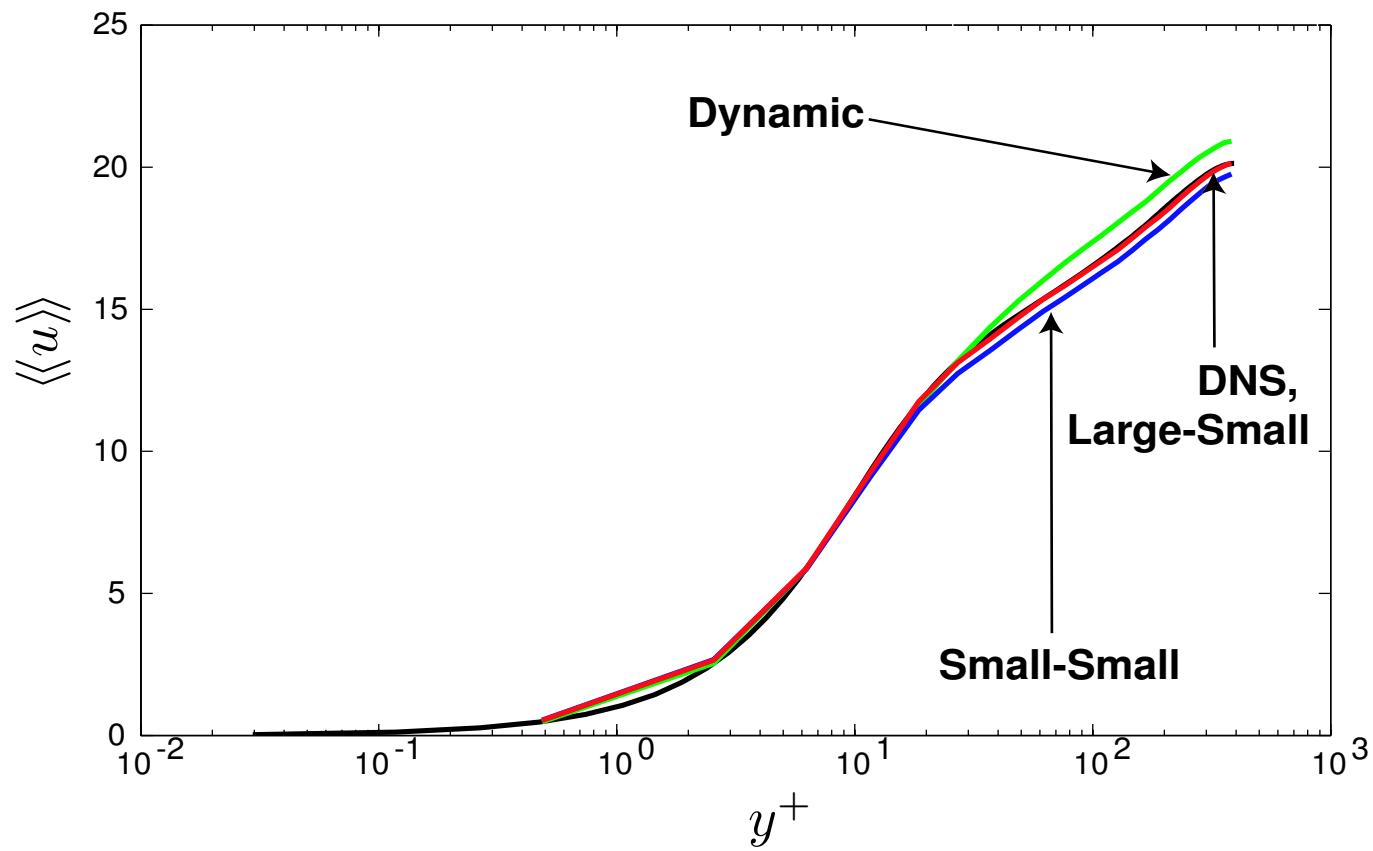
# $Re_\tau = 395$ : Calibration study

$L_x = 2\pi$	$\Delta x^+ \approx 77$	$N'_x = 32$
$L_y = 2$	$\Delta y^+ \in (1, 38)$	$N'_y = 32$
$L_z = 2\pi/3$	$\Delta z^+ \approx 26$	$N'_z = 32$

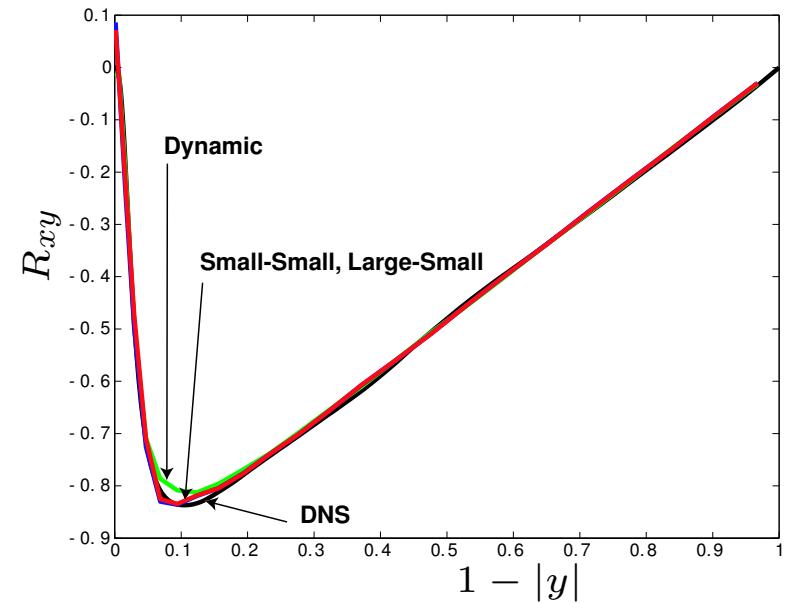
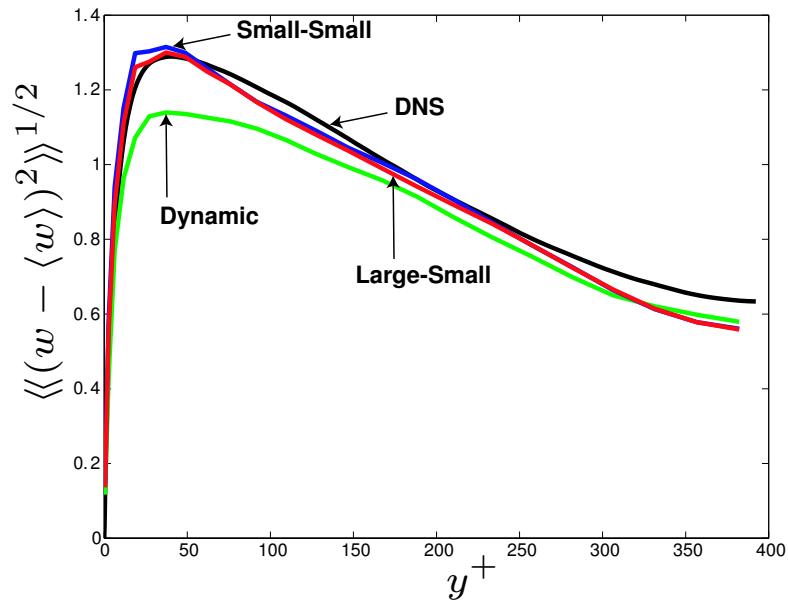
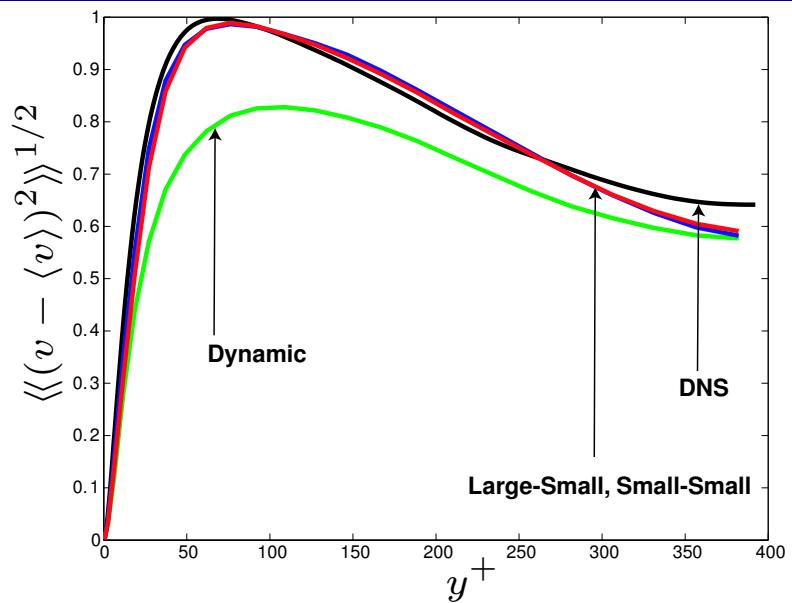
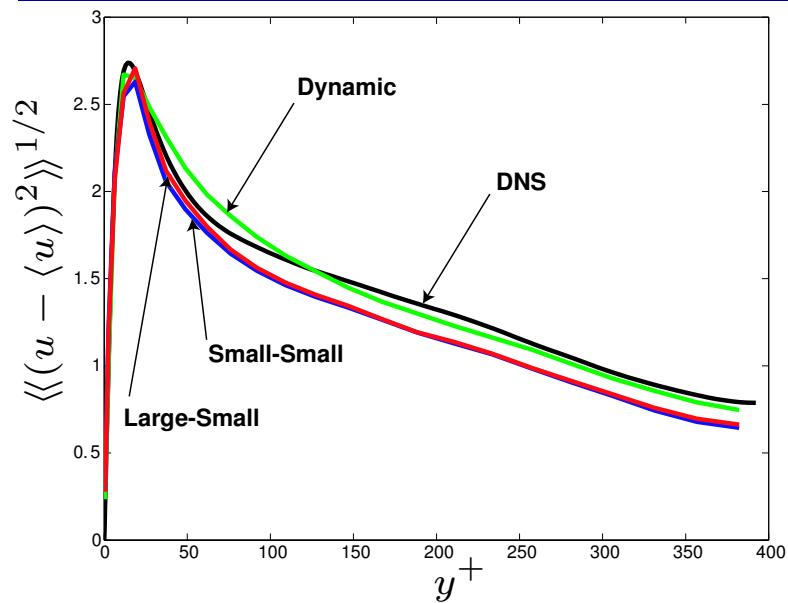


## $Re_\tau = 395$ : Comparison

$L_x = 2\pi$	$\Delta x^+ \approx 77$	$\overline{N}_x/N'_x = 22/32$
$L_y = 2$	$\Delta y^+ \in (1, 38)$	$\overline{N}_y/N'_y = 22/32$
$L_z = 4\pi/3$	$\Delta z^+ \approx 26$	$\overline{N}_z/N'_z = 22/32$

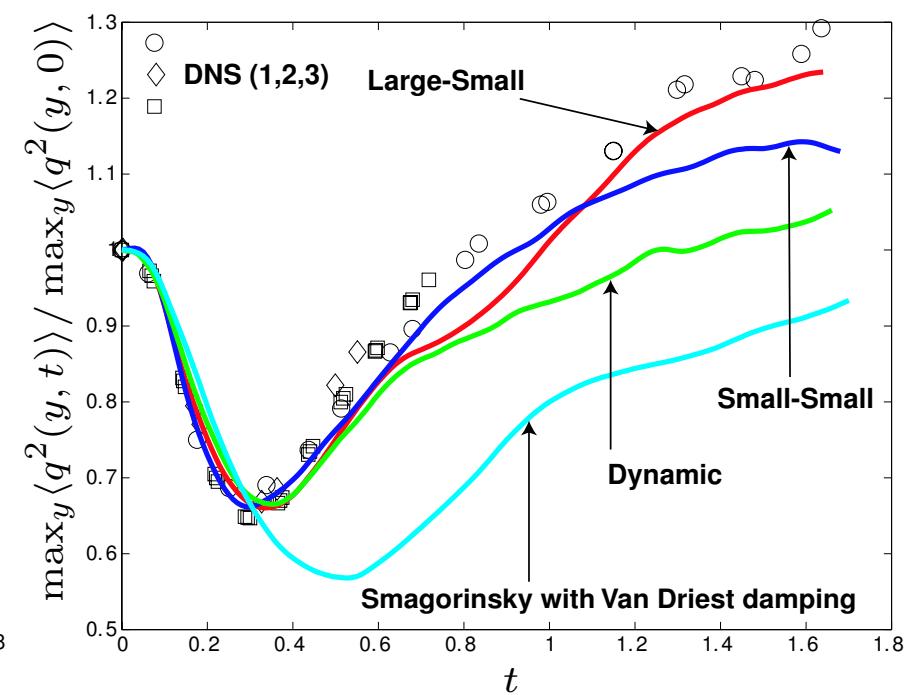
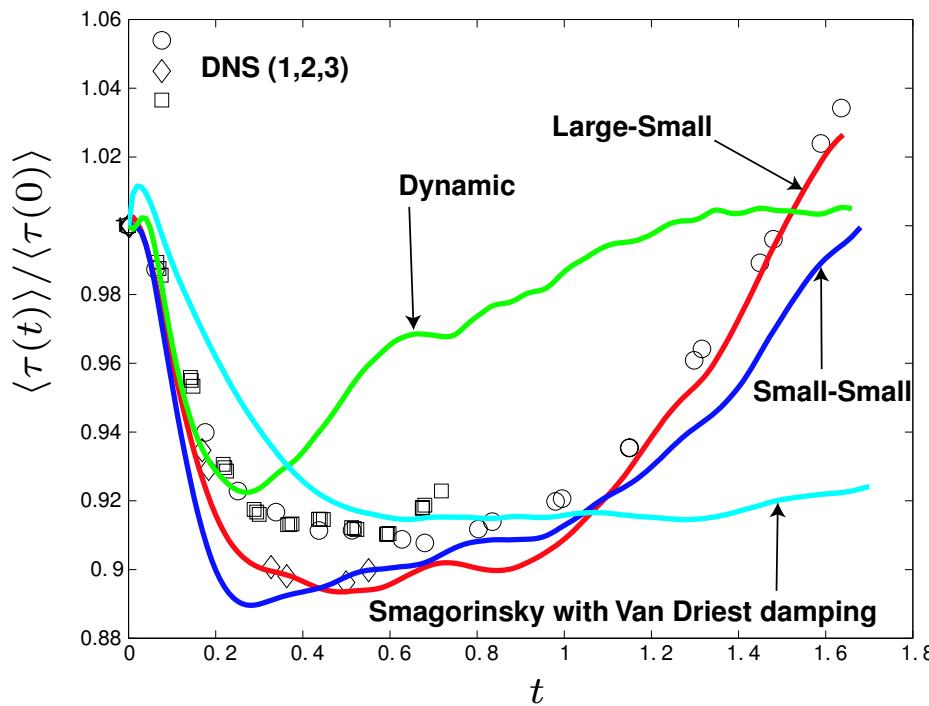


# $Re_\tau = 395$ : Comparison



# Non-equilibrium Channel Flow

- Start from a fully-developed 2D flow
- Shear the lower wall in  $-z$  direction
- DNS from Coleman, Kim & Le, 1996
- Average over 16 realizations



# Bypass transition on a flat plate

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- **Numerical method**

- Modified version of a code written by Pierce and Jacobs (Stanford U.)
- Second order FD on a staggered grid
- Fractional step method with spectral pressure solver
- Turbulent inflow conditions (Jacobs & Durbin, 2000, 2001)

- **Comparisons**

- DNS
- No model
- Dynamic Smagorinsky
- Multiscale: dynamic

# Bypass transition on a flat plate

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$$Re_\delta = \frac{U_{bulk} \delta_{inlet}}{\nu} = 795 \quad L_x = 620 \delta_{inlet} \quad \overline{N}_x/N'_x = 192/384$$

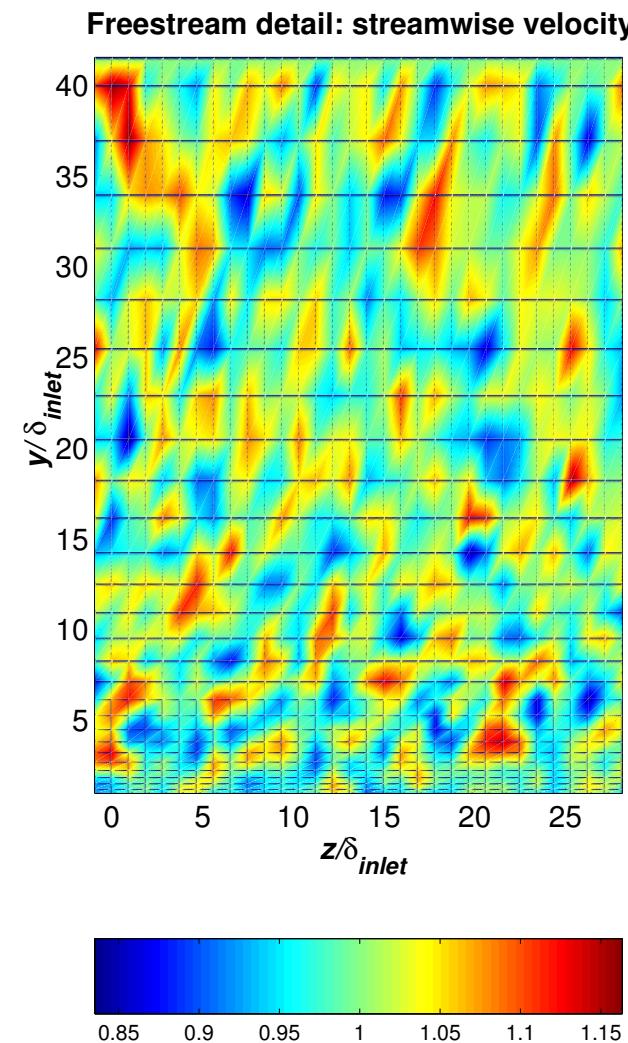
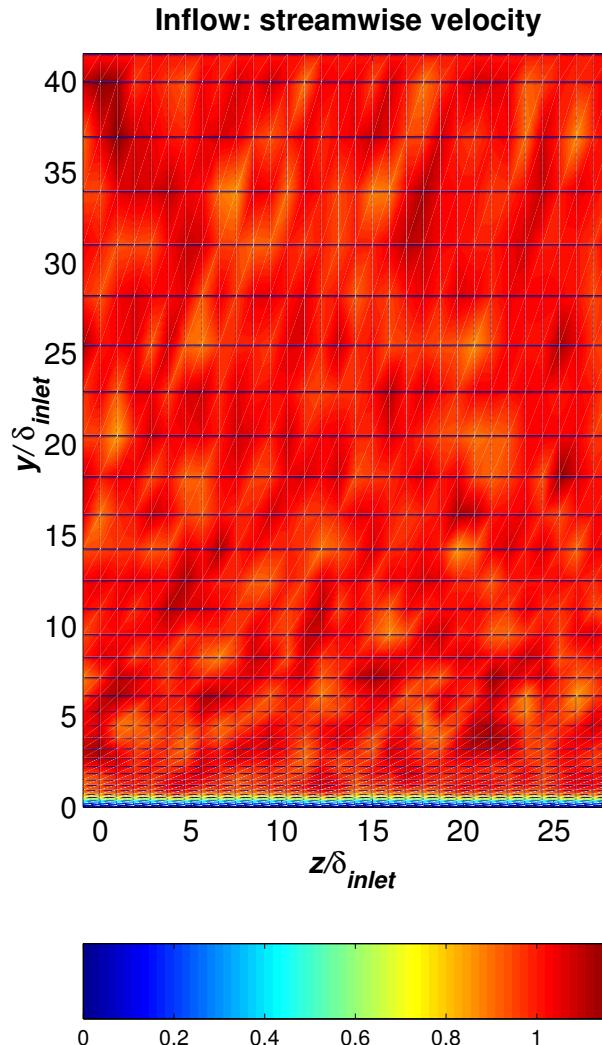
$$Re_\theta = \frac{U_{bulk} \theta_{inlet}}{\nu} = 27.8 \quad L_y = 40 \delta_{inlet} \quad \overline{N}_y/N'_y = 16/32$$

$$Tu = 3.5 \% \quad L_z = 30 \delta_{inlet} \quad \overline{N}_z/N'_z = 16/32$$

	$N_x$	$N_y$	$N_z$	$\Delta_x^+$	$\Delta_{y_{min}}^+$	$\Delta_z^+$	CPU time
DNS	2048	180	192	8 – 11	~ 0.1	4 – 6	~ 64000 cpu hrs.
LES	384	32	32	44 – 59	1.5 – 2	25 – 35	~ 48 cpu hrs.

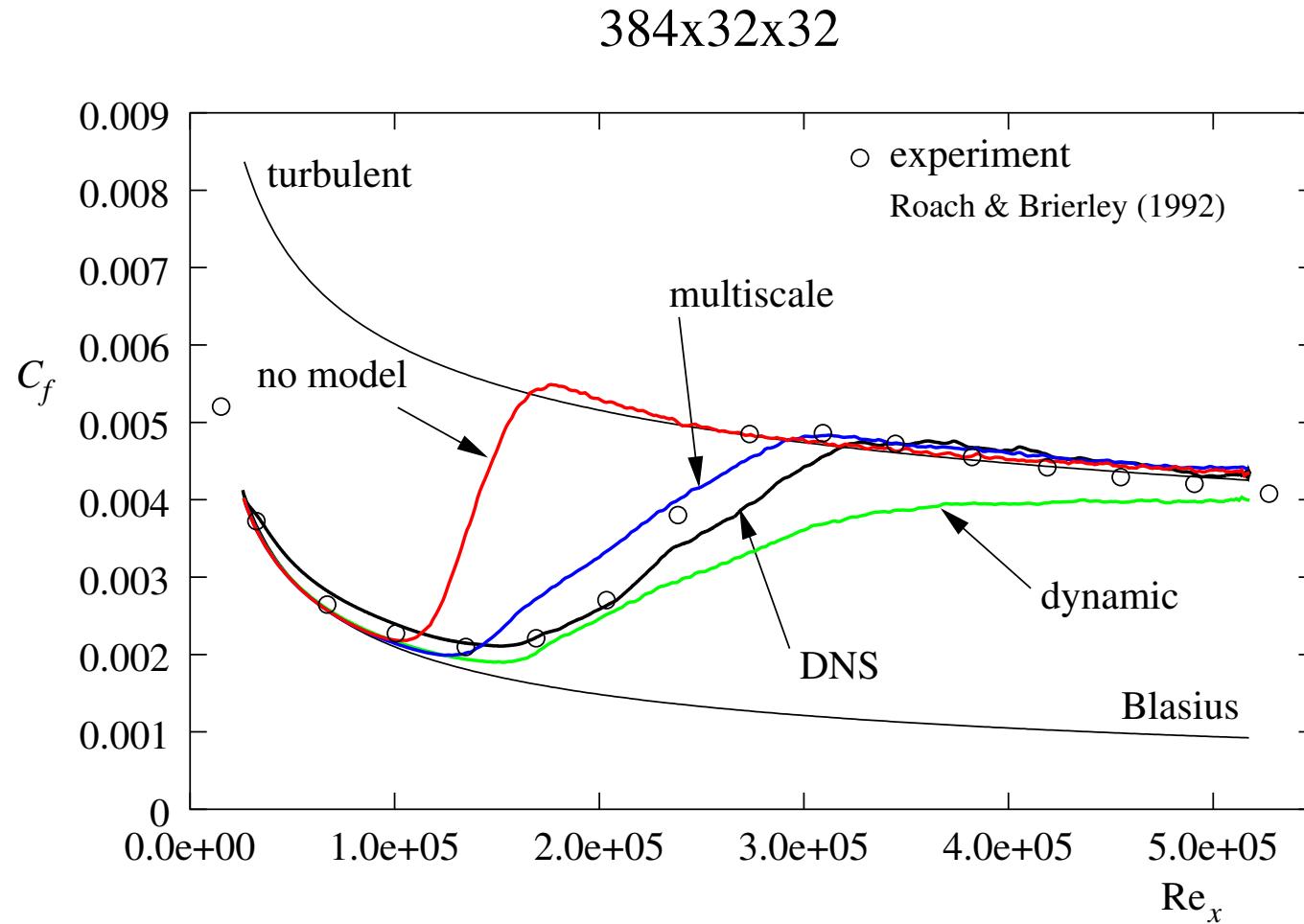
# Bypass transition on a flat plate: inflow conditions

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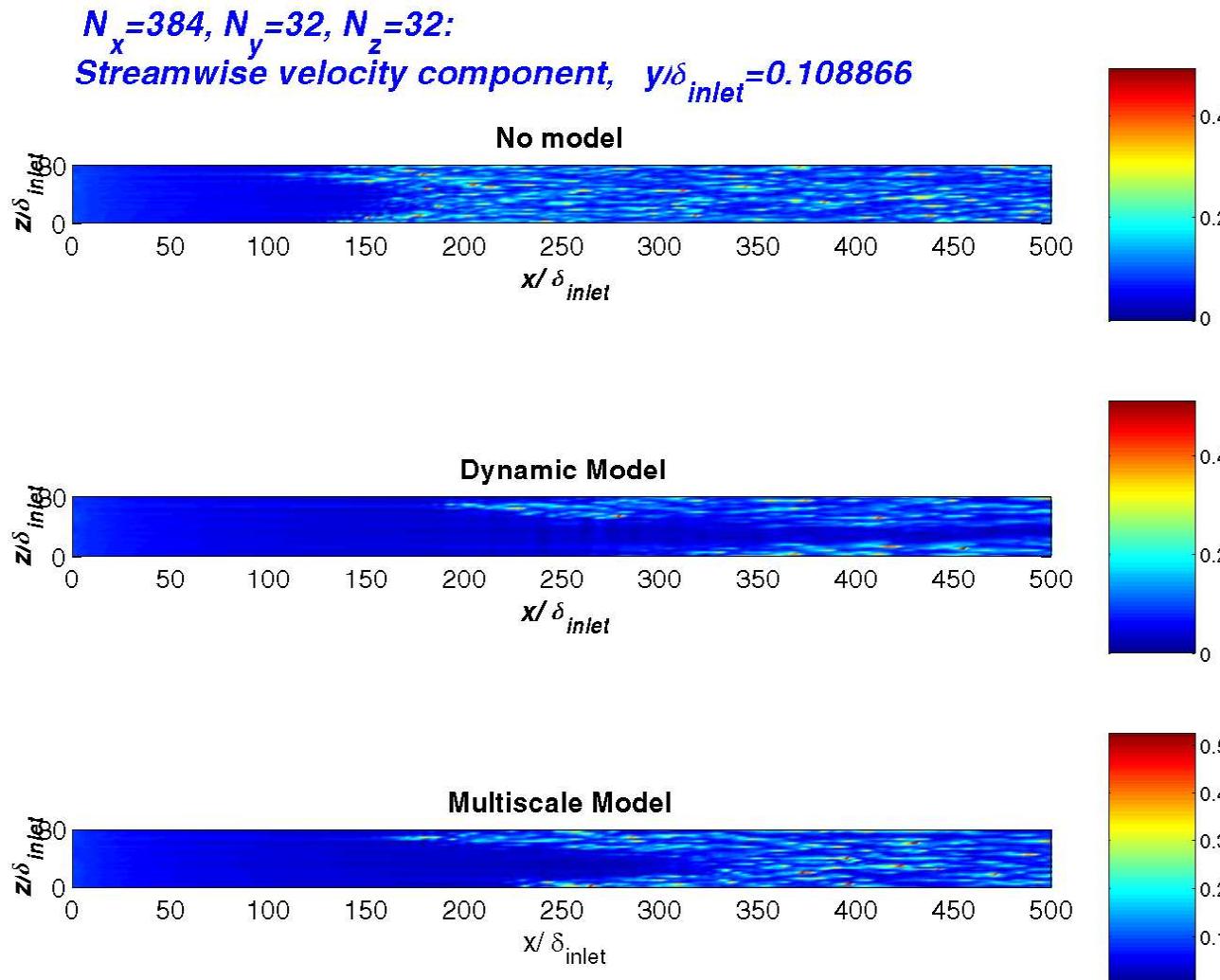


# Bypass transition on a flat plate: skin friction

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# Bypass transition on a flat plate



# Concluding Remarks

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## So far

- A variational multiscale formulation
- Treat the effect of subgrid-scales on large and small resolved scales differently
- Tested this variational multiscale formulation for
  - Homogeneous isotropic flows (*Physics of Fluids*, **13**, 505-512, 2001)
  - Channel flows (*Physics of Fluids*, **13**, 1784-1799, 2001)
- Effect of the size of the partition  $\bar{k}/k'$
- Performs *particularly* well on non-equilibrium flows and produces very accurate fluctuating quantities, of importance in acoustics

# Concluding Remarks

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## Current and future work

- Dynamic calculation of  $C'_s$
- Finite differences (J. Gullbrand)
- Implementation on unstructured meshes: finite elements/ hierarchical basis (K. Jansen), finite volumes/ multigrid operators (C. Farhat)
- Spectral/finite volumes (S. Collis)
- Applications to more challenging problems