

From Symmetric to Quasi-symmetric

Nantel Bergeron (York University)
Canadian Research Chair in
Mathematics

outline of the talk

$$\text{Sym} \quad \hookrightarrow \quad \mathbb{Q}[x_1, x_2, \dots, x_n]$$

Symmetric polynomials:

$$P(x_1, x_2, \dots, x_n) = P(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$$

Where in Math do we see that?

outline of the talk

$$\text{Sym} \hookrightarrow \text{QSym} \hookrightarrow \mathbb{Q}[x_1, x_2, \dots, x_n]$$

Symmetric polynomials:

$$P(x_1, x_2, \dots, x_n) = P(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$$

Where in Math do we see that?

Quasi-symmetric polynomials:

$$\sigma^* X_I^\alpha = X_{\sigma \cdot I}^\alpha$$

$$P(x_1, x_2, \dots, x_n) = \sigma^* P(x_1, x_2, \dots, x_n)$$

Where in Math do we see that?

more realistic

outline of the talk

$$\text{Sym} \hookrightarrow^{\text{QSym}} \mathbb{Q}[x_1, x_2, \dots, x_n]$$

Symmetric polynomials:

$$P(x_1, x_2, \dots, x_n) = P(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$$

Where in Math do we see that?

Quasi-symmetric polynomials:

$$\sigma * X_I^\alpha = X_{\sigma, I}^\alpha$$

$$P(x_1, x_2, \dots, x_n) = \sigma * P(x_1, x_2, \dots, x_n)$$

Where in Math do we see that?

Roots of a polynomial

$$P(z) = z^2 + 1z - 2 = (z - 1)(z + 2)$$

$$1 = -1 + 2$$

$$-2 = (-1)(2)$$

$$P(z) = z^2 + b z + c = (z - x_1)(z - x_2)$$

$$b = -x_1 - x_2$$

$$c = (-x_1)(-x_2)$$

$$x_1 = \frac{-b + \sqrt{b^2 - 4c}}{2}$$

$$x_2 = \frac{-b - \sqrt{b^2 - 4c}}{2}$$

Roots of a polynomial

$$P(z) = z^n - e_1 z^{n-1} + e_2 z^{n-2} - \dots + (-1)^n e_n$$

$$= (z - x_1)(z - x_2) \dots (z - x_n)$$

$$e_1 = x_1 + x_2 + \dots + x_n$$

$$e_2 = x_1 x_2 + x_1 x_3 + x_2 x_3 + \dots + x_{i_1} x_{i_2} + \dots + x_{n-1} x_n$$

⋮

$$e_k = \sum_{i_1 < i_2 < \dots < i_k} x_{i_1} x_{i_2} \dots x_{i_k}$$

⋮

$$e_n = x_1 x_2 \dots x_n$$

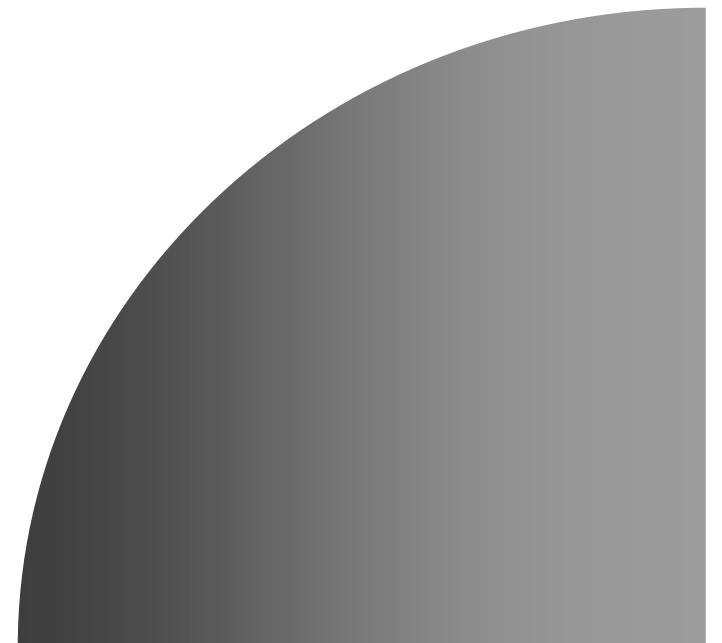
$i_1 < i_2$

Roots of a polynomial

$$P(z) = z^n - e_1 z^{n-1} + e_2 z^{n-2} - \dots + (-1)^n e_n$$

$$= (z - x_1) (z - x_2) \dots (z - x_n)$$

$$e_k = \sum_{i_1 < i_2 < \dots < i_k} x_{i_1} x_{i_2} \dots x_{i_k}$$



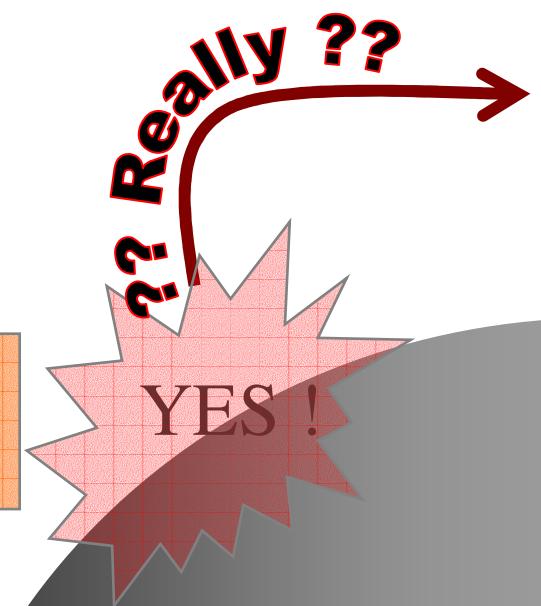
Roots of a polynomial

$$P(z) = z^n - e_1 z^{n-1} + e_2 z^{n-2} - \dots + (-1)^n e_n$$

$$= (z - x_1) (z - x_2) \dots (z - x_n)$$

$$e_k = \sum_{i_1 < i_2 < \dots < i_k} x_{i_1} x_{i_2} \dots x_{i_k}$$

$$D = \prod_{p < q} (x_p - x_q)^2 = f(e_1, e_2, \dots, e_n)$$



Why do we care to know this?

$D = 0 \longleftrightarrow P(z)$ has two equal roots

Sym: Symmetric Polynomials

Variables: x_1, x_2, \dots, x_n

Permutations of n :

$$\sigma: \{1, 2, \dots, n\} \longrightarrow \{1, 2, \dots, n\} \quad \text{bijection}$$

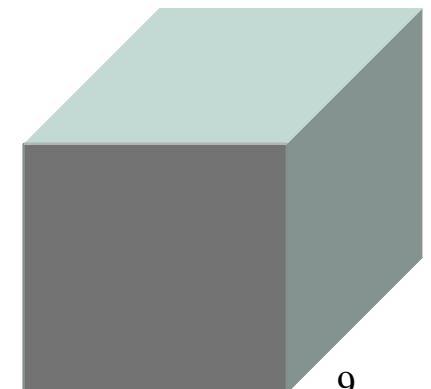
Action of symmetric group on polynomials

$$\sigma \cdot P(x_1, x_2, \dots, x_n) = P(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$$

Example $P(x_1, x_2, x_3) = x_1 + x_2 x_3 - 4 x_2$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$\sigma \cdot P(x_1, x_2, x_3) = x_3 + x_1 x_2 - 4 x_1$$



Sym: Symmetric Polynomials

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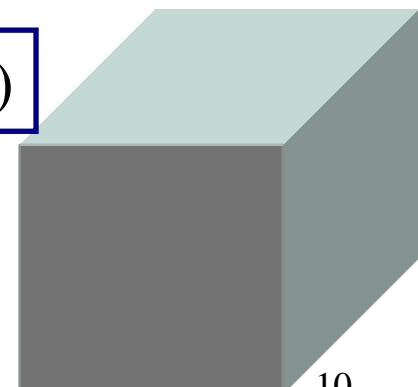
Action of symmetric group on polynomials

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Symmetric Polynomials

$$P(x_1, x_2, \dots, x_n) = P(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$$

For all permutations σ



Sym: Symmetric Polynomials

Symmetric Polynomials

$$P(x_1, x_2, \dots, x_n) = P(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$$

Example $e_1 = x_1 + x_2 + \dots + x_n$

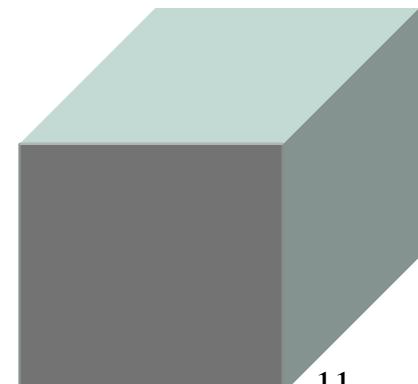
$e_2 = x_1 x_2 + x_1 x_3 + x_2 x_3 + \dots + x_{i_1} x_{i_2} + \dots + x_{n-1} x_n$

$e_1^2 e_2$

Any polynomials in $\underbrace{e_1, e_2, \dots, e_n}_{\text{Elementary Symmetric polynomials}}$



Elementary Symmetric polynomials



Sym: Symmetric Polynomials

Symmetric Polynomials

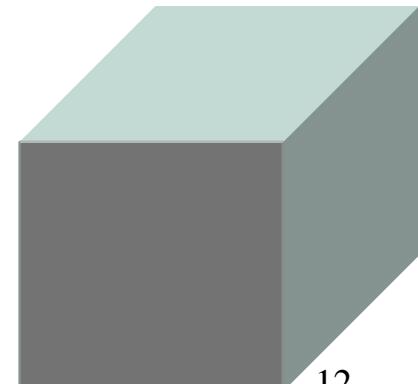
$$P(x_1, x_2, \dots, x_n) = P(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$$

Set of all symmetric polynomials

Invariant under the action
of the symmetric group

$$\text{Sym}_n = \underbrace{\mathbb{Q}[x_1, x_2, \dots, x_n]}_{S_n} = \{ P \mid \sigma. P = P \}$$

Set of all polynomials with
rational coefficients

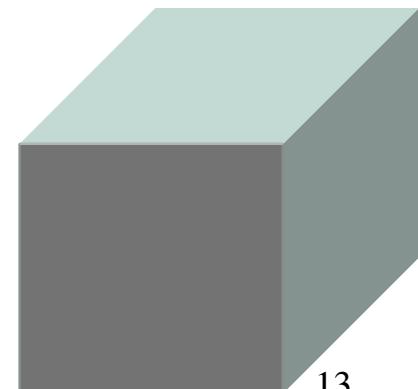


Sym: Symmetric Polynomials

$$Sym_n = \mathbb{Q}[x_1, x_2, \dots, x_n]^{S_n} = \{ P \mid \sigma. P = P \}$$

Fundamental Theorem (Newton)

$$Sym_n = \mathbb{Q}[e_1, e_2, \dots, e_n]$$



Sym: Symmetric Polynomials

Fundamental Theorem (Newton)

$$Sym_n = \mathbb{Q}[e_1, e_2, \dots, e_n]$$

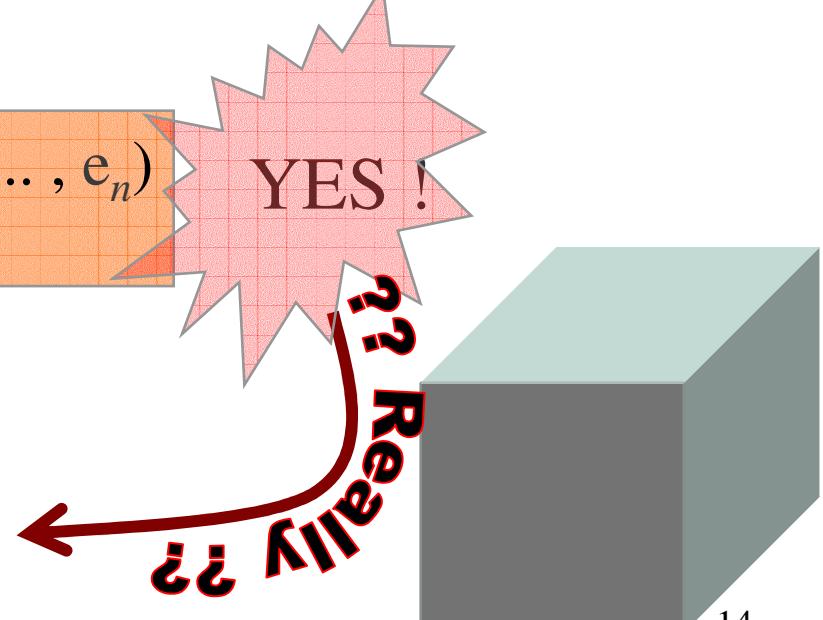
$$P(z) = z^n - e_1 z^{n-1} + e_2 z^{n-2} - \dots + (-1)^n e_n$$

$$= (z - x_1)(z - x_2) \dots (z - x_n)$$

$$D = \prod_{p < q} (x_p - x_q)^2 = f(e_1, e_2, \dots, e_n)$$

YES !

Yes!... D is Symmetric.



Sym: Symmetric Polynomials

Fundamental Theorem (Newton)

$$Sym_n = \mathbb{Q}[e_1, e_2, \dots, e_n]$$

$$n=2$$

$$P(z) = z^2 - e_1 z + e_2 = (z - x_1)(z - x_2)$$

$$D = (x_1 - x_2)^2 = x_1^2 - 2x_1x_2 + x_2^2$$

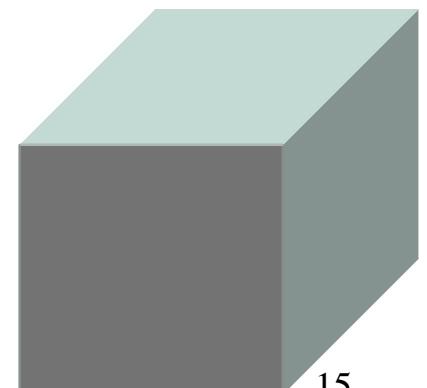
$$e_1^2 = (x_1 + x_2)^2 = x_1^2 + 2x_1x_2 + x_2^2$$

$$D - e_1^2 = -4x_1x_2$$

$$D - e_1^2 + 4e_2 = 0$$

$$D = e_1^2 - 4e_2$$

$$\frac{-b \pm \sqrt{b^2 - 4c}}{2}$$



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Nov 2002

Sym: Symmetric Polynomials

Fundamental Theorem (Newton)

$$Sym_n = \mathbb{Q}[e_1, e_2, \dots, e_n]$$

$n=3$

$$P(z) = z^3 - e_1 z^3 + e_2 z - e_3 = (z - x_1)(z - x_2)(z - x_3)$$

$$D = (x_1 - x_2)^2 (x_1 - x_3)^2 (x_2 - x_3)^2 = x_1^4 x_2^2 - 2 x_1^4 x_2 x_3 + x_1^4 x_3^2 + \dots$$

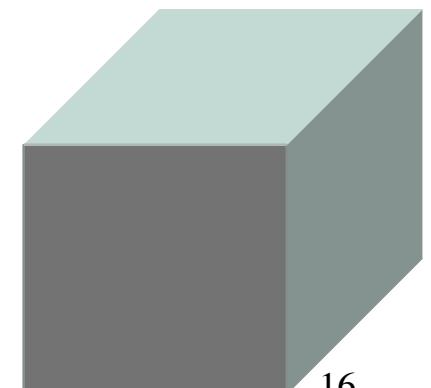
$$e_1^2 e_2^2 = x_1^4 x_2^2 + 2 x_1^4 x_2 x_3 + x_1^4 x_3^2 + \dots$$

$$D - e_1^2 e_2^2 = -4 x_1^4 x_2 x_3 - 4 x_1^3 x_2^3 + \dots$$

$$D - e_1^2 e_2^2 + 4 e_1^3 e_3 = -4 x_1^3 x_2^3 + \dots$$

⋮

$$D = e_1^2 e_2^2 - 4 e_1^3 e_3 - 4 e_2^3 + 18 e_1 e_2 e_3 - 27 e_3^2$$



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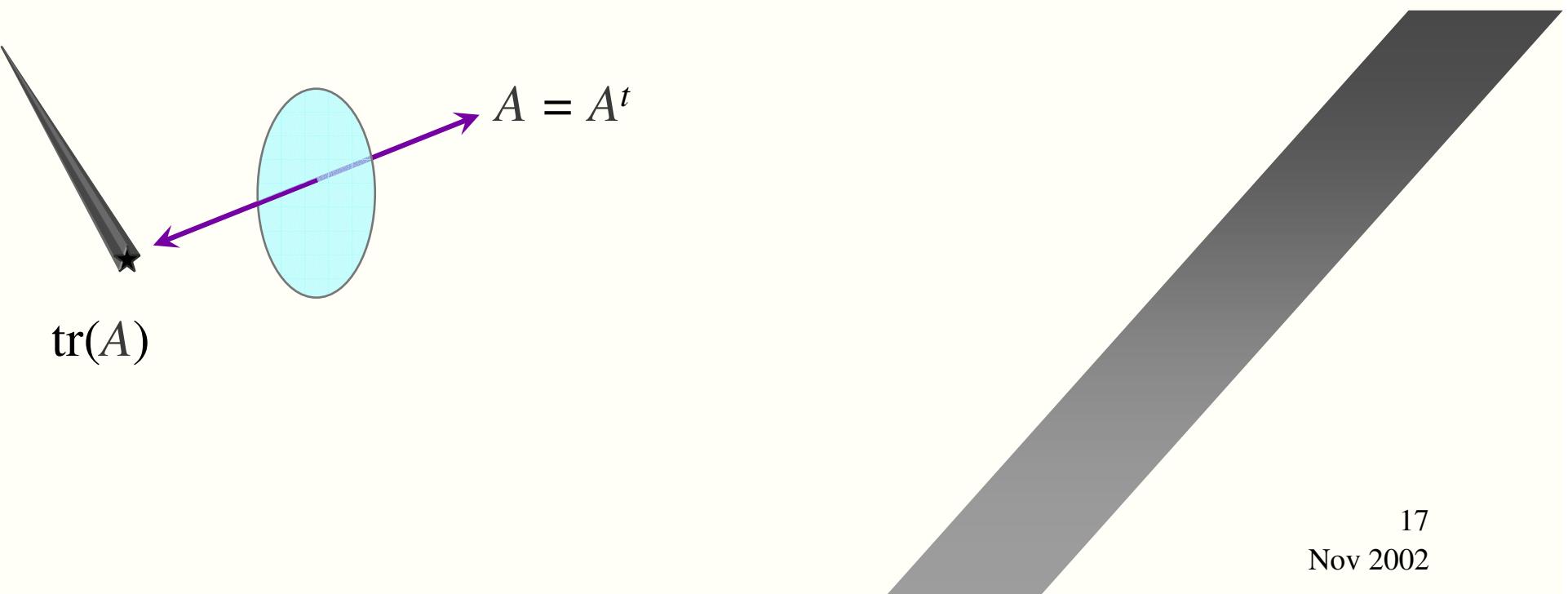
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A Story

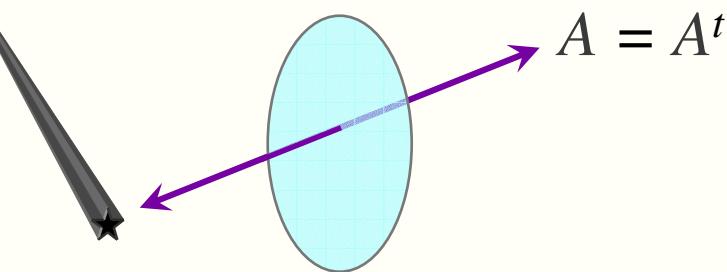
You have a physicist friend that has a theory about nature...

He tells you that particles must be like symmetric matrices but we can only measure (see) their traces.

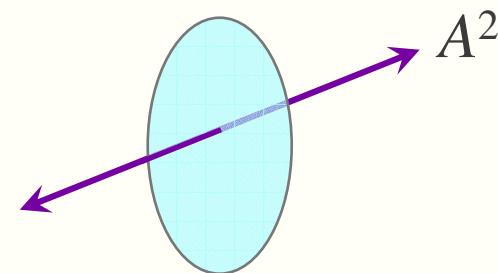
You start thinking.... Well, if the matrices are symmetric, then you remember from linear algebra that this means all the eigenvalues are real, so at least we will measure something.



A Story

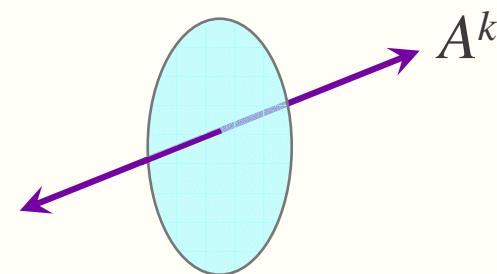


$$\text{tr}(A) = \lambda_1 + \lambda_2 + \dots + \lambda_n$$



$$\text{tr}(A^k) = \lambda_1^k + \lambda_2^k + \dots + \lambda_n^k$$

⋮



Back to the Story

$$\left. \begin{array}{l} \text{tr}(A) = \lambda_1 + \lambda_2 + \dots + \lambda_n = p_1 \\ \text{tr}(A^2) = \lambda_1^2 + \lambda_2^2 + \dots + \lambda_n^2 = p_2 \\ \vdots \\ \text{tr}(A^k) = \lambda_1^k + \lambda_2^k + \dots + \lambda_n^k = p_k \end{array} \right\}$$

Power sum symmetric
Polynomials

$$Sym_n = \mathbb{Q}[p_1, p_2, \dots, p_n]$$

Another Fundamental Theorem

If we find (measure) all p_1, p_2, \dots, p_n then we can get

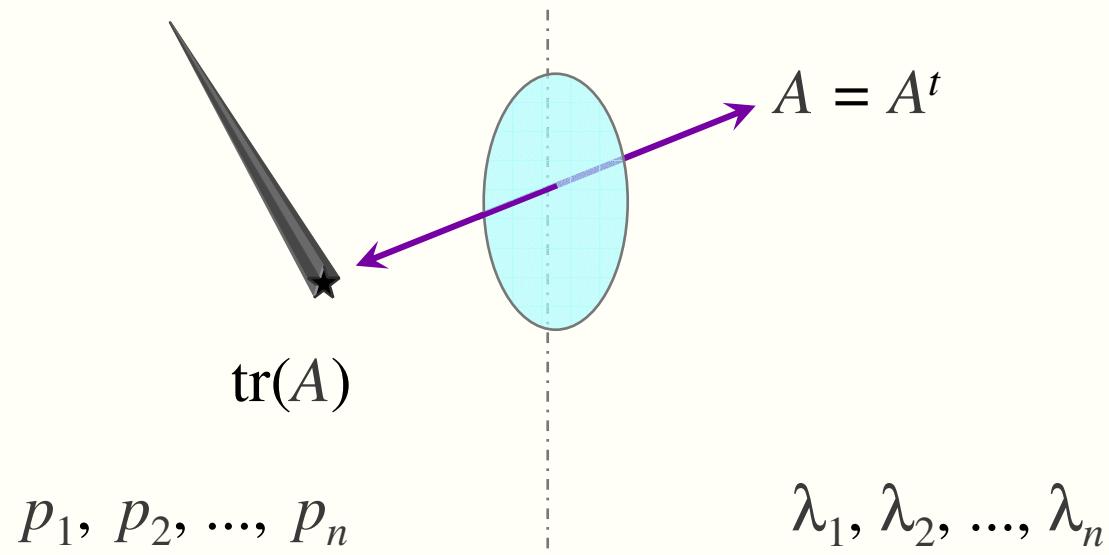
e_1, e_2, \dots, e_n .

From this we get the characteristic polynomial

$$P(z) = z^n - e_1 z^{n-1} + e_2 z^{n-2} - \dots + (-1)^n e_n$$

Factorizing, we get $\lambda_1, \lambda_2, \dots, \lambda_n$

Back to the Story



p_1, p_2, \dots, p_n $\lambda_1, \lambda_2, \dots, \lambda_n$

$$\text{Sym}_n = \mathbb{Q}[p_1, p_2, \dots, p_n]$$

$$Sym_n = \mathbb{Q}[e_1, e_2, \dots, e_n] = \mathbb{Q}[p_1, p_2, \dots, p_n]$$

Algebra

Representation Theory

Mathematical Physics

Topology

Galois Theory

Combinatorics

• • •

Geometry

Where in Math do we see that?

$$Sym_n = \mathbb{Q}[e_1, e_2, \dots, e_n] = \mathbb{Q}[p_1, p_2, \dots, p_n]$$

Sym_n is a ring, that is a vector space with a multiplication

Bases:

Monomial symmetric polynomials: m_λ

Monomials: $X^\eta = x_1^{\eta_1} x_2^{\eta_2} \dots x_n^{\eta_n}$

$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0.$

$\eta_i \geq 0$

Monomial symmetric polynomial: $m_\eta = \sum_{\sigma \text{ in } S_n} \sigma.X^\eta$

But for any σ in S_n : $m_\lambda = m_{\sigma.\lambda}$ So we can always assume

Example: $m_{(2,1,1)} = 2(x_1 x_2 x_3 + x_2 x_1 x_3 + x_1 x_2 x_3) = m_{(1,2,1)}$
 $= m_{(1,1,2)}$

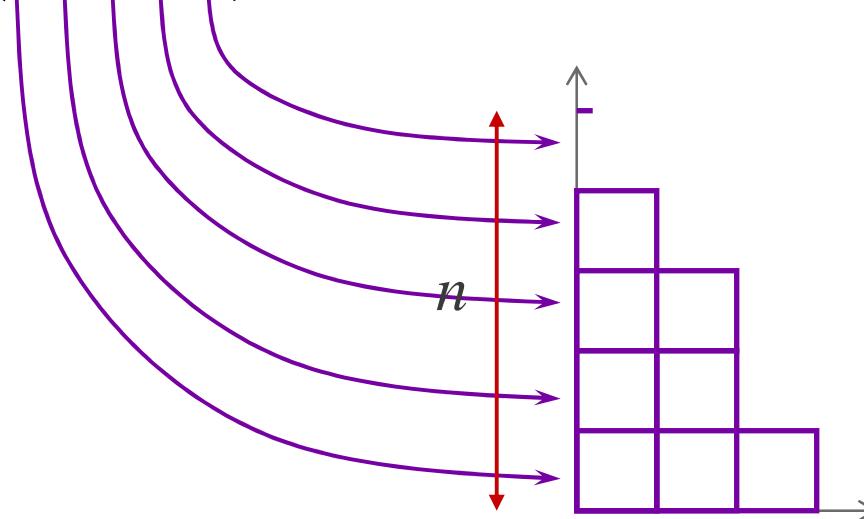
$$Sym_n = \mathbb{Q}[e_1, e_2, \dots, e_n] = \mathbb{Q}[p_1, p_2, \dots, p_n]$$

Bases:

Monomial symmetric polynomials: m_λ

$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$.

Example: $\lambda = (3, 2, 2, 1, 0)$



$$Sym_n = \mathbb{Q}[e_1, e_2, \dots, e_n] = \mathbb{Q}[p_1, p_2, \dots, p_n]$$

Bases:

Monomial symmetric polynomials: m_λ

Elementary symmetric polynomials: e_λ

Monomial in the e_1, e_2, \dots, e_n can be described by

$e_\lambda = e_{\lambda_1} e_{\lambda_2} \dots e_{\lambda_k}$ where $n \geq \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0$,
and $k \geq 0$.

$$Sym_n = \mathbb{Q}[e_1, e_2, \dots, e_n] = \mathbb{Q}[p_1, p_2, \dots, p_n]$$

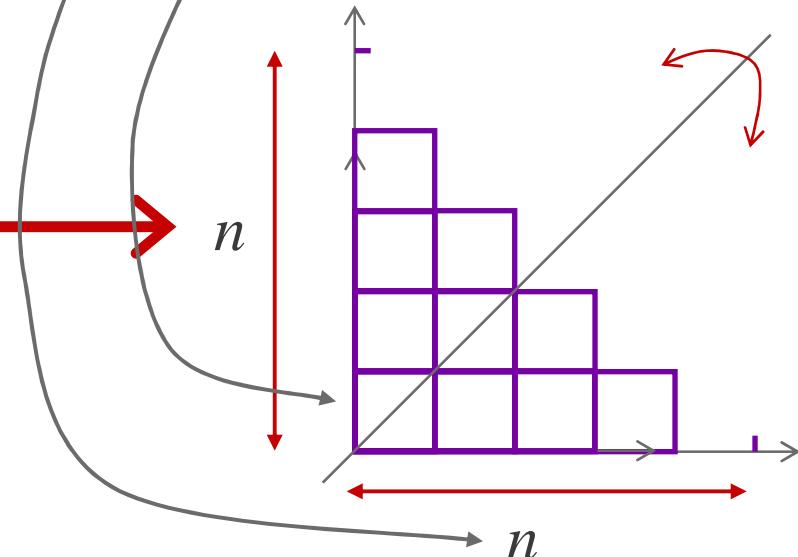
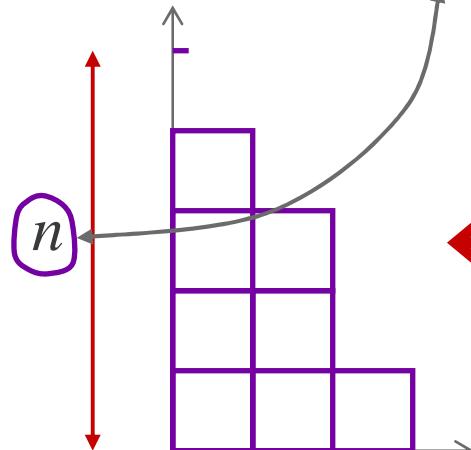
Bases:

Monomial symmetric polynomials: m_λ

Elementary symmetric polynomials: e_λ

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0.$$

$$n \geq \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0, k \geq 0.$$



$$Sym_n = \mathbb{Q}[e_1, e_2, \dots, e_n] = \mathbb{Q}[p_1, p_2, \dots, p_n]$$

Bases:

Monomial symmetric polynomials: m_λ

Elementary symmetric polynomials: e_λ

Power sum symmetric polynomials: p_λ

Homogeneous symmetric polynomials: h_λ

Schur symmetric polynomials: s_λ

$$h_k \text{ coefficient of } z^k \text{ in } \prod_{i=1}^n \frac{1}{1 - x_i z}$$

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \dots h_{\lambda_k}$$

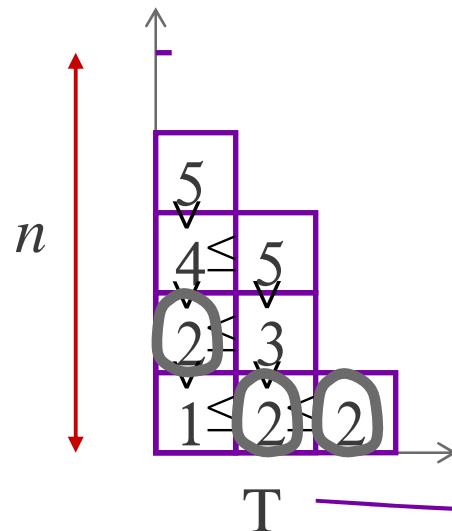
$$Sym_n = \mathbb{Q}[e_1, e_2, \dots, e_n] = \mathbb{Q}[p_1, p_2, \dots, p_n]$$

Schur symmetric polynomials: s_λ

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0.$$

Young tableau: $T: \lambda \rightarrow \{1, 2, \dots, n\}$

Weakly increasing in rows, Strictly increasing in columns



x^T defined to be $\prod_{i=1}^n x_i^{T^{-1}(i)}$

$$x^T = x_1^1 x_2^3 x_3^1 x_4^2 x_5^2$$

$$Sym_n = \mathbb{Q}[e_1, e_2, \dots, e_n] = \mathbb{Q}[p_1, p_2, \dots, p_n]$$

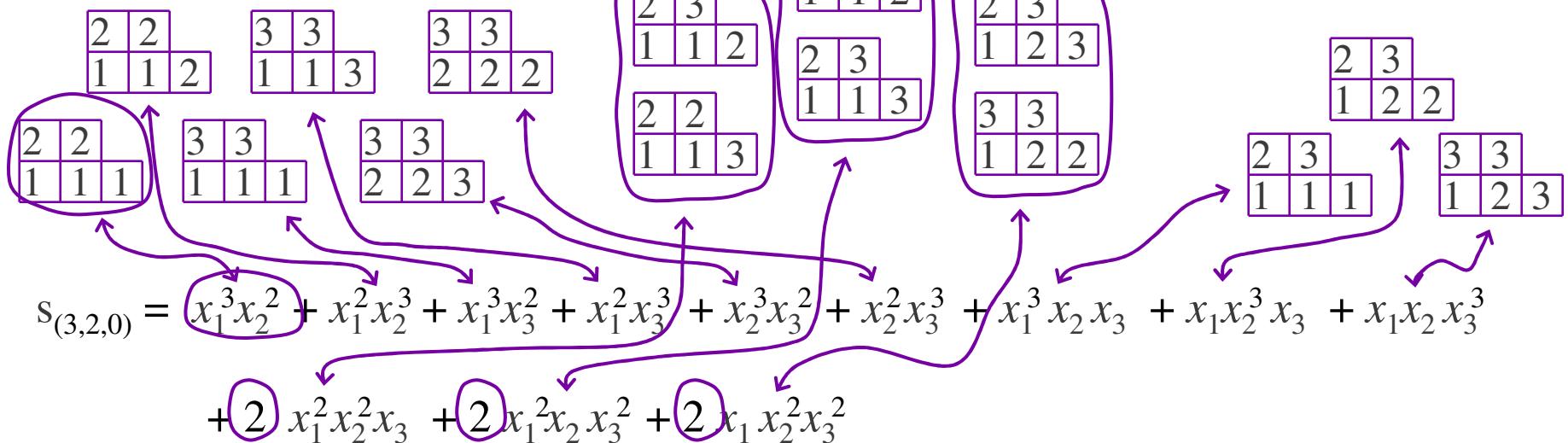
Schur symmetric polynomials: s_λ

$$= \sum_{T: \lambda \rightarrow \{1, 2, \dots, n\}} x^T$$

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0.$$

Young tableau: $T: \lambda \rightarrow \{1, 2, \dots, n\}$, $x^T = \prod_{i=1}^n x_i^{T^{-1}(i)}$

Example: $\lambda = (3, 2, 0)$



$$Sym_n = \mathbb{Q}[e_1, e_2, \dots, e_n] = \mathbb{Q}[p_1, p_2, \dots, p_n]$$

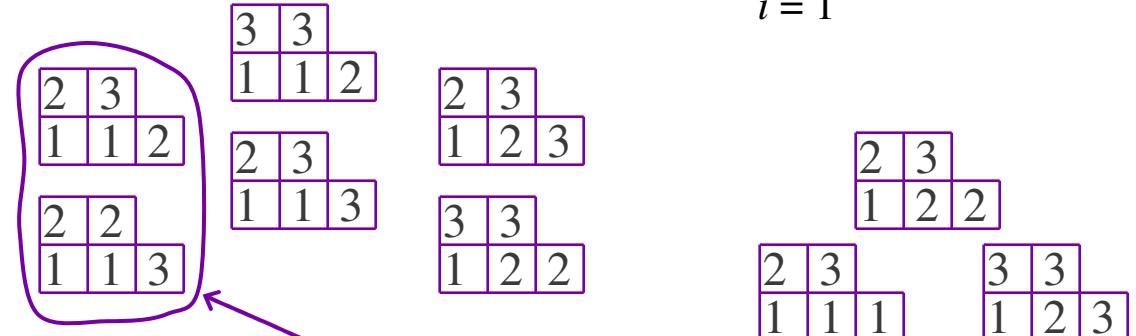
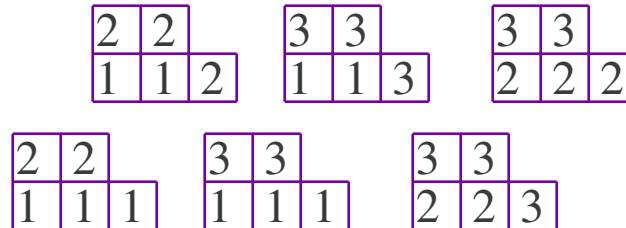
Schur symmetric polynomials: s_λ

$$= \sum_{T: \lambda \rightarrow \{1, 2, \dots, n\}} x^T$$

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Young tableau: $T: \lambda \rightarrow \{1, 2, \dots, n\}$, $x^T = \prod_{i=1}^n x_i^{T^{-1}(i)}$

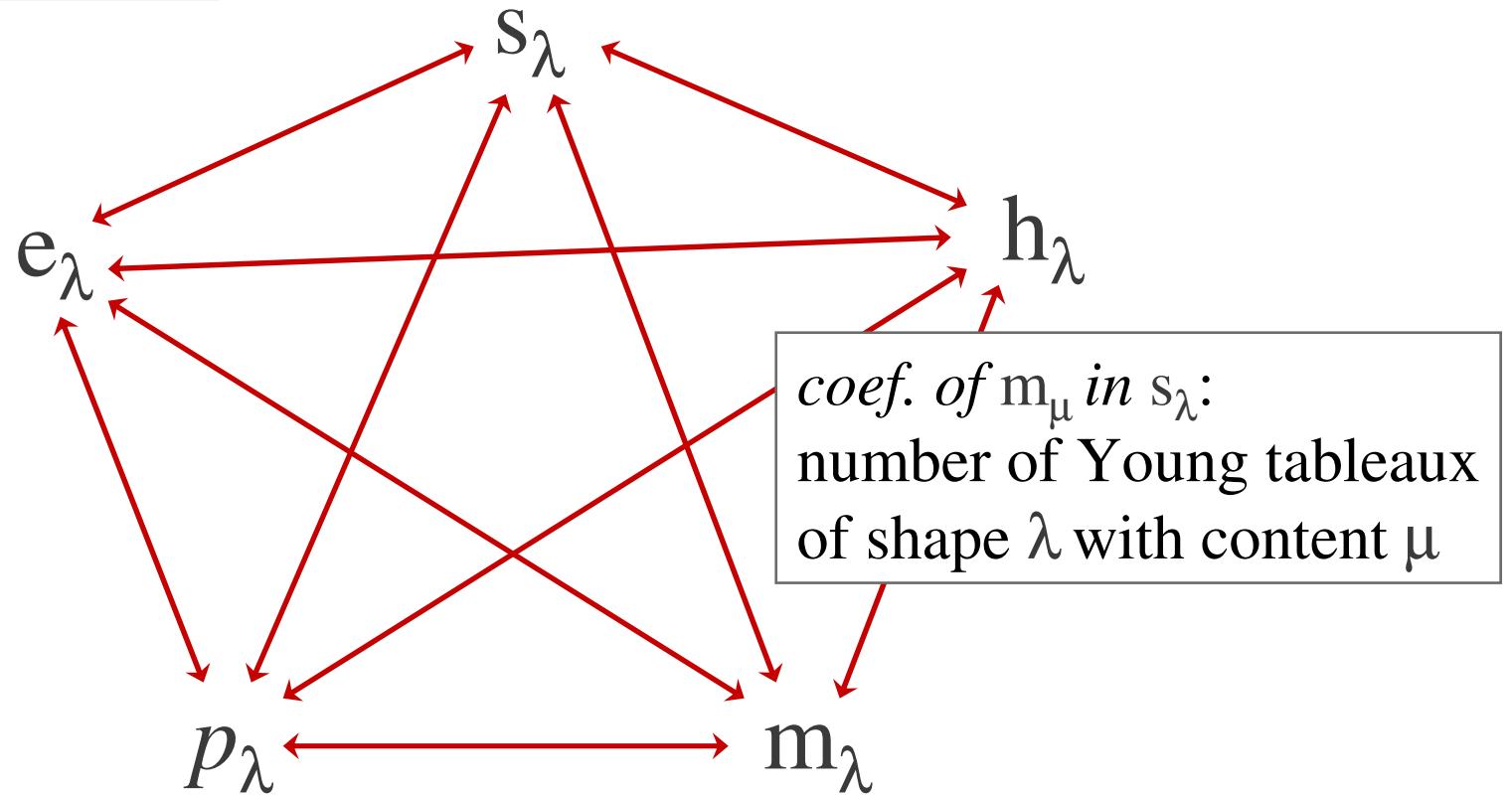
Example: $\lambda = (3, 2, 0)$



$$\begin{aligned}
 s_{(3,2,0)} &= \underbrace{x_1^3 x_2^2 + x_1^2 x_2^3 + x_1^3 x_3^2 + x_1^2 x_3^3 + x_2^3 x_3^2 + x_2^2 x_3^3}_{+ x_1^3 x_2 x_3 + x_1 x_2^3 x_3 + x_1 x_2 x_3^3} \\
 &\quad + \underbrace{+ 2 x_1^2 x_2^2 x_3 + 2 x_1^2 x_2 x_3^2 + 2 x_1 x_2^2 x_3^2}_{= m_{(3,2,0)} + m_{(3,1,1)} + 2m_{(2,2,1)}}
 \end{aligned}$$

$$Sym_n = \mathbb{Q}[e_1, e_2, \dots, e_n] = \mathbb{Q}[p_1, p_2, \dots, p_n]$$

Change of basis



$$Sym_n = \mathbb{Q}[e_1, e_2, \dots, e_n] = \mathbb{Q}[p_1, p_2, \dots, p_n]$$

Multiplicative Structure

$$s_\lambda s_\mu = \sum c_{\lambda\mu}^v s_v$$

s_v is a linear basis of Sym_n .
So $s_\lambda s_\mu$ can be express as a linear combination of the s_v

*These number are positive integers,
are wonderful to investigate,
and have “physical” meaning.*

Quasi-symmetric polynomials: QSym

Variables: x_1, x_2, \dots, x_n

Compositions

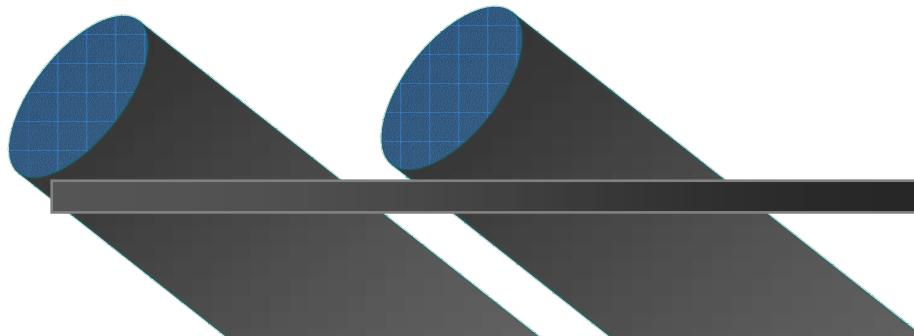
$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k), \quad \alpha_i > 0 \quad \text{and} \quad n \geq k = \ell(\alpha) \geq 0.$$

Monomials

$$X_I^\alpha = x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k} \quad I = \{i_1 < i_2 < \cdots < i_k\}$$

example:

$$x_2^3 x_3^1 x_5^4 \longleftrightarrow I = \{2, 3, 5\} \quad \text{and} \quad \alpha = (3, 1, 4)$$



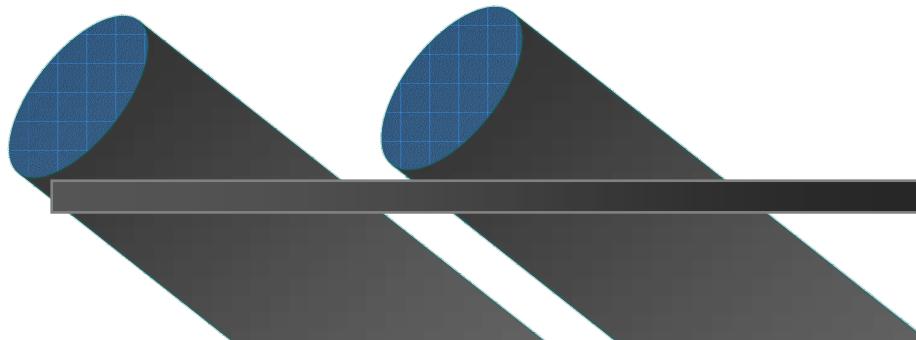
Quasi-symmetric polynomials: QSym

Monomial quasi-symmetric polynomial indexed by α

$$M_\alpha(X) = \sum_{\substack{I \subseteq \{1, 2, \dots, n\} \\ |I| = \ell(\alpha)}} X_I^\alpha$$

Example: $n = 4$ and $\alpha = (3, 1, 4)$ $X = x_1, x_2, x_3, x_4$

$$I \subseteq \{1, 2, 3, 4\} \text{ and } |I| = 3$$
$$M_{(3, 1, 4)} = x_{1 2 3}^{3 1 4} + x_{1 2 4}^{3 1 4} + x_{1 3 4}^{3 1 4} + x_{2 3 4}^{3 1 4}$$



Quasi-symmetric polynomials: QSym

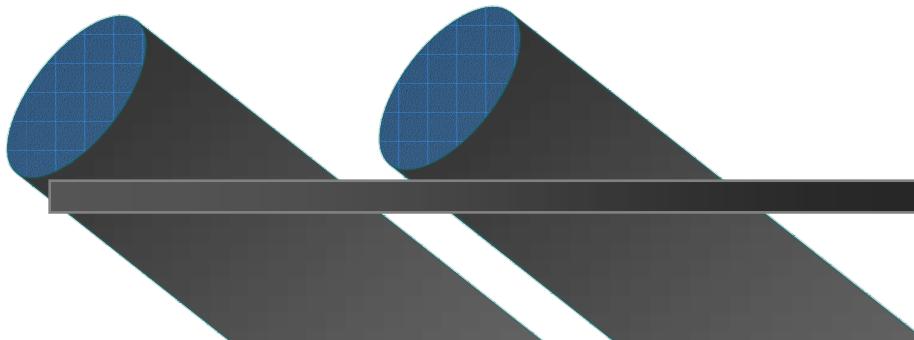
Monomial quasi-symmetric polynomial

$$M_\alpha(X) = \sum_{\substack{I \subseteq \{1, 2, \dots, n\} \\ |I| = \ell(\alpha)}} X_I^\alpha$$

Sym \subset QSym

$$m_\lambda = \sum_{\alpha \text{ in the orbit of } \lambda} M_\alpha$$

Example: $m_{(3,2,2)} = M_{(3,2,2)} + M_{(2,3,2)} + M_{(2,2,3)}$



Quasi-symmetric polynomials: QSym

Monomial quasi-symmetric polynomial

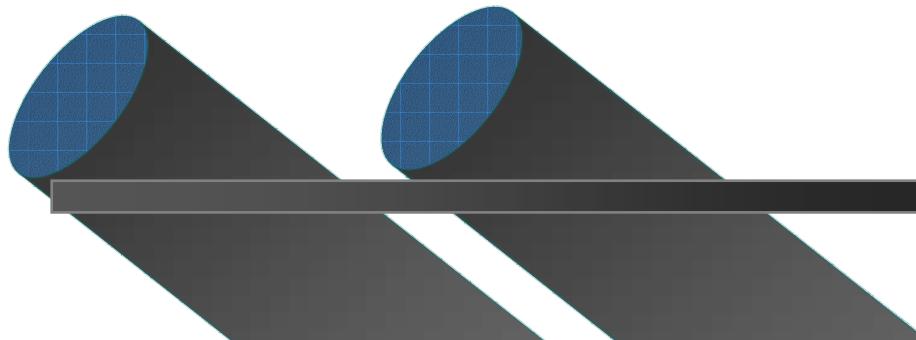
$$M_\alpha(X) = \sum_{\substack{I \subseteq \{1, 2, \dots, n\} \\ |I| = \ell(\alpha)}} X_I^\alpha$$

Different action of Symmetric group on monomial

$$\sigma^* X_I^\alpha = X_{\sigma I}^\alpha$$

The Ring of Quasi-symmetric polynomials

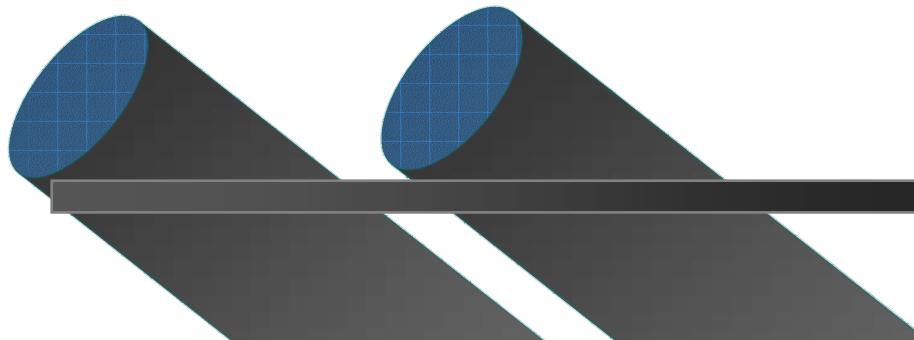
$$QSym = \{ P(X) \mid \sigma^* P = P \}$$



Quasi-symmetric polynomials: QSym

To study further Quasi-symmetric polynomials:

- Bases
- multiplicative structure
- change of basis
- Applications (where in math do we see that)



Multiplication in QSym

$$F_\alpha F_\beta = \sum_{\gamma \text{ is a shuffle of } \alpha \text{ and } \beta} F_\gamma$$

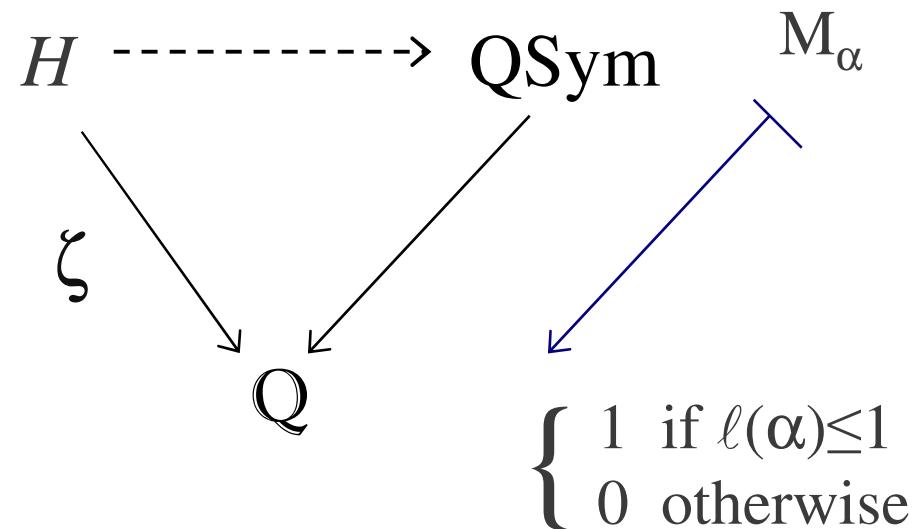
$$\begin{array}{c} \bullet 2 \quad \bullet 4 \\ | \qquad | \\ \bullet 1 \quad \bullet 5 \end{array} \longrightarrow \begin{array}{ccccccc} \bullet 4 & \bullet 4 & \bullet 2 & \bullet 4 & \bullet 2 & \bullet 2 & \bullet 2 \\ | & | & | & | & | & | & | \\ \bullet 5 & \bullet 2 & \bullet 4 & \bullet 5 & \bullet 2 & \bullet 4 & \bullet 1 \\ | & | & | & | & | & | & | \\ \bullet 2 & \bullet 5 & \bullet 1 & \bullet 1 & \bullet 1 & \bullet 5 & \bullet 5 \\ | & | & | & | & | & | & | \\ \bullet 1 & \bullet 1 & \bullet 1 & \bullet 1 & \bullet 5 & \bullet 1 & \bullet 5 \end{array}$$
$$F_{(2)} F_{(1,1)} = F_{(3,1)} + F_{(2,2)} + F_{(2,1,1)} + F_{(1,3)} + F_{(1,2,1)} + F_{(1,1,2)}$$

Universal Property of QSym

- $n \rightarrow \infty$, QSym is Hopf Algebra with

$$\Delta(M_\alpha) = \sum_{\alpha=\beta\gamma} M_\beta \otimes M_\gamma$$

Universal Property: Given any H and $\zeta : H \rightarrow Q$



More Properties of QSym

(and no time to discuss them deeply)

- Temperley-Lieb invariants: $\text{QSym} = \mathbb{Q}[x_1, x_2, \dots, x_n]^{\text{TL}_n(1)}$
 $\text{TL}_n(1)$ is the Temperley-Lieb algebra $\leftrightarrow \mathbb{Q}[\text{S}_n] / \text{action's kernel}$
 - Temperley-Lieb “covariants”
 $\dim(\text{TL}_n) = \dim(\mathbb{Q}[x_1, x_2, \dots, x_n] / \langle \text{QSym}^+ \rangle)$
 - The maximal Eulerian Hopf subalgebra of QSym
is the Peak algebra of Stembridge...
 - $\circ\circ\circ$ (and so much more!)
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