On projective $\mathcal{E}$-generators and premonadic functors

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Toronto 2002

$U : \mathbf{A} \to \mathbf{Set}$ is <u>premonadic</u> if

it is a right-adjoint and $\mathbf{A}$ is equivalent to a full (reflective) subcategory of $\mathbf{Set}^{\mathbf{T}}$

$$\begin{tikzcd} \mathbf{A} \arrow[rr, "K"] \arrow[dr, "U"'] & & \mathbf{Set}^{\mathbf{T}} \arrow[dl, "U^{\mathbf{T}}"] \\ & \mathbf{Set} & \end{tikzcd}$$

$\mathbf{A}$ is monadic over $\mathbf{Set}$	$\Longleftrightarrow$	i. $\mathbf{A}$ is exact ii. $\mathbf{A}$ has finite limits iii. $\mathbf{A}$ has copowers of P iv. P is an $\mathcal{E}$-generator of $\mathbf{A}$, for $\mathcal{E} = \{$reg. epis.$\}$ v. P is projective
$\Downarrow$		$\Downarrow$
$\mathbf{A}$ is premonadic over $\mathbf{Set}$		$\mathbf{A}$ has copowers of P $\mathbf{A}$ has a projective $\mathcal{E}$-generator P, for some class $\mathcal{E}$

$\mathcal{E} \subseteq \mathsf{Epi}(\mathbf{A})$ <u>coreflective</u> class if

$\mathcal{E}(B) \hookrightarrow B \downarrow \mathbf{A}$ is a left adjoint

If $\mathbf{A}$ has pushouts and $\mathcal{E}$ is pushout-stable, $\mathcal{E}$ is a coreflective class iff $\mathcal{E} \hookrightarrow \mathsf{Mor}(\mathbf{A})$ is a left adjoint

$$\begin{tikzcd}
{} \arrow[r, "e' \in \mathcal{E}"] \arrow[d, "g"'] & {} \arrow[dd, "h"] \arrow[ddl, "f"] \\
{} \arrow[d, "e"'] \arrow[dr] & \\
{} \arrow[r, "m"'] & {}
\end{tikzcd}$$

$\mathcal{E}$ pushout-stable coreflective class, $\mathbf{A}$ has pushouts:

$\xrightarrow{e}\xrightarrow{m}$ $\mathcal{E}$-factorization m split epimorphism $\Longrightarrow m$ is an isomorphism

$$\Updownarrow$$

$\mathcal{E}$ is closed under the composition with split epimorphisms from the left

$$\Updownarrow$$

$\mathcal{E}$ is closed under the composition with split epimorphisms

From now on:

$\mathbf{A}$ is cocomplete and has pullbacks

$\mathcal{E} \subseteq \mathsf{Epi}(\mathbf{A})$ is a pushout-stable coreflective class, closed under the composition with split epimorphisms

For $\mathcal{E} \subseteq \mathsf{Epi}(\mathbf{A})$, the stabilization of $\mathcal{E}$ is given by

$$\mathsf{St}(\mathcal{E}) = \left\{ f \in \mathsf{Mor}(\mathbf{A}) \mid \begin{array}{l} \text{the pullback of } f \text{ along any} \\ \text{morphism belongs to } \mathcal{E} \end{array} \right\}$$

$\mathcal{E}$ and $\mathsf{St}(\mathcal{E})$ are strongly right-cancellable.

$$\begin{array}{l} r \cdot s \in \mathcal{E} \ \Rightarrow \ r \in \mathcal{E} \\ r \cdot s \in \mathsf{St}(\mathcal{E}) \ \Rightarrow \ r \in \mathsf{St}(\mathcal{E}) \end{array}$$

P is an <u>$\mathcal{E}$-generator</u> of $\mathbf{A}$ if, for each $A \in \mathbf{A}$, the morphism ε_A belongs to $\mathcal{E}$.

$$\begin{tikzcd}
\coprod\limits_{\hom(P,A)} P \arrow[rr, "\varepsilon_A"] & & A \\
& P \arrow[ul, "\nu_f"] \arrow[ur, "f"'] &
\end{tikzcd}$$

P is a <u>projective $\mathcal{E}$-generator</u> if it is an $\mathcal{E}$-generator and is $\mathsf{St}(\mathcal{E})$-projective

$$\begin{tikzcd}
X \arrow[rr, "e \in \mathsf{St}(\mathcal{E})"] & & Y \\
& P \arrow[ul] \arrow[ur, "f"'] &
\end{tikzcd}$$

For $A \in \mathbf{A}$,
$\mathsf{Proj}(A) = \{\, f \in \mathsf{Mor}(\mathbf{A}) \mid \ \mathsf{hom}(A, f) \text{ is surjective } \}$

$\mathsf{Proj}(A)$ is pullback-stable.

If P is a projective $\mathcal{E}$-generator, then $\mathsf{St}(\mathcal{E}) = \mathsf{Proj}(P)$.

$\mathbb{C}(P) :=$ colimit-closure of P in $\mathbf{A}$

$A \perp f \;:\Leftrightarrow\; \mathsf{hom}(A, f)$ iso

P is a projective $\mathcal{E}$-generator of $\mathbf{A}$
$\mathbb{C}(P)$ is coreflective in $\mathbf{A}$

Then

$$\begin{aligned}\mathbb{C}(P) &= \text{smallest } \mathcal{E}\text{-correflective subcategory of } \mathbf{A} \\ &= \{A \in \mathbf{A} \mid A \perp f,\ f \in \mathsf{St}(\mathcal{E}) \cap \mathsf{Mono}(\mathbf{A})\}\end{aligned}$$

$\mathbf{A}$ monadic over $\mathbf{Set}$, $\quad \mathcal{E} = \mathsf{RegEpi}(\mathbf{A})$
$P = F\{*\}$, $\quad \mathsf{St}(\mathcal{E}) = \mathcal{E} = \mathsf{Proj}(P)$
$\mathbb{C}(P) = \mathbf{A}$

$\mathbf{A} = \mathbf{Cat}$, $\quad \mathcal{E} = \mathsf{RegEpi}(\mathbf{A})$
$P = \{0 \to 1 \to 2\}$, $\quad \mathsf{St}(\mathcal{E}) = \mathsf{Proj}(P) \neq \mathcal{E}$
$\mathbb{C}(P) = \mathbf{A}$

$\mathbf{A} = \mathbf{Cat}$, $\quad \mathcal{E} = \mathsf{ExtEpi}(\mathbf{A})$
$P = \{0 \to 1\}$, $\quad \mathsf{St}(\mathcal{E}) = \mathsf{Proj}(P) \neq \mathcal{E}$
$\mathbb{C}(P) = \mathbf{A}$

$\mathbf{A} = \mathbf{PreOrd}$, $\quad \mathcal{E} = \mathsf{RegEpi}(\mathbf{A}) = \mathsf{ExtEpi}(\mathbf{A})$
$P = \{0 \to 1\}$, $\quad \mathsf{St}(\mathcal{E}) = \mathsf{Proj}(P) \neq \mathcal{E}$
$\mathbb{C}(P) = \mathbf{A}$

P an injective $\mathcal{M}$-generator

$\mathrm{St}(\mathcal{M}) = \left\{ f \in \mathrm{Mor}(\mathbf{A}) \mid \begin{array}{l} \text{the pushout of } f \text{ along any mor-} \\ \text{phism belongs to } \mathcal{M} \end{array} \right\}$

$\mathbb{L}(P) =$ limit closure of P in $\mathbf{A}$

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$\mathbf{A} = \mathbf{Set}$, $\quad \mathcal{M} = \{\mathsf{monos}\}$, $\quad P = \{0, 1\}$

$\mathsf{Inj}(P) = \mathsf{St}(\mathcal{M}) = \mathcal{M}$, $\quad \mathbb{L}(P) = \mathbf{Set}$

$\mathbf{A} = \mathbf{Top}$, $\quad \mathcal{M} = \{\mathsf{embeddings}\}$

$P = (\{0, 1, 2\}, < \{0\} >)$, $\quad \mathsf{Inj}(P) = \mathsf{St}(\mathcal{M}) = \mathcal{M}$

$\mathbb{L}(P) = \mathbf{Top}$

$\mathbf{A} = \mathbf{0}\text{-}\mathsf{dim}\mathbf{Top}$, $\quad \mathcal{M} = \{\mathsf{embeddings}\}$

$P = (\{0, 1, 2\}, < \{0\}, \{1, 2\} >)$

$\mathcal{M} \neq \mathsf{St}(\mathcal{M}) = \mathsf{Inj}(P)$

$\mathbb{L}(P) = \mathbf{A}$

$\mathbf{A} = \mathbf{Ab}$, $\quad \mathcal{M} = \{\ \mathsf{monos}\}$

$\mathsf{St}(\mathcal{M}) = \mathcal{M} = \mathsf{Inj}(P)$ $\quad$ with $P = \prod\limits_{n \in \mathbf{N}} \mathbb{Q}/n\mathbb{Z}$

$\mathbb{L}(P) = \mathbf{Ab}$

$\mathbf{A} = \mathbf{TFAb}$, $\quad \mathcal{M} = \{\text{monos}\}$
$\mathsf{St}(\mathcal{M}) = \mathcal{M} = \mathsf{Inj}(\mathbb{Q})$
$\mathbb{L}(\mathbb{Q}) = \mathbf{DivTFAb}$

$\mathbf{A} = \mathbf{Top_0}$, $\quad \mathcal{M} = \{\text{embeddings}\}$
$\mathsf{Inj}(S)$=$\mathsf{St}(\mathcal{M})$=$\mathcal{M}$ with S =Sierpiński space
$\mathbb{L}(S) = \mathbf{Sob}$

$\mathbf{A} = \mathbf{Tych}$, $\quad \mathcal{M} = \{\text{embeddings}\}$
$\mathsf{St}(\mathcal{M}) = \mathsf{Inj}(I) \neq \mathcal{M}$ with I the unit interval
$\mathbb{L}(I) = \mathbf{CompHaus}$

$\mathbf{A} = \mathbf{Met}_\infty$, $\quad \mathcal{M} = \{\text{isometric injective maps}\}$
$\mathcal{M} = \mathsf{St}(\mathcal{M}) = \mathsf{Inj}([0,\infty])$
$\mathbb{L}([0,\infty]) = \mathbf{ComplMet}_\infty$

If $\mathbf{A}$ has enough St($\mathcal{E}$)-projectives, then, for some subcategory $\mathbf{B}$ of $\mathbf{A}$,

$$\mathsf{St}(\mathcal{E}) = \mathsf{Proj}(\mathbf{B})\,.$$

If $\mathsf{St}(\mathcal{E}) = \mathsf{Proj}(\mathbf{B})$ for some $\mathcal{E}$-coreflective subcategory $\mathbf{B}$ of $\mathbf{A}$, then

$\mathbf{A}$ has St($\mathcal{E}$)-projective hulls.

$\mathcal{F} \subseteq$ Epi$(\mathbf{A})$ is saturated if, for each $f \in \mathcal{F}$, the coequalizer of the kernel pair of f belongs to $\mathcal{F}$.

If there is a projective $\mathcal{E}$-generator,

St$(\mathcal{E})$ is saturated

iff $\left.\begin{array}{l}\begin{tikzcd} \bullet \arrow[r, "r"] \arrow[d, "s"'] & \bullet \arrow[d, "e"] \\ \bullet \arrow[r, "e"] & \bullet \end{tikzcd} \\ c = \mathsf{coeq}(r, s) \\ d \cdot c = e \end{array}\right\} \Longrightarrow d$ is a monomorphism

$\mathcal{E} = \{$regular epimorphisms$\}$ $\Rightarrow$ St$(\mathcal{E})$ is saturated

In $\mathbf{Cat}$, for $\mathcal{E} = \{$extremal epimorphisms$\}$, St$(\mathcal{E})$ is not saturated.

$\mathbf{A}$ is monadic $\iff$ over $\mathbf{Set}$	i. $\mathbf{A}$ is exact ii. $\mathbf{A}$ has finite limits iii. $\mathbf{A}$ has copowers of P iv. P is a regular generator $\mathbf{A}$ v. P is projective

For $\mathcal{E} = \{$reg. epis.$\}$:
St$(\mathcal{E}) = \mathcal{E}$
P is a projective $\mathcal{E}$-generator
$\mathbf{A} = \mathbb{C}(P)$

> If
> P is a projective $\mathcal{E}$-generator of $\mathbf{A}$
> St$(\mathcal{E})$ is saturated
> $\mathbb{C}(P)$ coreflective in $\mathbf{A}$
>
> then
>
> hom$(P, -) : \mathbb{C}(P) \to \mathbf{Set}$ is premonadic

$\mathbb{C}(P)$ is equivalent to a reflective subcategory of $\mathbf{Set}^{\mathbf{T}}$

P is a projective dense $\mathcal{E}$-generator of $\mathbf{A}$ if it is a projective $\mathcal{E}$-generator and $\mathbb{C}(P) = \mathbf{A}$.

If $\mathbf{A}$ has a projective dense $\mathcal{E}$-generator P and $\mathrm{St}(\mathcal{E})$ is saturated then $\mathrm{hom}(P, -) : \mathbf{A} \to \mathbf{Set}$ is premonadic.

hom$(0 \to 1, -) : \mathbf{PreOrd} \to \mathbf{Set}$ is premonadic

$\mathbf{PreOrd} \ \hookrightarrow \ \mathbf{M\text{-}Set}$

hom$(0 \to 1 \to 2, -) : \mathbf{Cat} \to \mathbf{Set}$ is premonadic

hom$(0 \to 1, -) : \mathbf{Cat} \to \mathbf{Set}$ is not premonadic

hom$(-, S) : \mathbf{Sob^{op}} \to \mathbf{Set}$

hom$(, P) : 0$-dim$\mathbf{Top^{op}} \to \mathbf{Set}$,
for $P = (\{0, 1, 2\}, < \{0\}, \{1, 2\} >)$

hom$(, \mathbb{Q}) : (\mathbf{DivTFAb})^{\mathbf{op}} \to \mathbf{Set}$

hom$(-, [0, \infty]) : (\mathbf{ComplMet}_{\infty})^{\mathbf{op}} \to \mathbf{Set}$

are premonadic

Let $\mathbf{A}$ be cocomplete and cowellpowered.
If $U : \mathbf{A} \to \mathbf{Set}$ is a premonadic functor
then
there is a coreflective class $\mathcal{E}$ and a dense $\mathcal{E}$-generator P of $\mathbf{A}$ such that Proj$(P) \subseteq$ St$(\mathcal{E})$.