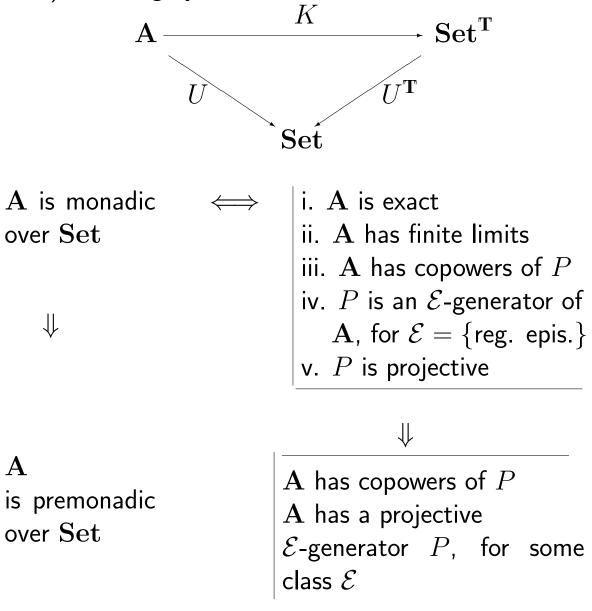
## On projective *E*-generators and premonadic functors

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Toronto 2002

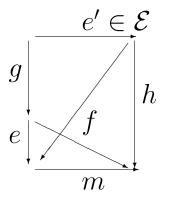
## $U:\mathbf{A}\to\mathbf{Set}$ is premonadic if

it is a right-adjoint and A is equivalent to a full (reflective) subcategory of  $\mathbf{Set}^{\mathrm{T}}$ 

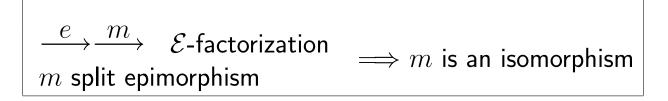


$$\begin{split} \mathcal{E} &\subseteq \mathsf{Epi}(\mathbf{A}) \quad \underline{\text{coreflective}} \text{ class if} \\ \mathcal{E}(B) &\hookrightarrow B \downarrow \mathbf{A} \text{ is a left adjoint} \end{split}$$

If A has pushouts and  $\mathcal{E}$  is pushout-stable,  $\mathcal{E}$  is a core-flective class iff  $\mathcal{E} \hookrightarrow \mathsf{Mor}(\mathbf{A})$  is a left adjoint



 ${\cal E}$  pushout-stable coreflective class,  ${\bf A}$  has pushouts:



## $\bigcirc$

 $\ensuremath{\mathcal{E}}$  is closed under the composition with split epimorphisms from the left

## $\uparrow$

 $\ensuremath{\mathcal{E}}$  is closed under the composition with split epimorphisms

From now on:

 ${\bf A}$  is cocomplete and has pullbacks

 $\mathcal{E}\subseteq {\sf Epi}({f A})$  is a pushout-stable coreflective class, closed under the composition with split epimorphisms

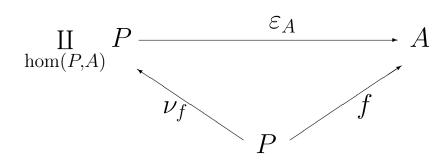
For  $\mathcal{E} \subseteq \mathsf{Epi}(\mathbf{A})$ , the <u>stabilization of  $\mathcal{E}$ </u> is given by

 $\mathsf{St}(\mathcal{E}) = \left\{ f \in \mathsf{Mor}(\mathbf{A}) \mid \begin{array}{c} \mathsf{the pullback of } f \text{ along any} \\ \mathsf{morphism belongs to } \mathcal{E} \end{array} \right\}$ 

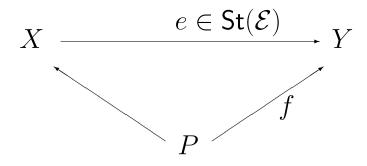
 $\boldsymbol{\mathcal{E}}$  and  $\mathsf{St}(\boldsymbol{\mathcal{E}})$  are strongly right-cancellable.

$$\begin{array}{l} r \cdot s \in \mathcal{E} \ \Rightarrow \ r \in \mathcal{E} \\ r \cdot s \in \mathsf{St}(\mathcal{E}) \ \Rightarrow \ r \in \mathsf{St}(\mathcal{E}) \end{array}$$

*P* is an  $\underline{\mathcal{E}}$ -generator of **A** if, for each  $A \in \mathbf{A}$ , the morphism  $\varepsilon_A$  belongs to  $\mathcal{E}$ .



P is a projective  $\mathcal E\text{-generator}$  if it is an  $\mathcal E\text{-generator}$  and is  $\mathsf{St}(\mathcal E)\text{-projective}$ 



For  $A \in \mathbf{A}$ ,  $\operatorname{Proj}(A) = \{ f \in \operatorname{Mor}(\mathbf{A}) \mid \operatorname{hom}(A, f) \text{ is surjective } \}$ 

Proj(A) is pullback-stable.

If P is a projective  $\mathcal{E}$ -generator, then  $St(\mathcal{E}) = Proj(P)$ .

 $\mathbb{C}(P) := \text{colimit-closure of } P \text{ in } \mathbf{A}$ 

 $A \perp f :\Leftrightarrow \mathsf{hom}(A, f)$  iso

P is a projective  $\mathcal E\text{-generator}$  of  $\mathbf A$   $\mathbb C(P)$  is coreflective in  $\mathbf A$ 

Then

 $\mathbb{C}(P) = \text{smallest } \mathcal{E}\text{-correflective subcategory of } \mathbf{A} \\ = \{A \in \mathbf{A} \mid A \bot f, \ f \in \mathsf{St}(\mathcal{E}) \cap \mathsf{Mono}(\mathbf{A})\}$ 

$$\begin{array}{ll} \mathbf{A} \text{ monadic over } \mathbf{Set}, & \mathcal{E} = \mathsf{RegEpi}(\mathbf{A}) \\ P = F\{*\}, & \mathsf{St}(\mathcal{E}) = \mathcal{E} = \mathsf{Proj}(P) \\ \mathbb{C}(P) = \mathbf{A} \end{array}$$

$$\begin{array}{ll} \mathbf{A} = \mathbf{Cat}, & \mathcal{E} = \mathsf{RegEpi}(\mathbf{A}) \\ P = \{ 0 \rightarrow 1 \rightarrow 2 \}, & \mathsf{St}(\mathcal{E}) = \mathsf{Proj}(P) \neq \mathcal{E} \\ \mathbb{C}(P) = \mathbf{A} \end{array}$$

$$\begin{array}{ll} \mathbf{A} = \mathbf{Cat}, & \mathcal{E} = \mathsf{ExtEpi}(\mathbf{A}) \\ P = \{ 0 \rightarrow 1 \}, & \mathsf{St}(\mathcal{E}) = \mathsf{Proj}(P) \neq \mathcal{E} \\ \mathbb{C}(P) = \mathbf{A} \end{array}$$

$$\begin{array}{ll} \mathbf{A} = \mathbf{PreOrd}, & \mathcal{E} = \mathsf{RegEpi}(\mathbf{A}) = \mathsf{ExtEpi}(\mathbf{A}) \\ P = \{ 0 \rightarrow 1 \}, & \mathsf{St}(\mathcal{E}) = \mathsf{Proj}(P) \neq \mathcal{E} \\ \mathbb{C}(P) = \mathbf{A} \end{array}$$

 $P \text{ an injective } \mathcal{M}\text{-generator}$   $\operatorname{St}(\mathcal{M}) = \left\{ f \in \operatorname{Mor}(\mathbf{A}) \mid \begin{array}{c} \text{the pushout of } f \text{ along any mor-} \\ \text{phism belongs to } \mathcal{M} \end{array} \right\}$  $\mathbb{L}(P) = \text{limit closure of } P \text{ in } \mathbf{A}$ 

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$$\mathbf{A} = \mathbf{Top}, \quad \mathcal{M} = \{ \mathsf{embeddings} \}$$
$$P = (\{0, 1, 2\}, <\{0\} >), \quad \mathsf{Inj}(P) = \mathsf{St}(\mathcal{M}) = \mathcal{M}$$
$$\mathbb{L}(P) = \mathbf{Top}$$

$$\begin{split} \mathbf{A} &= \mathbf{0}\text{-dim}\mathbf{Top}, \quad \mathcal{M} = \{\text{embeddings}\}\\ P &= (\{0, 1, 2\}, <\{0\}, \{1, 2\} >)\\ \mathcal{M} &\neq \mathsf{St}(\mathcal{M}) = \mathsf{Inj}(P)\\ \mathbb{L}(P) &= \mathbf{A} \end{split}$$

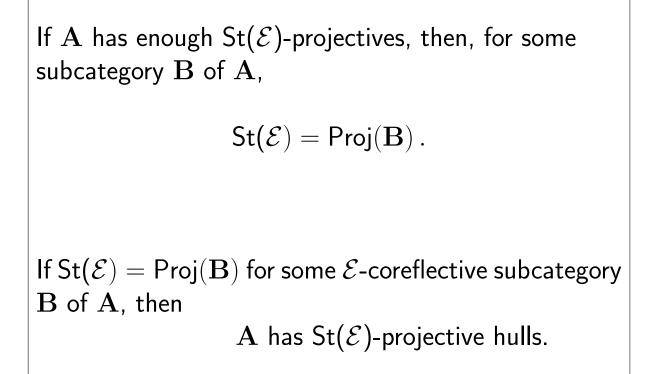
$$\begin{split} \mathbf{A} &= \mathbf{A}\mathbf{b}, \quad \mathcal{M} = \{ \text{ monos} \} \\ \mathsf{St}(\mathcal{M}) &= \mathcal{M} = \mathsf{Inj}(P) \quad \text{with } P = \underset{n \in \mathbb{N}}{\Pi} \mathbb{Q}/n\mathbb{Z} \\ \mathbb{L}(P) &= \mathbf{A}\mathbf{b} \end{split}$$

$$\begin{split} \mathbf{A} &= \mathbf{TFAb}, \quad \mathcal{M} = \{\mathsf{monos}\} \\ \mathsf{St}(\mathcal{M}) &= \mathcal{M} = \mathsf{Inj}(\mathbb{Q}) \\ \mathbb{L}(\mathbb{Q}) &= \mathbf{DivTFAb} \end{split}$$

$$\mathbf{A} = \mathbf{Top_0}, \quad \mathcal{M} = \{ \text{embeddings} \}$$
  
$$\mathsf{Inj}(S) = \mathsf{St}(\mathcal{M}) = \mathcal{M} \quad \text{with } S = \mathsf{Sierpiński space}$$
  
$$\mathbb{L}(S) = \mathbf{Sob}$$

$$\begin{split} \mathbf{A} &= \mathbf{Tych}, \quad \mathcal{M} = \{ \mathsf{embeddings} \} \\ \mathsf{St}(\mathcal{M}) &= \mathsf{Inj}(I) \neq \mathcal{M} \quad \mathsf{with} \ I \ \mathsf{the} \ \mathsf{unit} \ \mathsf{interval} \\ \mathbb{L}(I) &= \mathbf{CompHaus} \end{split}$$

 $\begin{aligned} \mathbf{A} &= \mathbf{Met}_{\infty}, \quad \mathcal{M} &= \{ \text{isometric injective maps} \} \\ \mathcal{M} &= \mathsf{St}(\mathcal{M}) = \mathsf{Inj}([0,\infty]) \\ \mathbb{L}([0,\infty]) &= \mathbf{ComplMet}_{\infty} \end{aligned}$ 



 $\mathcal{F} \subseteq \operatorname{Epi}(\mathbf{A})$  is <u>saturated</u> if, for each  $f \in \mathcal{F}$ , the coequalizer of the kernel pair of f belongs to  $\mathcal{F}$ .

If there is a projective 
$$\mathcal{E}$$
-generator,  
St( $\mathcal{E}$ ) is saturated  

$$\begin{array}{c} \bullet & \bullet \\ s & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \\ c = \operatorname{coeq}(r, s) \\ d \cdot c = e \end{array} \right\} \Longrightarrow d \text{ is a monomorphism}$$

 $\mathcal{E} = \{ \mathsf{regular epimorphisms} \} \Rightarrow \mathsf{St}(\mathcal{E}) \text{ is saturated}$ 

In Cat, for  $\mathcal{E} = \{\text{extremal epimorphisms}\}, St(\mathcal{E})$  is not saturated.

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A is monadic\iffi. A is exactover Setii. A has finite limitsiii. A has copowers of Piv. P is a regular generator Av. P is projectiveFor \mathcal{E} = \{ \text{reg. epis.} \}:St(\mathcal{E}) = \mathcal{E}P is a projective \mathcal{E}-generator\mathbf{A} = \mathbb{C}(P)
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If

P is a projective \mathcal{E}-generator of \mathbf{A}

\operatorname{St}(\mathcal{E}) is saturated

\mathbb{C}(P) coreflective in \mathbf{A}

then

\operatorname{hom}(P, -) : \mathbb{C}(P) \to \operatorname{Set} is premonadic
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 $\mathbb{C}(P)$  is equivalent to a reflective subcategory of  $\mathbf{Set}^{\mathbf{T}}$ 

P is a projective dense  $\mathcal{E}$ -generator of  $\mathbf{A}$  if it is a projective  $\mathcal{E}$ -generator and  $\mathbb{C}(P) = \mathbf{A}$ .

If A has a projective dense  $\mathcal{E}$ -generator P and  $St(\mathcal{E})$  is saturated then  $hom(P, -) : \mathbf{A} \to \mathbf{Set}$  is premonadic.

 $hom(0 \rightarrow 1, -) : \mathbf{PreOrd} \rightarrow \mathbf{Set} \text{ is premonadic}$  $\mathbf{PreOrd} \hookrightarrow \mathbf{M}\text{-}\mathbf{Set}$ 

 $hom(0 \rightarrow 1 \rightarrow 2, -) : \mathbf{Cat} \rightarrow \mathbf{Set} \text{ is premonadic}$  $hom(0 \rightarrow 1, -) : \mathbf{Cat} \rightarrow \mathbf{Set} \text{ is not premonadic}$ 

 $| \mathsf{hom}(-,S) : \mathbf{Sob^{op}} \to \mathbf{Set}$ 

hom(,P): 0-dimTop<sup>op</sup>  $\rightarrow$  Set, for  $P = (\{0, 1, 2\}, <\{0\}, \{1, 2\} >)$ 

 $| \mathsf{hom}(\mathbb{Q}) : (\mathbf{DivTFAb})^{\mathbf{op}} \to \mathbf{Set}$ 

 $|\operatorname{\mathsf{hom}}(-,[0,\infty]):(\operatorname{\mathbf{ComplMet}}_{\infty})^{\operatorname{\mathbf{op}}}\to\operatorname{\mathbf{Set}}$ 

are premonadic

Let A be cocomplete and cowellpowered. If  $U : \mathbf{A} \to \mathbf{Set}$  is a premonadic functor then there is a coreflective class  $\mathcal{E}$  and a dense  $\mathcal{E}$ -generator P of A such that  $\operatorname{Proj}(P) \subseteq \operatorname{St}(\mathcal{E})$ .