

# OVERVIEW OF TOPOLOGICAL DESCENT THEORY

$\Phi : \mathcal{C}^{\text{op}} \rightarrow \mathcal{CAT}$  a pseudofunctor

$p : E \rightarrow B$  in  $\mathcal{C}$

$\text{Eq}(p)$

$$E \times_B E \times_B E \begin{array}{c} \xrightarrow{\pi_{12}} \\ \xrightarrow[\pi_{23}]{\pi_{13}} \\ \end{array} E \times_B E \begin{array}{c} \xrightarrow{\pi_1} \\ \xleftarrow[\pi_2]{\delta} \\ \end{array} E$$

$\text{Des}_\Phi(p)$  = internal actions of  $\text{Eq}(p)$  on  $\mathcal{CAT}$

$$\begin{array}{ccc} \Phi(B) & \xrightarrow{K} & \text{Des}_\Phi(p) \\ & \searrow \Phi(p) & \swarrow Up \\ & \Phi(E) & \end{array}$$

$p$   $\Phi$ -descent if  $K$  is full and faithful

$p$  effective  $\Phi$ -descent if  $K$  is an equivalence

$$\mathcal{C} = \mathcal{T}op$$

$\mathbb{E}$  a pullback stable class of continuous maps

$\Phi(B) = \mathbb{E}(B)$  full subcategory of  $\mathcal{T}op \downarrow B$

$\Phi(p)$  induced by  $p^* : \mathcal{T}op \downarrow B \rightarrow \mathcal{T}op \downarrow E$

$$\begin{array}{ccc} \mathbb{E}(B) & \xrightarrow{K} & \mathbf{Des}_{\mathbb{E}}(p) \\ & \searrow \Phi(p) & \swarrow U^p \\ & \mathbb{E}(E) & \end{array}$$

$\mathbb{E}$ -descent, effective  $\mathbb{E}$ -descent

$\mathbb{E} = \text{all maps}$ , descent reduces to monadicity

$$\begin{array}{ccc} \mathcal{T}op \downarrow B & \xrightarrow{K} & \mathbf{Des}(p) \cong (\mathcal{T}op \downarrow E)^{\mathbb{T}} \\ & \searrow p^* & \swarrow U^{\mathbb{T}} \\ & \mathcal{T}op \downarrow E & \end{array}$$

$\mathbb{T}$  being the monad induced by  $p ! \dashv p^*$

[Bénabou and Roubaud, 70] [Beck, unpublished]

$\mathbb{T}$ -algebras are triples

$$(C, \gamma : C \rightarrow E, \xi : E \times_B C \rightarrow C)$$

such that, for  $\xi(e, c) = e \cdot c$ ,

$$\gamma(e \cdot c) = e, \quad \gamma(c) \cdot c = c \quad \text{and} \quad e \cdot (e' \cdot c) = e \cdot c.$$

The class of effective descent morphisms in  $\mathcal{T}_{op}$   
**properly contains** the classes of

- locally sectionable [Janelidze and Tholen, 94]
- open surjective [Moerdijk 90; Sobral 91]
- proper surjective [Vermeulen, 94; Moerdijk 90]
- triquotient maps [Plewe, 97]

and **has good properties**: e.g. it is

- stable under pullbacks [Sobral and Tholen, 91]
- closed for composition [Reiterman, Sobral, Tholen, 93]
- closed for products [Clementino and Hofmann, 00]

$p$  is a descent morphism  $\Leftrightarrow p$  is a universal quotient  
 [Janelidze and Tholen, 1991]

$p : E \rightarrow B$  is a universal quotient  $\Leftrightarrow$

(i)  $p^{-1}(b) \subseteq \bigcup_{i \in I} U_i \Rightarrow b \in \text{Int}(p(U_{i_1}) \cup \dots \cup p(U_{i_k}))$   
 for some finite subset of  $I$ .

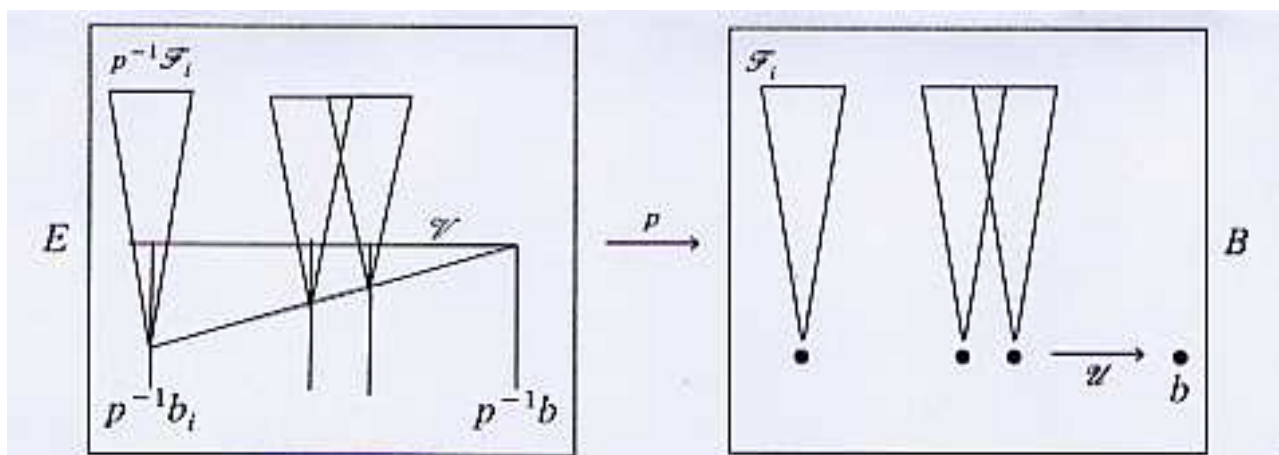
$\Leftrightarrow$

(ii)  $\mathcal{U} \rightarrow b \Rightarrow \exists \mathcal{V} \rightarrow e \in p^{-1}(b)$  such that  $p(\mathcal{V}) = \mathcal{U}$ .

[Day and Kelly, 1970]

$p$  is an effective descent morphism  $\Leftrightarrow$  every crest of ultrafilters in  $B$  has a lifting along  $p$ .

[Reiterman and Tholen, 1994]



In 1991 Jan Reiterman constructs the first example of a non-effective descent morphism.

$$\begin{array}{ccccc}
 E \times_B C & \xrightarrow[\xi]{\pi_2} & C & \xrightarrow{q} & A \\
 & & \downarrow \gamma & (1) & \downarrow \alpha \\
 & & E & \xrightarrow{p} & B
 \end{array}$$

$(C, \gamma, \xi) \in \text{Des}(p)$  and  $q = \text{coeq}(\pi_2, \xi)$

descent situation defining  $A$

In  $\mathcal{T}op$ , a quotient  $p$  is an effective descent morphism if and only if, for every descent situation defining the space  $A$ , (1) is a pullback.

[Sobral, 1991]

Finite example of a non-effective descent morphism.

[Sobral, 1995]

Finite example of a surjective effective étale-descent morphism which is not an effective descent morphism.

[Sobral, 1994]

$$\mathcal{F}inTop \cong \mathcal{F}inPreord$$

$$(X, \mathcal{O}) \longmapsto (X, \rightarrow)$$

$$y \rightarrow x \text{ if } y \in \bigcap \{U \mid x \in U \in \mathcal{O}(X)\}$$

In  $\mathcal{F}inTop$

$$\begin{array}{c} E \\ \downarrow p \\ B \end{array}$$

descent morphism  $\Leftrightarrow$

$$\begin{array}{ccc} e_1 & \longrightarrow & e_0 \\ | & & | \\ b_1 & \longrightarrow & b_0 \end{array}$$

$$\begin{array}{c} E \\ \downarrow p \\ B \end{array}$$

effective descent morphism  $\Leftrightarrow$

$$\begin{array}{ccccc} e_2 & \longrightarrow & e_1 & \longrightarrow & e_0 \\ | & & | & & | \\ b_2 & \longrightarrow & b_1 & \longrightarrow & b_0 \end{array}$$

[Janelidze and Sobral, 1999]

and then the description of pullback stable regular epimorphisms and the Reiterman-Tholen theorem are the appropriate infinite versions of these theorems.

In  $\mathcal{FinTop}$

$$\begin{array}{c} E \\ \downarrow p \\ B \end{array} \quad \text{triquotient} \Leftrightarrow \begin{array}{ccccccc} e_n & \longrightarrow & e_{n-1} & \longrightarrow & \cdots & \longrightarrow & e_1 & \longrightarrow & e_0 \\ | & & | & & | & & | & & | \\ b_n & \longrightarrow & b_{n-1} & \longrightarrow & \cdots & \longrightarrow & b_1 & \longrightarrow & b_0 \end{array}$$

$\Rightarrow$  [Janelidze and Sobral, 99]

$\Leftarrow$  [Clementino, 00]

Characterization of triquotients in  $\mathcal{T}op$

[Clementino and Hofmann, 00]

$$\mathcal{F}inDLat^{\text{op}} \sim \mathcal{F}inOrd$$

In  $\mathcal{F}inDLat$

$D$   
 $\downarrow m$   
 $B$

codescent morphism

$\Leftrightarrow$

$I_1 \subseteq I_2$   
 $\downarrow$   
 $J_1 \subseteq J_2$

$m^{-1}(J_k) = I_k, \quad k = 1, 2$

$D$   
 $\downarrow m$   
 $B$

effective codescent morph.

$\Leftrightarrow$

$I_1 \subseteq I_2 \subseteq I_3$   
 $\downarrow$   
 $J_1 \subseteq J_2 \subseteq J_3$

$m^{-1}(J_k) = I_k, \quad k = 1, 2, 3$

## (effective) $\mathbb{E}$ -descent

$\mathbb{E} = \{\text{bijective continuous maps}\}$

(global) descent =  $\mathbb{E}$ -descent

(global) effective descent = effective  $\mathbb{E}$ -descent

morphisms which are stable under pullback

[Sobral, 1994]

$\mathbb{E} = \{\text{étale maps}\}$

$\mathbb{E}$ -descent [Janelidze and Tholen, 1994]

effective  $\mathbb{E}$ -descent (open problem)

effective  $\mathbb{E}$ -descent for finite topological spaces

[Janelidze and Sobral, 2001]

In *FinTop*

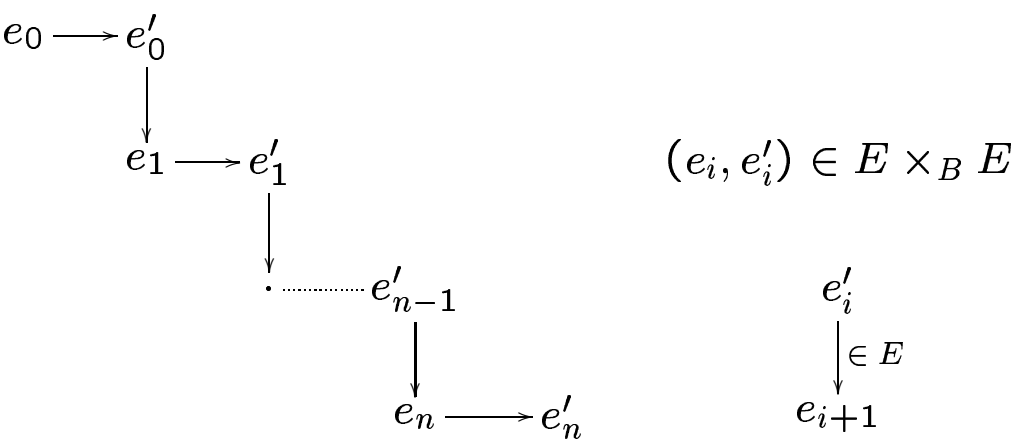
$f : A \rightarrow B$  is an étale map  $\Leftrightarrow f$  is a discrete fibration

$$\mathbb{E}(B) \cong \mathcal{S}et^{B^{op}}$$

$$\begin{array}{ccc}
 E & \xrightarrow{p} & B \\
 \searrow \psi & & \nearrow \varphi \\
 & Z(\mathsf{Eq}(p)) &
 \end{array}$$

$Z \dashv S : \mathcal{C}at \rightarrow \mathcal{D}ouble\mathcal{C}at$

$$Z(\mathsf{Eq}(p)) = \overline{E} / \sim$$



$\sim$  the smallest equivalence relation which contains

- $((\alpha, \beta), \alpha\beta)$  if  $\alpha, \beta \in E \times_B E$  or  $\alpha, \beta \in E$
- pairs forming squares

$$\begin{array}{ccc} \mathbb{E}(B) & \xrightarrow{K} & \mathbf{Des}_{\mathbb{E}}(p) \\ & \searrow p^* \quad \swarrow Up & \\ & \mathbb{E}(E) & \end{array}$$

transforms into

$$\begin{array}{ccc} \mathcal{S}et^{B^{op}} & \xrightarrow{\mathcal{S}et\varphi^{op}} & \mathcal{S}et^{Z(\mathbf{Eq}(p))^{op}} \\ & \searrow \mathcal{S}et p^{op} \quad \swarrow \mathcal{S}et\psi^{op} & \\ & \mathcal{S}et^{E^{op}} & \end{array}$$

$p$  étale-descent  $\Leftrightarrow \varphi^{op}$  lax epimorphism

$p$  effective étale-descent  $\Leftrightarrow \varphi^{op}$  equivalence

In  $\mathcal{F}in\mathcal{T}op$

$p : E \rightarrow B$  effective étale-descent morphism  $\Leftrightarrow$

(i)  $p : E \rightarrow p(E)$  is a quotient map;

(ii)  $Z(\mathbf{Eq}(p))$  is a preorder;

(iii)  $p$  is essentially surjective on objects.

[Janelidze and Sobral, 01]

In  $\mathcal{Cat}$ ,

$P : \mathcal{E} \rightarrow \mathcal{B}$  is a **lax epimorphism**

if  $(-)^P : [\mathcal{B}, \mathcal{C}] \rightarrow [\mathcal{E}, \mathcal{C}]$  is full and faithful,

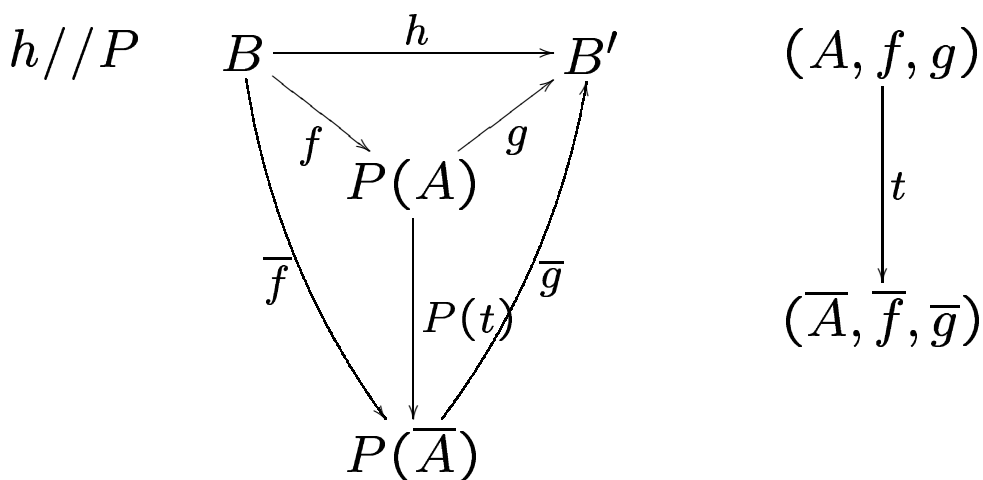
for every small category  $\mathcal{C}$ .

$P : \mathcal{E} \rightarrow \mathcal{B}$  **lax epimorphism**

$\Leftrightarrow P^* = (-)^P : [\mathcal{B}, \mathcal{Set}] \rightarrow [\mathcal{E}, \mathcal{Set}]$  full and faithful

$\Leftrightarrow h//P$  connected for all morphisms  $h$  of  $\mathcal{B}$

[Adámek, el Bashir, Sobral and Velebil, 01]



$$\begin{array}{ccc}
 \mathcal{E} & \xrightarrow{P} & \mathcal{B} \\
 \searrow \psi & & \nearrow \varphi \\
 & Z(\text{Eq}(p)) &
 \end{array}
 \quad \mathbb{E} = \{\text{discrete fibrations}\} \text{ or } \{\text{discrete op-fibrations}\}$$

$$P \text{ } \mathbb{E}\text{-descent} \Leftrightarrow \varphi \text{ lax epimorphism}$$

$$P \text{ effective } \mathbb{E}\text{-descent} \Leftrightarrow$$

$$\overline{\varphi} : \text{Cauchy}C(Z(\text{Eq}(p))) \rightarrow \text{Cauchy}C(B) \text{ is an equivalence}$$

$$\begin{array}{ccc}
 \text{Set}^{\mathcal{B}} & \xrightarrow{\text{Set}^{\varphi}} & \text{Set}^{Z(\text{Eq}(p))} \\
 \searrow P^* & & \swarrow \text{Set}^{\psi} \\
 & \text{Set}^{\mathcal{E}} &
 \end{array}$$

$$P^* = (-)P \text{ monadic} \Leftrightarrow$$

Every object of  $\mathcal{B}$  is a retract of an object in  $P[\mathcal{E}]$

[Adámek, el Bashir, Sobral and Velebil, 01]

$$P^* \text{ monadic} \not\Rightarrow P \text{ is } \mathbb{E}\text{-descent}$$