

# Extension Theory for Mal'tsev Varieties

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## OUTLINE

- Mal'tsev varieties
- Centrality
- Conventions
- Monadic cohomology
- Seeded simplicial objects
- Growing modules from seeds
- Congruences plant seeds
- Obstructions
- Unobstructed extensions

## MAL'TSEV VARIETIES

- $(x, x, y)P = y = (y, x, x)P$
- Congruences permute
- Congruence join is relational product
- Reflexive subalgebras of squares are congruences

Examples: Groups, rings, Lie algebras, Heyting algebras, orthomodular lattices, quasigroups.

## CENTRALITY

$\gamma$  centralises  $\beta$  iff have centreing congruence  $(\gamma|\beta)$  on  $\beta$  satisfying:

**(C0)**  $(x, y) (\gamma|\beta) (x', y') \Rightarrow x \gamma y$

**(C1)**  $\forall (x, y) \in \beta,$   
 $\pi^0 : (x, y)^{(\gamma|\beta)} \rightarrow x^\gamma; (x', y') \mapsto x'$  bijects

**(C2) (RR)**  $\forall (x, y) \in \gamma, (x, x) (\gamma|\beta) (y, y)$

**(RS)**  $(x, y) (\gamma|\beta) (x', y') \Rightarrow (y, x) (\gamma|\beta) (y', x')$

**(RT)**  $(x, y) (\gamma|\beta) (x', y')$  and  
 $(y, z) (\gamma|\beta) (y', z') \Rightarrow (x, z) (\gamma|\beta) (x', z')$

In a Mal'tsev variety: Centreing congruences are unique, and: Largest congruence  $\eta(\alpha)$  centralising  $\alpha$  exists uniquely.

## CONVENTIONS

Simplicial object  $B^*$  (augmented):

$$\begin{array}{ccccccc}
 & & \xrightarrow{\varepsilon_3^0} & & & & \\
 & \xrightarrow{\varepsilon_3^1} & & \xrightarrow{\varepsilon_2^0} & & & \\
 \dots & \xrightarrow{\varepsilon_3^2} & B^2 & \xrightarrow{\varepsilon_2^1} & B^1 & \xrightarrow{\varepsilon_1^0} & B^0 \\
 & \xleftarrow{\delta_2^0} & & \xleftarrow{\delta_1^0} & & & \\
 & & \xleftarrow{\delta_2^1} & & & &
 \end{array}$$

Extend truncated simplicial objects by tacking on simplicial kernels, and omit “truncated”.

In Mal'tsev variety  $\mathfrak{V}$ ,  $R$ -module  $E \rightarrow R$  is an abelian group in  $\mathfrak{V}/R$ .

Examples:  $\alpha^{(\alpha|\alpha)} \rightarrow T^\alpha$  for self-centralising  $\alpha$ ;  
 $R \ltimes M \rightarrow R$  for  $R$ -module  $M$ .  
 Pull back  $B^0$ -module to  $B^*$ -module.

## MONADIC COHOMOLOGY

Given  $R$ -module  $E \rightarrow R$  in  $\mathfrak{V}$ :

- Let  $AG$  be free  $\mathfrak{V}$ -algebra over  $\{\{a\} \mid a \in A\}$
- Let  $\varepsilon_n^j : RG^n \rightarrow RG^{n-1}$  delete  $j$ -th brackets
- Let  $\delta_n^j : RG^n \rightarrow RG^{n+1}$  insert  $j$ -th brackets
- Write  $\text{Der}(RG^n, E)$  for abelian group  
 $\mathfrak{V}/R(RG^n \rightarrow R, E \rightarrow R)$
- $d_n : \text{Der}(RG^n, E) \rightarrow \text{Der}(RG^{n+1}, E);$   
 $f \mapsto \sum_{i=0}^n (-)^i \varepsilon_{n+1}^i f$
- $H^n(R, E) = \text{Ker}(d_n)/\text{Im}(d_{n-1}),$   
cocycles/coboundaries

## SEEDED SIMPLICIAL OBJECTS

$B^*$  is *seeded* if:

- truncated at  $B^2$
- $(\varepsilon_2^0, \varepsilon_2^1) : B^2 \rightarrow \ker(\varepsilon_1^0)$  surjects
- $\varepsilon_1^0 : B^1 \rightarrow B^0$  surjects
- $\ker(\varepsilon_2^0 : B^2 \rightarrow B^1) = \eta(\ker(\varepsilon_2^1 : B^2 \rightarrow B^1))$

## GROWING MODULES FROM SEEDS

Seeded simplicial object  $B^*$ :

Let  $C$  be equalizer of  $B^2 \xrightarrow{\begin{smallmatrix} \varepsilon_2^0 \\ \varepsilon_2^1 \end{smallmatrix}} B^1$ .

Define  $V$  on  $C$  by  $c V c'$  iff  
 $((c\varepsilon_2^0\delta_1^0, c), (c'\varepsilon_2^0\delta_1^0, c'))$  lies in  
 $(\ker \varepsilon_2^0 \circ \ker \varepsilon_2^1 | \ker \varepsilon_2^0 \cap \ker \varepsilon_2^1)$ .

Module  $C^V \rightarrow B^0; c^V \mapsto c\varepsilon_2^0\varepsilon_1^0$  is *grown* by  $B^*$ .

Note  $B^0$ -module  $C^V$  is isomorphic to  
 $(\ker \varepsilon_2^0 \cap \ker \varepsilon_2^1)^{(\ker \varepsilon_2^0 \circ \ker \varepsilon_2^1 | \ker \varepsilon_2^0 \cap \ker \varepsilon_2^1)}$ .

## CONGRUENCES PLANT SEEDS

Congruence  $\alpha$  on  $\mathfrak{V}$ -algebra  $T$  *plants* seeded simplicial object

$$\alpha^{(\eta(\alpha)|\alpha)} \xrightarrow{\quad} T^{\eta(\alpha)} \rightarrow T^{\alpha \circ \eta(\alpha)}$$

growing  $T^{\alpha \circ \eta(\alpha)}$ -module

$$(\alpha \cap \eta(\alpha))^{(\alpha \circ \eta(\alpha)|\alpha \cap \eta(\alpha))} \rightarrow T^{\alpha \circ \eta(\alpha)}.$$

## OBSTRUCTIONS

Given seeded simplicial object  $B^*$ , and regular epi  $p^0 : R \rightarrow B^0$ , define an obstruction:

Extend  $p$  to simplicial map  $p^* : RG^* \rightarrow B^*$ .

Module  $C^V$  grown by  $B^*$  becomes module over  $RG^*$ .

Define  $D = (\varepsilon_3^0, \varepsilon_3^1, \varepsilon_3^2)P^V : B^3 \rightarrow C^V$ .

Then the *obstruction* to  $p^0$  is the cohomology class of  $p^3 D : RG^3 \rightarrow C^V$  in  $H^3(R, C^V)$ .

## UNOSTRUCTURED EXTENSIONS

**THEOREM** [“Mal’cev Varieties,” 623]

Given seeded simplicial object  $B^*$ , and regular epi  $p^0 : R \rightarrow B^0$ , there is a congruence  $\alpha$  on an algebra  $T$  planting  $B^*$  and realizing  $p^0 : R \rightarrow B^0$  as  $T^\alpha \rightarrow T^{\alpha \circ \eta(\alpha)}$  iff  $p^0$  is unobstructed.

**PROOF**

“only if”: diagram chase in

$$\begin{array}{ccccccc}
 \Rightarrow & RG^2 & \Rightarrow & RG & \rightarrow & R \\
 & \downarrow & & \downarrow & & \downarrow \\
 \Rightarrow & \alpha & \Rightarrow & T & \rightarrow & R \\
 & \downarrow & & \downarrow & & \downarrow \\
 \Rightarrow & \alpha^{(\eta(\alpha)|\alpha)} & \Rightarrow & T^{\eta(\alpha)} & \rightarrow & T^{\alpha \circ \eta(\alpha)}
 \end{array}$$

“if”: obtain  $T$  as a quotient of the pullback of

$$\begin{array}{ccc}
 & RG & \\
 & \downarrow p_1 & \\
 B^2 & \xrightarrow{\varepsilon_2^1} & B^1
 \end{array}$$