A quasibialgebra H is an associative k-algebra equipped with a non-coassociative comultiplication $\Delta\colon H \to H \otimes H$,
a counit $\varepsilon\colon H \to k$, and an invertible element $\phi \in H \otimes H \otimes H$

such that the category ${}_H\mathcal{M}$ of left H-modules is a monoidal category with respect to tensor product over k,

$$h \cdot (v \otimes w) = h_{(1)}v \otimes h_{(2)}w \text{ for } V, W \in {}_H\mathcal{M}, v \in V, w \in W$$

$$\begin{aligned} \Phi\colon (U \otimes V) \otimes W &\to U \otimes (V \otimes W) \\ u \otimes v \otimes w &\mapsto \phi \cdot (u \otimes v \otimes w) \\ &= \phi^{(1)}u \otimes \phi^{(2)}v \otimes \phi^{(3)}w \end{aligned}$$

The element ϕ, the associator, has to satisfy suitable axioms so that we get a monoidal category as stated:

But we vow never to use these at all!!

...though they're not hard to find, really:

$$(H \otimes \Delta)\Delta(h) \cdot \phi = \phi \cdot (\Delta \otimes H)\Delta(h)$$

$$\begin{aligned}(H \otimes H \otimes \Delta)(\phi) \cdot (\Delta \otimes H \otimes H)(\phi) \\ = (1 \otimes \phi) \cdot (H \otimes \Delta \otimes H)(\phi) \cdot (\phi \otimes 1)\end{aligned}$$

$$(\varepsilon \otimes H)\Delta(h) = h = (H \otimes \varepsilon)\Delta(h)$$

$$(H \otimes \varepsilon \otimes H)(\phi) = 1$$

Why not use the axioms?

- Calculations with ϕ and the axioms are complicated and not very conceptual.
- There's extra notation for the "components" $\phi = \phi^{(1)} \otimes \phi^{(2)} \otimes \phi^{(3)}$ and the inverse $\phi^{-1} = \phi^{(-1)} \otimes \phi^{(-2)} \otimes \phi^{(-3)}$.
- Especially bad mess if we need several copies of ϕ...
- We can expect to get away without the mess!

What shall we use instead?

We will use the monoidal category structure of ${}_H\mathcal{M}$, which ought to contain all there is to know about the quasibialgebra H.

We will also use the monoidal category structure of ${}_H\mathcal{M}_H$, where the associator isomorphism is given by

$$\begin{aligned}\Phi\colon (U\otimes V)\otimes W &\to U\otimes (V\otimes W)\\ u\otimes v\otimes w &\mapsto \phi(u\otimes v\otimes w)\phi^{-1}\\ &= \phi^{(1)}u\phi^{(-1)}\otimes \phi^{(2)}v\phi^{(-2)}\otimes \phi^{(3)}w\phi^{(-3)}.\end{aligned}$$

(For once, we actually use the axioms of ϕ here.)

The key observation:

H is a coassociative coalgebra within the monoidal category ${}_H\mathcal{M}_H$.

This is nothing but the modified coassociativity axiom

$$
\begin{aligned}
(H \otimes \Delta)\Delta(h) &= \phi \cdot (\Delta \otimes H)\Delta(h) \cdot \phi^{-1} \\
&= \Phi(\Delta \otimes H)\Delta(h)
\end{aligned}
$$

$$
\begin{tikzcd}
H \arrow[rr, "\Delta"] \arrow[dd, "\Delta"'] & & H \otimes H \arrow[d, "\Delta \otimes H"] \\
& & (H \otimes H) \otimes H \arrow[dl, "\Phi"'] \\
H \otimes H \arrow[r, "H \otimes \Delta"] & H \otimes (H \otimes H) &
\end{tikzcd}
$$

First target, “classical” Hopf case

Theorem 1 (The structure theorem for Hopf modules). *Let H be a Hopf algebra.*

A category equivalence $\mathcal{M}_k \cong \mathcal{M}_H^H$ is given by

$V \mapsto V \otimes H_\bullet^\bullet$

Definition 2. A Hopf module $M \in \mathcal{M}_H^H$ is a right H-module and -comodule such that $(mh)_{(0)} \otimes (mh)_{(1)} = m_{(0)}h_{(1)} \otimes m_{(1)}h_{(2)}$ for $m \in M$ and $h \in H$.

Obviously, this makes no sense in the quasi-Hopf case.

Note, though, that the Hopf module condition says that

$M \to M_\bullet \otimes H_\bullet$ is an H-module map,

or shorter: M is an H-comodule in $\mathcal{M}_H$.

(Quasi-)Hopf (bi-)modules

Definition 3. Let H be a quasi-bialgebra.
A Hopf module in ${}_H\mathcal{M}_H^H$ is an H-comodule within the monoidal category ${}_H\mathcal{M}_H$.

Theorem 4. *Let H be a finite quasi-bialgebra. The following are equivalent:*

1. *H is a quasi-Hopf algebra.*
2. *The functor $\mathcal{R}\colon {}_H\mathcal{M} \to {}_H\mathcal{M}_H^H$ is a category equivalence.*
3. *The category ${}_H\mathcal{M}_{\mathrm{f.d.}}$ of finite left H-modules is rigid.*

(1)⇒(3) is already in Drinfeld (without "finite").

(1)⇒(2) is due to Hausser and Nill (without "finite").

(3)⇒(1) for bialgebras is due to Ulbrich.

We shall discuss (2)⇔(3)⇒(1), but first...

The dual case

It goes without saying that there is a dual notion of a coquasibialgebra, involving $\phi\colon H \otimes H \otimes H \to k$ instead of $\phi \in H \otimes H \otimes H$, and a nonassociative multiplication in place of a non-coassociative comultiplication.

For a coquasibialgebra H, both $\mathcal{M}^H$ and ${}^H\mathcal{M}^H$ are monoidal categories,

H is an associative algebra in ${}^H\mathcal{M}^H$,

one can define Hopf modules in ${}^H_H\mathcal{M}^H$,

and a functor $\mathcal{L}\colon \mathcal{M}^H \to {}^H_H\mathcal{M}^H$.

Theorem 4*: *Let H be a coquasibialgebra. The following are equivalent:*

1. *H is a coquasi-Hopf algebra.*
2. *The functor $\mathcal{L}\colon \mathcal{M}^H \to {}^H_H\mathcal{M}^H$ is a category equivalence.*
3. *The category $\mathcal{M}^H_{\mathrm{f.d.}}$ of finite right H-comodules is rigid.*

(1)⇒(3) is formally dual to Drinfeld.

(3)⇒(1) is due to Ulbrich for bialgebras.

(1)⇒(2) can be proved by arguments formally dual to those of Hausser and Nill.

(2)⇔(3) can be proved by formally dual arguments to those we shall give below for Theorem 4.

But (3)⇒(1) is false for $\dim H = \infty$.

Recall that a monoidal category $\mathcal{C}$ is rigid if for all $V \in \mathcal{C}$ there exists a dual object $(V^\vee, \mathrm{ev}, \mathrm{db})$,

where $V^\vee \in \mathcal{C}$, $\mathrm{ev}\colon V^\vee \otimes V \to I$ and $\mathrm{db}\colon I \to V \otimes V^\vee$ satisfy

$$\left(V \xrightarrow{\mathrm{db}\otimes V} (V \otimes V^\vee) \otimes V \xrightarrow{\Phi} V \otimes (V^\vee \otimes V) \xrightarrow{V\otimes \mathrm{ev}} V\right) = \mathrm{id}$$

$$\left(V^\vee \xrightarrow{V^\vee\otimes \mathrm{db}} V^\vee \otimes (V \otimes V^\vee) \xrightarrow{\Phi} (V^\vee \otimes V) \otimes V^\vee \xrightarrow{\mathrm{ev}\otimes V^\vee} V^\vee\right) = \mathrm{id}$$

A quasiantipode for a quasibialgebra H is a triple (S, α, β) where S is an algebra anti-endomorphism of H and $\alpha, \beta \in H$ satisfy

$$S(h_{(1)})\alpha h_{(2)} = \varepsilon(h)\alpha \qquad h_{(1)}\beta S(h_{(2)}) = \varepsilon(h)\beta$$

$$\phi^{(1)}\beta S(\phi^{(2)})\alpha\phi^{(3)} = 1 \qquad S(\phi^{(-1)})\alpha\phi^{(-2)}\beta\phi^{(-3)} = 1$$

The definition of a coquasi-Hopf algebra is, of course, formally dual!

If H is a quasi-Hopf algebra (i.e. has a quasiantipode) then $V \in {}_H\mathcal{M}_{\text{f.d.}}$ has left dual $V^\vee = V^*$ with module structure via S, evaluation and coevaluation

$$V^* \otimes V \to k \qquad\qquad k \to V \otimes V^*$$

$$\varphi \otimes v \mapsto \varphi(\alpha v) \qquad\qquad 1 \to \beta v_i \otimes v^i$$

In particular, ${}_H\mathcal{M}_{\text{f.d.}}$ is rigid when H is a quasi-Hopf algebra, and $\dim(V^\vee) = \dim(V)$ for all $V \in {}_H\mathcal{M}_{\text{f.d.}}$.

Of course, the same holds true for coquasi-Hopf algebras!

However, there is an example of a coquasibialgebra H such that $\mathcal{M}^H_{\text{f.d.}}$ is rigid, and there is $V \in \mathcal{M}^H_{\text{f.d.}}$ with $\dim(V^\vee) \neq \dim(V)$.

In particular, H is not a coquasi-Hopf algebra.

...now, back to business!

To prepare for Theorem 4 we have to establish a nice functor

$$\mathcal{R}\colon {}_H\mathcal{M} \to {}_H\mathcal{M}_H^H$$

...by just using generalities on monoidal categories:
Since H is a coalgebra in the monoidal category ${}_H\mathcal{M}_H$,
the underlying functor ${}_H\mathcal{M}_H^H \to {}_H\mathcal{M}_H$
forgetting the comodule structure of an H-comodule in ${}_H\mathcal{M}_H$
has a right adjoint.

$$\begin{array}{ccccccc} & & & {}_\bullet P_\bullet & \mapsto & {}_\bullet P_\bullet \otimes {}_\bullet H_\bullet^\bullet & \\ \mathcal{R} := (& {}_H\mathcal{M} & \to & {}_H\mathcal{M}_H & \xrightarrow{\tilde{R}} & {}_H\mathcal{M}_H^H &) \\ & V & \mapsto & {}_\bullet V_\varepsilon & \mapsto & {}_\bullet V \otimes {}_\bullet H_\bullet^\bullet & \end{array}$$

Now we will sketch a proof of

Theorem 5. *Let H be a finite quasibialgebra. Then the following are equivalent:*

1. *The functor $\mathcal{R}\colon {}_H\mathcal{M} \to {}_H\mathcal{M}_H^H$ is an equivalence.*
2. *The category ${}_H\mathcal{M}_{\mathrm{f.d.}}$ is rigid.*

- We start by an observation on ${}_H\mathcal{M}_H^H$ which is **not** a general categorical fact. The category ${}_H\mathcal{M}_H^H$ is a monoidal category with respect to the tensor product over H; i.e. for $M, N \in {}_H\mathcal{M}_H^H$ we have $M \otimes_H N \in {}_H\mathcal{M}_H^H$ with the codiagonal comodule structure. The associator is trivial!

- Moreover, the functor $\mathcal{R}$ is a monoidal functor. More generally $\mathcal{R}(V) \otimes_H M \cong (V \otimes H) \otimes_H M \cong {}_{\bullet}V \otimes {}_{\bullet}M$:
for all $V \in {}_H\mathcal{M}$ and $M \in {}_H\mathcal{M}_H^H$. Really!

- It is easy to check that $\mathcal{R}$ is fully faithful and exact. So the question is whether it is essentially surjective. The image of $\mathcal{R}$ is closed under limits, colimits, and tensor products.

- We can form cotensor products in the monoidal category $\mathcal{C} = {}_H\mathcal{M}_H$. So for $M \in {}_H\mathcal{M}_H^H$ we have $M \cong M \,\square_H\, H$ which amounts to an equalizer

$$0 \to {}_\bullet M_\bullet^\bullet \to {}_\bullet M_\bullet \otimes {}_\bullet H_\bullet^\bullet \rightrightarrows ({}_\bullet M_\bullet \otimes {}_\bullet H_\bullet) \otimes {}_\bullet H_\bullet^\bullet$$

Thus we need only know when objects of the form $P \otimes H$ with $P \in {}_H\mathcal{M}_H$, are in the image of $\mathcal{R}$.

- For $P \in {}_H\mathcal{M}_H$ we have $P \cong P \otimes_H H$; that is, we have a coequalizer

$$ {}_\bullet P \otimes H \otimes H_\bullet \rightrightarrows {}_\bullet P \otimes H_\bullet \to {}_\bullet P_\bullet \to 0. $$

So we need only check when $P \otimes H$ is in the image of $\mathcal{R}$ in the case $P = {}_\bullet V \otimes U_\bullet$ for $V \in {}_H\mathcal{M}$ and $U \in \mathcal{M}_H$.

- Since $({}_\bullet V \otimes U_\bullet) \otimes {}_\bullet H_\bullet^\bullet \cong ({}_\bullet V \otimes {}_\bullet H_\bullet^\bullet) \otimes_H (U_\bullet \otimes {}_\bullet H_\bullet^\bullet)$ and $\mathcal{R}$ is monoidal, we actually need only check that $U_\bullet \otimes {}_\bullet H_\bullet^\bullet$ is in the image of $\mathcal{R}$ when $U \in \mathcal{M}_H$.
- We may assume U is finite-dimensional, so $U = V^*$ for some $V \in {}_H\mathcal{M}_{\text{f.d.}}$.

Claim: All such $V^* \otimes H$ are in the image of $\mathcal{R}$ iff ${}_H\mathcal{M}_{\text{f.d.}}$ is rigid.

To prove this, one shows that $V^* \otimes H$ is something's dual.

Lemma: Let $V \in {}_H\mathcal{M}_{\text{f.d.}}$. Then $(V^*)_\bullet \otimes {}_\bullet H^\bullet_\bullet$ is a dual object of ${}_\bullet V \otimes {}_\bullet H^\bullet_\bullet$ in ${}_H\mathcal{M}^H_H$. Evaluation and coevaluation are given according to the identification

$$V^* \otimes H \cong \operatorname{Hom}_{-H}(V \otimes H, H)$$

of $V^* \otimes H$ with the dual of the finitely generated projective right H-module $V \otimes H$.

End of proof of Theorem 5: If $V \in {}_H\mathcal{M}_{\text{f.d.}}$ has a dual $V^\vee$, then $V^* \otimes H \cong \mathcal{R}(V^\vee)$, since monoidal functors preserve duals.

If $\mathcal{R}$ is an equivalence, the Lemma shows that any finite $M \in {}_H\mathcal{M}^H_H$ has a dual, hence so does any $V \in {}_H\mathcal{M}_{\text{f.d.}}$.

If H is a quasi-Hopf algebra, we learn more from the proof: In this case, for $V \in {}_H\mathcal{M}$ and $U = V^* \in \mathcal{M}_H$ we have $V^\vee = {}_S U$, hence $U_\bullet \otimes {}_\bullet H_\bullet^\bullet \cong \mathcal{R}({}_S U)$, and further, for $V \in {}_H\mathcal{M}$,

$$\begin{aligned}({}_\bullet V \otimes U_\bullet) \otimes {}_\bullet H_\bullet^\bullet &\cong ({}_\bullet V \otimes {}_\bullet H_\bullet^\bullet) \underset{H}{\otimes} (U_\bullet \otimes {}_\bullet H_\bullet^\bullet) \\ &\cong \mathcal{R}(V) \underset{H}{\otimes} \mathcal{R}({}_S U) \\ &\cong \mathcal{R}(V \otimes {}_S U) \\ &\cong \mathcal{R}({}_{\mathrm{ad}}({}_\bullet V \otimes U_\bullet))\end{aligned}$$

Where, for $P \in {}_H\mathcal{M}_H$, we define ${}_{\mathrm{ad}}P \in {}_H\mathcal{M}$ to have the action $h \rightharpoonup p = h_{(1)} p S(h_{(2)})$. More generally, we see that ${}_\bullet P_\bullet \otimes {}_\bullet H_\bullet^\bullet \cong \mathcal{R}({}_{\mathrm{ad}}P)$

Next target: The Double

...first, the ordinary Hopf picture:

If H is a finite Hopf algebra, its double $D(H)$ is a quasitriangular Hopf algebra, with underlying vector space $H^* \otimes H$.

The module category ${}_{D(H)}\mathcal{M}$ is isomorphic to the center $\mathcal{Z}({}_H\mathcal{M})$.

By definition, objects of $\mathcal{Z}(\mathcal{C})$ are objects of $\mathcal{C}$ plus a specified way of commuting them past objects of $\mathcal{C}$.

So $\mathcal{Z}({}_H\mathcal{M}) \ni (V, \sigma_{V,-})$, where $\sigma_{VX} : V \otimes X \to X \otimes V$ is an isomorphism, natural in $X \in {}_H\mathcal{M}$. There are axioms, of course!

Without axioms, any $\sigma_{V,-}$ as above has to have the form

$$\sigma_{VX} : V \otimes X \ni v \otimes x \mapsto v_{(-1)} \cdot x \otimes v_{(0)} \in X \otimes V$$

For some "coaction" $V \ni v \mapsto v_{(-1)} \otimes v_{(0)} \in H \otimes V$.

To actually define an object in $\mathcal{Z}({}_H\mathcal{M})$, the "coaction" has to turn V into a Yetter-Drinfeld module $V \in {}^H_H\mathcal{YD}$.

To get from here to the center, one turns the coaction into an action of H^*.
And one turns the Yetter-Drinfeld condition into a commutation relation between the actions of H and H^*.
And one turns this into the definition of an algebra structure on $H^* \otimes H$. ....and that's the Drinfeld double.

There's an obvious reason why it can't work like that in the quasi-Hopf case:

H^* isn't even an associative algebra.

So how is it going to embed into an associative $H^* \otimes H$?

...of course one can try...

In fact everything works just fine in the quasi-Hopf case, except for the last two steps. Even the Yetter-Drinfeld condition stays the same:

$$h_{(1)}v_{(-1)} \otimes h_{(2)}v_{(0)} = (h_{(1)}v)_{(-1)}h_{(2)} \otimes (h_{(1)}v)_{(0)}.$$

In the Hopf case, one turns this into

$$h_{(1)}v_{(-1)}S(h_{(3)}) \otimes h_{(2)}v_{(0)} = (hv)_{(-1)} \otimes (hv)_{(0)}$$

coact first act first

That won't work for the quasi-Hopf case (Antipodes are too complicated)

However, everything has been fixed by Hausser and Nill...by rather unwieldy calculations.

We'll do an approach with less calculations and more categories...

Let $\mathcal{C}$ be a monoidal category. A $\mathcal{C}$-actegory is a category $\mathcal{D}$ on which $\mathcal{C}$ acts. This means that there is a functor $\diamond\colon \mathcal{C} \times \mathcal{D} \to \mathcal{D}$ and a natural isomorphism $\Psi\colon (P \otimes Q) \diamond V \to P \diamond (Q \diamond V)$, where $P, Q \in \mathcal{C}$ and $V \in \mathcal{D}$, which is coherent.

If C is a coalgebra in $\mathcal{C}$, then surely the comodule category $\mathcal{C}^C$ is a $\mathcal{C}$-actegory. Just take $P \diamond V = P \otimes V^\bullet$.

So in particular ${}_H\mathcal{M}_H^H$ is a ${}_H\mathcal{M}_H$-actegory.

But ${}_H\mathcal{M}_H^H \cong {}_H\mathcal{M}$, so ${}_H\mathcal{M}$ is a ${}_H\mathcal{M}_H$-actegory, too.

The action $\diamond$ comes via the equivalence, so, for $P \in {}_H\mathcal{M}_H, V \in {}_H\mathcal{M}$,

$$\begin{aligned}\mathcal{R}(P \diamond V) &\cong P \otimes \mathcal{R}(V) \cong P \otimes (V \otimes H) \\ &\cong (P \otimes V) \otimes H \cong \mathcal{R}({}_{\mathrm{ad}}(P \otimes V))\end{aligned}$$

and $P \diamond V \cong {}_{\mathrm{ad}}(P \otimes V)$.

We can turn the action into a "representation", a monoidal functor

$$
\begin{tikzcd}
{}_H\mathcal{M}_H \arrow[r] \arrow[rr, bend left, "P \mapsto P \diamond H"] & \mathfrak{Fun}({}_H\mathcal{M}, {}_H\mathcal{M}) & {}_H\mathcal{M}_H \arrow[l, "\sim"', "\text{Watts}"] \\
\begin{array}{c}\text{tensor product} \\ \text{over } k \text{ with} \\ \text{nontrivial associator}\end{array} \arrow[u, dotted] & & \begin{array}{c}\text{tensor product} \\ \text{over } H \text{ with} \\ \text{trivial associator}\end{array} \arrow[u, dotted]
\end{tikzcd}
$$

This will turn any (co)algebra in ${}_H\mathcal{M}_H$ (a non-(co)associative ordinary (co)algebra) into a (co)algebra in ${}_H\mathcal{M}_H$, or an H-(co)ring.

In particular, this applies to the coalgebra $H \in {}_H\mathcal{M}_H$, giving an H-coring $H \diamond H$.

Or to the algebra $H^\vee \cong H^* \in {}_H\mathcal{M}_H$, giving an H-ring and hence k-algebra $H^\vee \diamond H \cong H^* \otimes H$.

It turns out that $D(H) = H^\vee \diamond H$ is the Drinfeld double.

For any $\mathcal{C}$-actegory $\mathcal{D}$ and any coalgebra C it makes sense to talk about the category ${}^C\mathcal{D}$ of C-comodules in $\mathcal{D}$. Use this formalism to calculate

$$ {}_{D(H)}\mathcal{M} = {}_{H^\vee \diamond H}\mathcal{M} \cong {}_{H^\vee}({}_H\mathcal{M}) \cong {}^H({}_H\mathcal{M}) \cong {}^H({}_H\mathcal{M}_H^H) \cong {}_H^H\mathcal{M}_H^H $$

and deduce that ${}_{D(H)}\mathcal{M}$ is a monoidal category, hence $D(H)$ is a quasibialgebra.