

Weak Categories in Additive 2-categories with kernels

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Internal Bicategories in Additive Categories with

$$\partial_0 \partial_1 = 0$$

2-cells

Composition

$$\left(x,y'-\partial_1\rho\left(y'\right)+y-\partial_1\lambda\left(y\right)-\partial_1\eta\left(x\right),z'-\rho\partial_1\left(z'\right)+z \right)$$

Associativity isomorphism

where

$$\alpha = \left(x, f_1(y'') + g_1 f_1(y') + g_1^2(y) + h(x) + g_1 h(x), \alpha_1(y) \right)$$

with

$$\begin{aligned}
\alpha_1(y'') &= \rho \partial_1 \rho(y'') - \rho(y'') \\
\alpha_2(y') &= \rho \partial_1 \lambda(y') - \lambda \partial_1 \rho(y') \\
\alpha_3(y) &= \lambda(y) - \lambda \partial_1 \lambda(y) - \eta \partial_0(y) + \rho \partial_1 \eta \partial_0(y) \\
\alpha_0(x) &= \rho \partial_1 \eta(x) - \lambda \partial_1 \eta(x) \\
f_1(y'') &= y'' - \partial_1 \rho(y'') \\
g_1 f_1(y') &= y' - \partial_1 \lambda(y') - \partial_1 \rho(y') + \partial_1 \lambda \partial_1 \rho(y') \\
g_1^2(y) &= y - 2 \partial_1 \lambda(y) + \partial_1 \lambda \partial_1 \lambda(y) \\
g_1 h(x) &= -\partial_1 \eta(x) + \partial_1 \lambda \partial_1 \eta(x)
\end{aligned}$$

Identity Isomorphisms

with

$$\lambda = (x, y - \partial_1 \lambda(y) - \partial_1 \eta(x), \lambda(y) + \eta(x))$$

and

$$\rho = \left(x, y - \partial_1 \rho(y) - \partial_1 \eta(x), \rho(y) + \eta(x) \right)$$

Weak Category

$$\begin{array}{c} C_2 \times_{C_1} C_2 \xrightarrow{m} C_2 \xrightleftharpoons[{}^c]{{}^d} {}^e C_1 \\ de = 1_{c_1} = ce \\ dm = d\pi_2 \\ cm = c\pi_1 \end{array}$$

$$\begin{array}{lcl} d\circ \lambda & = & 1_d=d\circ \rho \\ c\circ \lambda & = & 1_c=c\circ \rho \\ d\circ \alpha & = & 1_{d\pi_3}, c\circ \alpha=1_{c\pi_1} \end{array}$$

$$\lambda\circ e=\rho\circ e.$$

Weak Categories in Mor(Ab)

$$\delta_1 \lambda = 0$$

$$\delta_1 \rho = 0$$

$$\delta_1 \eta = 0$$

$$\begin{array}{ccc}
b_0 & \xrightarrow{(b_0 \quad a_0)} & b_0 + \delta_0(a_0) \\
\left(\begin{matrix} b_0 \\ b_1 \end{matrix} \right) \downarrow & \left(\begin{matrix} b_0 & a_0 \\ b_1 & a_1 \end{matrix} \right) & \downarrow \left(\begin{matrix} b_0 + \delta_0(a_0) \\ b_1 + \delta_0(a_1) \end{matrix} \right) \\
b_0 + \partial(b_1) & \xrightarrow{(b_0 + \partial(b_1) \quad a_0 + \partial(a_1))} & *
\end{array}$$

$$b_0 + \partial(b_1) + \delta_0(a_0 + \partial(a_1)) = b_0 + \delta_0(a_0) + \partial(b_1 + \delta_0(a_1))$$

2-Ab-category

- $[A, B] \times_{\text{hom}(A, B)} [A, B] \xrightarrow{m} [A, B] \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{e} \\ \xleftarrow{c} \end{array} \text{hom}(A, B)$ (diagonal groups)

$$[A, B] = \{\tau : f \longrightarrow g \mid f, g \in \text{hom}(A, B)\}$$

- horizontal composition is bilinear

Additive 2-category

- 2-Ab-category
- zero object
- biproducts

- $H(A, B) \xrightarrow{D} \text{hom}(A, B)$ (diagram of abelian groups)
- three associative bilinear composition laws

$$g\sigma, \tau\sigma, \tau f \in H(A, C)$$

$$\sigma \in H(A, B), \tau \in H(B, C), f \in \text{hom}(A, B), g \in \text{hom}(B, C)$$

$$\sigma 1_A = \sigma, 1_C \tau = \tau$$

$$\begin{aligned} D(\tau\sigma) &= D(\tau)D(\sigma) \\ D(\tau f) &= D(\tau)f \\ D(g\tau) &= gD(\tau) \end{aligned}$$

notation: $[\tau, \sigma] = D(\tau)\sigma - \tau D(\sigma)$

$$\mathbf{2\text{-}cell}$$

$$(\tau,f):f\longrightarrow g$$

$$\begin{array}{rcl}\tau & \in & H(A,B) \\ D\left(\tau\right) & = & g-f \\ (\sigma,g)\cdot(\tau,f) & = & (\sigma+\tau,f) \quad , \quad \left(\tau',f'\right)\circ(\tau,f)=\left(\tau'\tau+\tau'f-\tau f',f'\circ f\right) \\ \tau \in H\left(A_1\oplus A_2,B_1\oplus B_2\right) & & \tau=\left(\begin{array}{cc}\tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22}\end{array}\right),\tau_{ij}\end{array}$$

Weak Categories in Additive 2-Categories With Kernels

$$A \xrightarrow{\delta} B$$

$$\begin{aligned}\lambda, \rho &\in H(A, A) \\ \eta &\in H(B, A)\end{aligned}$$

$$\begin{aligned}\delta\lambda &= 0 \\ \delta\rho &= 0 \\ \delta\eta &= 0\end{aligned}$$

$$(1-D\rho)\,[\rho,\rho]=0$$

$$D\lambda\,[\rho,\rho]=D\rho\,[\lambda,\rho]+(1-D\rho)\,[\rho,\lambda]$$

$$(1-D\lambda)\,[\lambda,\rho]+D\lambda\,[\rho,\lambda]=D\rho\,[\lambda,\lambda]+(1-D\rho)\,[\rho,$$

$$(1-D\lambda)\,[\lambda,\lambda]=(1-D\lambda)\,[\lambda,\eta]$$

$$(1-D\rho-D\lambda)\,[\lambda,\eta]=(1-D\rho-D\lambda)\,[\rho,\eta]$$

Description of the weak category

$$A \oplus A \oplus B \xrightarrow{m} A \oplus B \xrightleftharpoons[\delta]{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} B$$

$$m = \begin{pmatrix} f & g & h \\ 0 & 0 & 1 \end{pmatrix}$$

$$g = 1 - D(\lambda)$$

$$f = 1 - D(\rho)$$

$$h = -D(\eta)$$

$$\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\alpha_1 \; = \; -f\rho$$

$$\alpha_2 \; = \; \lambda + g\rho - \rho - f\lambda$$

$$\alpha_3 \; = \; g\lambda - f\eta\delta$$

$$\alpha_0 \; = \; g\eta - f\eta$$

$$\lambda=\left(\begin{array}{cc}\lambda&\eta\\0&0\end{array}\right)\qquad\qquad\rho=\left(\begin{array}{cc}\rho&\eta\\0&0\end{array}\right)$$

The Category of all Weak Categories

Weak Categories

Weak Functors

Horizontal Weak Natural Transformations

$$\tau : C_1 \longrightarrow C'_2$$

$$\begin{aligned} d'\tau &= F_1 \\ c'\tau &= G_1 \\ m' \langle \tau c, F_2 \rangle &= m' \langle G_2, \tau d \rangle \end{aligned}$$

Vertical Natural Transformation

$$\theta_2 : F_2 \longrightarrow G_2 , \theta_1 : F_1 \longrightarrow G_1$$

$$\begin{aligned} d' \circ \theta_2 &= \theta_1 \circ d \\ c' \circ \theta_2 &= \theta_1 \circ c \end{aligned}$$

$$\cdot \xrightarrow{\mu_F} \cdot \qquad \cdot \xrightarrow{\varepsilon_F} \cdot$$

$$m' \circ \left(\theta_2 \times_{\theta_1} \theta_2 \right) \downarrow \qquad \qquad \qquad \downarrow \theta_2 \circ m \qquad \qquad e' \circ \theta_1 \downarrow \qquad \qquad \qquad \downarrow \theta_2 \circ e$$

$$\cdot \xrightarrow{\mu_G} \cdot \qquad \cdot \xrightarrow{\varepsilon_G} \cdot$$

Square Modifications

$$\begin{array}{ccc} F & \xrightarrow{\tau} & G \\ \theta \downarrow & \Phi & \downarrow \theta' \\ H & \xrightarrow{\tau'} & K \end{array}$$

$$\Phi : \tau \longrightarrow \tau'$$

$$d'\circ\Phi\;\;=\;\;\theta_1$$

$$c'\circ\Phi\;\;=\;\;\theta'_1$$

$$m'\left<\Phi\circ c,\theta_2\right>\;\;=\;\;m'\left<\theta'_2,\Phi\circ d\right>$$

Weak functors in additive 2-categories with kernels

$$\begin{aligned} F_B &: B \longrightarrow B' \\ F_A &: A \longrightarrow A' \end{aligned}$$

$$\varepsilon \in H(B, A')$$

$$\delta' \varepsilon = 0$$

$$[F_A\rho, \rho] = [\rho', F_A\rho]$$

$$[\rho', \varepsilon] \delta + [\rho', F_A\lambda] + [F_A\lambda, \rho] = [\lambda', F_A\rho] + [F_A\rho,$$

$$[F_A\lambda, \lambda] + [\lambda', F_A\eta] \delta = [\lambda', F_A\lambda] + [F_A\lambda, \eta] \delta$$

$$\left[\rho',\varepsilon\right]+\left[\rho',F_A\eta\right]+\left[F_A\lambda,\eta\right]=\left[\lambda',\varepsilon\right]+\left[\lambda',F_A\eta\right]+\left[F_A\lambda',\eta\right]$$

Description of the weak functor

$$F_1 = F_B \quad F_2 = \begin{pmatrix} F_A & D(\varepsilon) \\ 0 & F_B \end{pmatrix}$$

$$\mu = \begin{pmatrix} \mu_1 & \mu_2 & \mu_3 \\ 0 & 0 & 0 \\ \varepsilon \\ 0 \end{pmatrix}$$

$$\begin{aligned} \mu_1 &= \rho' F_A - F_A \rho \\ \mu_2 &= \lambda' F_A - F_A \lambda - \varepsilon \delta + \rho' D \varepsilon \delta \\ \mu_3 &= \rho' D \varepsilon + \lambda' D \varepsilon + \eta' F_B - F_A \eta - \varepsilon \end{aligned}$$

Vertical Natural Transformation

$$\theta_2 \in H(A, A') \quad , \quad \theta_1 \in H(B, B')$$

such that

$$\begin{aligned} D(\theta_2) &= G_A - F_A \\ D(\theta_1) &= G_B - F_B \end{aligned}$$

$$\delta' \theta_2 = \theta_1 \delta$$

$$D(\rho') \theta_2 + (G_A - F_A) \rho = \theta_2 D(\rho) + \rho' (G_A - F_A)$$

$$\begin{aligned} [\rho', \varepsilon_G] \delta + D(\lambda') \theta_2 + (G_A - F_A) \lambda &= \\ &= [\rho', \varepsilon_F] \delta + \theta_2 D(\lambda) + \lambda' \end{aligned}$$

$$D\left(\eta'\right)\theta_1+\left(G_A-F_A\right)\eta+\left[\rho',\varepsilon_G\right]+\left[\lambda',\varepsilon_G\right]=\\=\theta_2D\left(\eta\right)+\eta'\left(G_B-F_B\right)+\left[\rho',\varepsilon_F\right]$$

$$\text{Horizontal weak natural transformation } \tau : F \longrightarrow G$$

$$\tau:B\longrightarrow A'\oplus B'$$

$$\tau = \left(\begin{array}{c} \tau \\ F_B \end{array}\right)$$

$$\delta \tau = G_B - F_B$$

$$\left(1-D\rho'\right)\tau\delta=\left(1-D\rho'\right)G_A-\left(1-D\lambda'\right)F_A$$

$$D\left(\lambda'-\rho'\right)\tau=D\left(\varepsilon_G-\varepsilon_F+\lambda'\varepsilon_F-\rho'\varepsilon_G\right)$$

Square Modifications

$$\begin{array}{ccc} F & \xrightarrow{\tau} & G \\ \theta \downarrow & \Phi & \downarrow \theta' \\ H & \xrightarrow{\tau'} & K \end{array}$$

$$\Phi \in H(B, A')$$

such that

$$D\Phi = \tau' - \tau$$

$$\delta'\Phi = \theta'_1 - \theta_1$$

$$f'\Phi\delta = f'\theta'_2 - g'\theta_2$$

$$\left(f'-g'\right)\Phi=f'\left(\varepsilon_K-\varepsilon_H\right)-g'\left(\varepsilon_G-\varepsilon_F\right)$$