On locally compact c-compact groups

Gábor Lukács
York University, Canada

Under supervision of Prof. Walter Tholen

By the well known Kuratowski-Mrówka theorem, a (Hausdorff) topological space X is compact if and only if for any (Hausdorff) topological space Y the projection $p_Y : X \times Y \to Y$ is closed.

All topological groups are assumed to be Hausdorff.

Using this theorem as a categorical definition, a topological group G is $\mathbf{c}$-***compact*** if for any topological group H the image of every closed subgroup of $G \times H$ under the projection $\pi_H : G \times H \to H$ is closed in H.

If G is c-compact, then all closed subgroups and quotients of G are c-compact.

If G is c-compact and $\varphi : G \to H$ is a continuous homomorphism, then $\varphi(G)$ is closed in H (as its graph is closed).

A group G is $\mathbf{h}$-***complete*** if for any continuous homomorphism $\varphi : G \to H$, $\varphi(G)$ is closed in H.

For T_3 topological spaces even a property weaker than h-completeness (called H-closedness) is sufficient to ensure compactness.

The question whether c-compact groups are compact has been open for over 15 years. Some known results:

- The product of any family of c-compact groups is c-compact (Clementino & Tholen, 1996).
- The product of any family of h-complete groups is h-complete (Dikranjan & Uspenskij, 1998).
- Every nilpotent h-complete topological groups is compact; every soluble c-compact group is compact (Dikranjan & Uspenskij, 1998).
- Every locally compact connected c-compact group is compact (Dikranjan & Uspenskij, 1998).

A group G is called *SIN* (Small Invariant Neighborhoods), if G has a base at e consisting of neighborhoods U such that $g^{-1}Ug = U$ for all $g \in G$.

- An SIN group is c-compact if and only if every closed subgroup of G is h-complete (Dikranjan & Uspenskij, 1998).

A topological group G is called *MAP* (Maximally Almost Periodic), if G admits a continuous injective homomorphism into a compact group $m : G \to K$, or alternatively, if the finite-dimensional representations of G separate points in G.

THEOREM I (GL, 2002). *If G is locally compact, c-compact and MAP then G is compact.*

In fact, THEOREM I follows from the following

THEOREM II (GL, 2002). *Let G be a locally compact group. The following statements are equivalent:*

(i) *G is SIN, the closed σ-compact subgroups of G and G itself are h-complete, and the countable discrete quotient M/N, where M is a closed subgroup of G and N is a closed normal subgroup of M, are MAP;*

(ii) *the open σ-compact subgroups of G are h-complete and MAP, and G is h-complete;*

(iii) *G is compact.*

Sketch of the proof. (ii) $\Rightarrow$ (iii) Locally compact σ-compact subgroup satisfy an open map theorem; using the MAP property, it is not hard to show that σ-compact open subgroups of G are compact.

This implies, by a result due to Pestov, that G is SIN.

If A is an infinite set in G, then A together with a neighborhood of e having compact closure generate a σ-compact open subgroup of G, which is thus compact. Therefore A has a limit point, which shows that G is countably compact.

Since G is SIN, it embeds into product of metrizable groups $M = \prod_{\alpha \in I} M_\alpha$. (by Graev's theorem). Since G is countably compact, $\pi_\alpha(G)$ is compact in M_α, because M_α is metrizable (where π_α is the canonical projection on M_α).

Thus, without loss of generality one may assume that the M_α are all compact, and since G is h-complete, it is closed in their product M, which is again compact. $\square$

Sketch of the proof. (i) $\Rightarrow$ (ii) Let H be an open σ-compact subgroup of G. Then H and all its closed subgroups are h-complete by (i), thus H is c-compact (because it is SIN).

Let C be the connected component of e in H. Since C is locally compact, connected and c-compact, it is compact. So it suffices to show that H/C is compact.The quotient H/C is totally-disconnected, locally compact, SIN and c-compact. Thus, compact-open normal subgroups form a base at e.

This means that H/C admits a continuous injection into the product of its discrete quotients. Each quotient is MAP and h-complete, by (i), and countable because H is σ-compact. So, by the open map theorem for locally compact groups, the injection of each discrete section into a compact group is actually an embedding. Therefore each discrete section is compact, and hence H/C is compact, because it embeds into the product of compact groups (again, by open map theorem). $\square$