Descent theory for lax algebras

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The basic setting.

Throughout we will assume that:

- $\mathbf{V}$ is a symmetric monoidal closed complete lattice, with tensor product $\otimes$ and unit I.
- (T, e, m) is a monad on **Set** lax-extended to $\mathrm{Mat}(\mathbf{V})$.

In **Top** and **Loc**:

- Every open surjection is an effective descent morphism (Moerdijk 90; Sobral 91).
- Every proper surjection is an effective descent morphism (Moerdijk 90, Vermeulen 94).
- Every triquotient map is an effective descent morphism (Plewe 97).

- Reiterman and Tholen (1994) characterized effective descent morphism in **Top**:

  Theorem. *A continuous map $p : E \to B$ is effective descent iff every crest of ultrafilters in B converging to $y \in B$ has a lifting along p which converges to some $x \in p^{-1}(y)$.*

$$
\begin{tikzcd}
E \arrow[d, "p"'] & & \mathfrak{A} \arrow[r, dashed] \arrow[d, dash, dashed] & \mathfrak{a} \arrow[r, dashed] \arrow[d, dash, dashed] & x \arrow[d, dash, dashed] \\
B & & \mathfrak{B} \arrow[r] & \mathfrak{b} \arrow[r] & y
\end{tikzcd}
$$

- In **FinTop** it reduces to (Janelidze, Sobral - 1999)

  Theorem. *A continuous map $p : E \to B$ is effective descent iff for every chain*

  $$b_2 \to b_1 \to b_0$$

  in B there exists

  $$e_2 \to e_1 \to e_0$$

  in E with $p(e_i) = b_i$, for $i = 0, 1, 2$.

$$\begin{tikzcd} E \arrow[d, "p"'] & {\color{blue}e_2} \arrow[r, dashed] \arrow[d, dash, dashed] & {\color{blue}e_1} \arrow[r, dashed] \arrow[d, dash, dashed] & {\color{blue}e_0} \arrow[d, dash, dashed] \\ B & b_2 \arrow[r] & b_1 \arrow[r] & b_0 \end{tikzcd}$$

Theorem. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be categories satisfying*

1. $\mathbf{B}$ *has pullbacks and coequalizers and* $\mathbf{A}$ *is a full subcategory of* $\mathbf{B}$ *closed under pullbacks;*
2. *every regular epi in* $\mathbf{B}$ *is effective descent;*
3. *every pb-stable regular epi in* $\mathbf{A}$ *is a regular epi in* $\mathbf{B}$.

Then a pb-stable regular epi $p : E \to B$ *in* $\mathbf{A}$ *is effective descent iff*

$$E \times_B A \in \mathbf{A} \Rightarrow A \in \mathbf{A}$$

holds for every pullback

$$\begin{tikzcd} E \times_B A \arrow[r, "p'"] \arrow[d, "f'"'] & A \arrow[d, "f"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

in $\mathbf{B}$.

We intend to apply this theorem to

$$\mathbf{A} = \mathrm{Alg}(T, e, m; \mathbf{V}) \hookrightarrow \mathrm{Alg}(T, e; \mathbf{V}) = \mathbf{B}.$$

Hence, in addition to our basic situation, we assume that:

- $\mathbf{V}$ is locally c.c. and the functor $T : \mathbf{Set} \to \mathbf{Set}$ has (BCP).
- For each set X, each $\mathfrak{x}_0 \in TX$ and each $x_0 \in X$ with $e_X(x_0) \neq \mathfrak{x}_0$ and each $\alpha \in \mathbf{V}$, the structure $a^{\alpha}_{\mathfrak{x}_0, x_0} : TX \nrightarrow X$ defined by

$$a^{\alpha}_{\mathfrak{x}_0, x_0}(\mathfrak{x}, x) = \begin{cases} \alpha & \text{if } \mathfrak{x} = \mathfrak{x}_0 \text{ and } x = x_0, \\ I & \text{if } \mathfrak{x} = e_X(x), \\ 0 & \text{else} \end{cases}$$

  is transitive.

The case $(T, e, m) = (\mathrm{Id}, \mathrm{id}, \mathrm{id})$.

Theorem. *For every effective descent morphism* $f : (X, a) \to (Y, b)$ *in* $\mathrm{Alg}(T, e, m; \mathbf{V})$,

$$b(y_2, y_1) \otimes b(y_1, y_0) = \bigvee_{\substack{x_i \in f^{-1}(y_i) \\ i=0,1,2}} a(x_2, x_1) \otimes a(x_1, x_0) \qquad (*)$$

for all $y_2, y_1, y_0 \in Y$.

Remark.

$$\left.\begin{array}{l} (*) \\ I \text{ terminal} \end{array}\right\} \Rightarrow \quad f \text{ is a pb-stable reg. epi.}$$

Theorem. *Assume that* $\mathbf{V} = \overline{\mathbb{R}}_+$ *or* $\otimes = \wedge$ *and let* $f : (X, a) \to (Y, b)$ *in* $\mathrm{Alg}(T, e, m; \mathbf{V})$. *The following are equivalent:*

- f *is effective descent.*
- *For all* $y_2, y_1, y_0 \in Y$,

$$b(y_2, y_1) \otimes b(y_1, y_0) = \bigvee_{\substack{x_i \in f^{-1}(y_i)\\ i=0,1,2}} a(x_2, x_1) \otimes a(x_1, x_0).$$

In particular, $f : (X, d_1) \to (Y, d_2)$ in **QMet** is effective descent iff

$$d_2(y_2, y_1) + d_2(y_1, y_0) = \inf\{d_1(x_2, x_1) + d_1(x_1, x_0) \mid x_i \in f^{-1}(y_i)\},$$

for all $y_2, y_1, y_0 \in Y$.

Theorem. $f : (X, d_1) \to (Y, d_2)$ *in* **Met** *is effective descent iff*

$$d_2(y_2, y_1) + d_2(y_1, y_0) = \inf\{d_1(x_2, x_1) + d_1(x_1, x_0) | \, x_i \in f^{-1}(y_i)\},$$

for all $y_2, y_1, y_0 \in Y$.

The general case: open and proper maps.

	FinTop	**Top**
$\begin{array}{c} E \\ \downarrow{\scriptstyle p} \\ B \end{array}$ proper	$\begin{tikzcd} x_0 \arrow[r, dashed] \arrow[d, dash] & x \arrow[d, dash, dashed] \\ p(x_0) \arrow[r] & y \end{tikzcd}$	$\begin{tikzcd} \mathfrak{a} \arrow[r, dashed] \arrow[d, dash] & x \arrow[d, dash, dashed] \\ p(\mathfrak{a}) \arrow[r] & y \end{tikzcd}$
$\begin{array}{c} E \\ \downarrow{\scriptstyle p} \\ B \end{array}$ perfect	$\begin{tikzcd} x_0 \arrow[r, dashed] \arrow[d, dash] \arrow[dr, phantom, "\text{(unique)}"] & x \arrow[d, dash, dashed] \\ p(x_0) \arrow[r] & y \end{tikzcd}$	$\begin{tikzcd} \mathfrak{a} \arrow[r, dashed] \arrow[d, dash] \arrow[dr, phantom, "\text{(unique)}"] & x \arrow[d, dash, dashed] \\ p(\mathfrak{a}) \arrow[r] & y \end{tikzcd}$

	FinTop	**Top**
$\begin{tikzcd} E \arrow[d, "p"'] \\ B \end{tikzcd}$ open	$\begin{tikzcd} \color{blue}{x} \arrow[r, dashed] \arrow[d, dash, dashed] & x_0 \arrow[d, dash] \\ y \arrow[r] & p(x_0) \end{tikzcd}$	$\begin{tikzcd} \color{blue}{\mathfrak{a}} \arrow[r, dashed] \arrow[d, dash, dashed] & x_0 \arrow[d, dash] \\ \mathfrak{b} \arrow[r] & p(x_0) \end{tikzcd}$
$\begin{tikzcd} E \arrow[d, "p"'] \\ B \end{tikzcd}$ loc. hom.	$\begin{tikzcd} \color{magenta}{x} \arrow[r, dashed] \arrow[d, dash, dashed] \arrow[dr, phantom, "\color{magenta}{\text{(unique)}}"] & x_0 \arrow[d, dash] \\ y \arrow[r] & p(x_0) \end{tikzcd}$	

Theorem. $p : E \to B$ *in* **Top** *is a local homeomorphism iff*

$$\begin{tikzcd}
E' \arrow[d, "p'"'] & & \mathfrak{a} \arrow[r, dashed] \arrow[d, dashed, dash] \arrow[dr, phantom, "\text{(unique)}"] & x_0 \arrow[d, dash] \\
B' & & \mathfrak{b} \arrow[r] & p'(x_0)
\end{tikzcd}$$

for each pullback p' of p.

Definition. A morphism $f : (X, a) \to (Y, b)$ is called *proper (open)* if the diagram

$$\begin{tikzcd}
TX \arrow[r, "Tf"] \arrow[d, "a"', "|" marking] & TY \arrow[d, "b", "|" marking] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

commutes (satisfies the Beck-Chevalley Property).

In addition to our basic situation we assume that

- T commutes with $(\_)^\circ$ and
- $T(f \cdot r) = Tf \cdot Tr$ for any map f.

Proposition. *Proper and open surjections are pb-stable regular epis.*

Proposition. *The class of proper (open) maps is pullback stable.*

Theorem. *Open and proper surjections are effective descent.*