

The basic setting.

Throughout we will assume that:

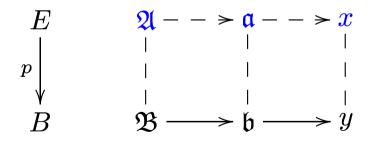
- V is a symmetric monoidal closed complete lattice, with tensor product \otimes and unit I.
- (T, e, m) is a monad on **Set** lax-extended to Mat(**V**).

In **Top** and **Loc**:

- Every open surjection is an effective descent morphism (Moerdijk 90; Sobral 91).
- Every proper surjection is an effective descent morphism (Moerdijk 90, Vermeulen 94).
- Every triquotient map is an effective descent morphism (Plewe 97).

• Reiterman and Tholen (1994) characterized effective descent morphism in **Top**:

Theorem. A continuous map $p: E \to B$ is effective descent iff every crest of ultrafilters in B converging to $y \in B$ has a lifting along p which converges to some $x \in p^{-1}(y)$.



• In **FinTop** it reduces to (Janelidze, Sobral - 1999) **Theorem.** A continuous map $p: E \to B$ is effective descent iff for every chain

$$b_2 \rightarrow b_1 \rightarrow b_0$$

in B there exists

$$e_2 \rightarrow e_1 \rightarrow e_0$$

in E with $p(e_i) = b_i$, for i = 0, 1, 2.

$$\begin{array}{c|cccc}
E & e_2 -- > e_1 -- > e_0 \\
\downarrow & & | & | & | \\
p & & | & | & | \\
\downarrow & & | & | & | \\
B & b_2 \longrightarrow b_1 \longrightarrow b_0
\end{array}$$

Theorem. Let A and B be categories satisfying

- 1. **B** has pullbacks and coequalizers and **A** is a full subcategory of **B** closed under pullbacks;
- 2. every regular epi in **B** is effective descent;
- 3. every pb-stable regular epi in **A** is a regular epi in **B**.

Then a pb-stable regular epi $p: E \to B$ in **A** is effective descent iff

$$E \times_B A \in \mathbf{A} \Rightarrow A \in \mathbf{A}$$

holds for every pullback

$$E \times_B A \xrightarrow{p'} A$$

$$f' \downarrow \qquad \qquad \downarrow f$$

$$E \xrightarrow{n} B$$

in \mathbf{B} .

We intend to apply this theorem to

$$\mathbf{A} = \operatorname{Alg}(T, e, m; \mathbf{V}) \hookrightarrow \operatorname{Alg}(T, e; \mathbf{V}) = \mathbf{B}.$$

Hence, in addition to our basic situation, we assume that:

- V is locally c.c. and the functor $T : \mathbf{Set} \to \mathbf{Set}$ has (BCP).
- For each set X, each $\mathfrak{x}_0 \in TX$ and each $x_0 \in X$ with $e_X(x_0) \neq \mathfrak{x}_0$ and each $\alpha \in \mathbf{V}$, the structure $a_{\mathfrak{x}_0,x_0}^{\alpha} : TX \nrightarrow X$ defined by

$$a_{\mathfrak{x}_0,x_0}^{\alpha}(\mathfrak{x},x) = \begin{cases} \alpha & \text{if } \mathfrak{x} = \mathfrak{x}_0 \text{ and } x = x_0, \\ I & \text{if } \mathfrak{x} = e_X(x), \\ 0 & \text{else} \end{cases}$$

is transitive.

The case (T, e, m) = (Id, id, id).

Theorem. For every effective descent morphism

$$f:(X,a)\to (Y,b)$$
 in $\mathrm{Alg}(T,e,m;\mathbf{V}),$

$$b(y_2, y_1) \otimes b(y_1, y_0) = \bigvee_{\substack{x_i \in f^{-1}(y_i) \\ i = 0, 1, 2}} a(x_2, x_1) \otimes a(x_1, x_0) \tag{*}$$

for all $y_2, y_1, y_0 \in Y$.

Remark.

$$\left. \begin{array}{c} (*) \\ I \text{ terminal} \end{array} \right\} \quad \Rightarrow \quad f \text{ is a pb-stable reg. epi.}$$

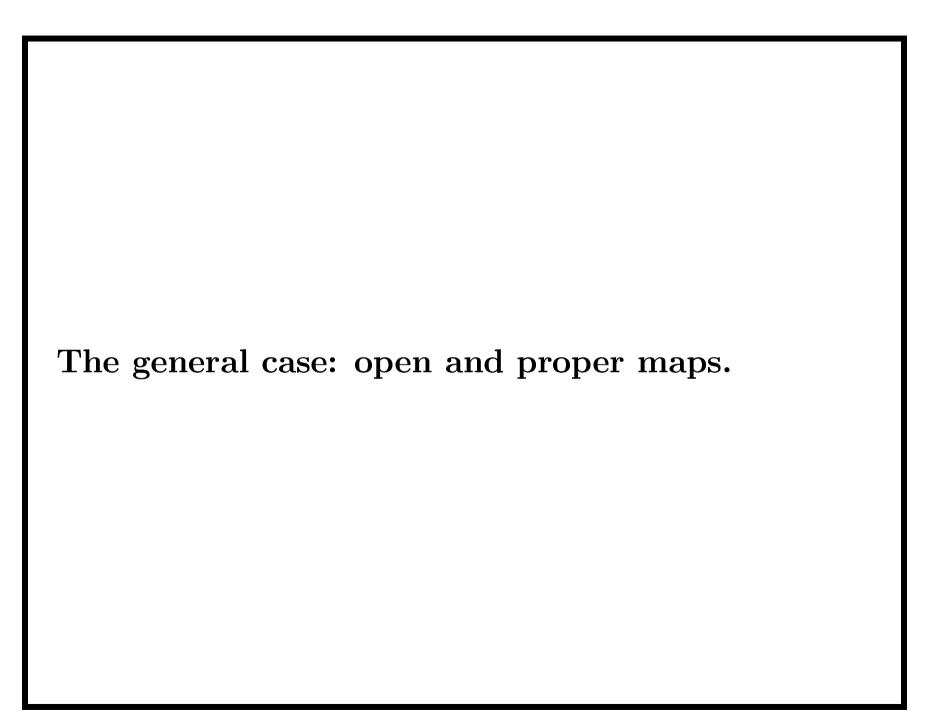
Theorem. Assume that $\mathbf{V} = \overline{\mathbb{R}}_+$ or $\otimes = \wedge$ and let $f: (X, a) \to (Y, b)$ in $Alg(T, e, m; \mathbf{V})$. The following are equivalent:

- f is effective descent.
- For all $y_2, y_1, y_0 \in Y$,

$$b(y_2, y_1) \otimes b(y_1, y_0) = \bigvee_{\substack{x_i \in f^{-1}(y_i) \\ i = 0, 1, 2}} a(x_2, x_1) \otimes a(x_1, x_0).$$

In particular, $f:(X,d_1) \to (Y,d_2)$ in **QMet** is effective descent iff $d_2(y_2,y_1) + d_2(y_1,y_0) = \inf\{d_1(x_2,x_1) + d_1(x_1,x_0) | x_i \in f^{-1}(y_i)\},$ for all $y_2,y_1,y_0 \in Y$.

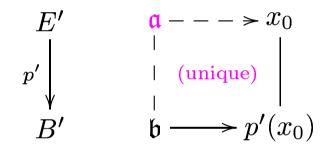
Theorem. $f:(X,d_1) \to (Y,d_2)$ in **Met** is effective descent iff $d_2(y_2,y_1) + d_2(y_1,y_0) = \inf\{d_1(x_2,x_1) + d_1(x_1,x_0) | x_i \in f^{-1}(y_i)\},$ for all $y_2,y_1,y_0 \in Y$.



	FinTop	Top
$\left egin{array}{c} E \\ p \\ \downarrow & \mathrm{proper} \\ B \end{array} \right $	$ \begin{array}{c c} x_0 > x \\ & & \\ & & \\ p(x_0) \longrightarrow y \end{array} $	$ \begin{array}{ccc} \mathfrak{a} > x \\ & & \\ & & \\ p(\mathfrak{a}) \longrightarrow y \end{array} $
$\left egin{array}{c} E \\ p \\ \downarrow & ext{perfect} \\ B \end{array} \right $	$x_0 > x$ $ \text{(unique)} $ $p(x_0) \longrightarrow y$	$ \begin{array}{c} \mathfrak{a} > x \\ $

		FinTop	Top
$\begin{bmatrix} E \\ p \\ \downarrow \\ B \end{bmatrix}$	open	$\begin{array}{c c} x > x_0 \\ \hline \\ \\ \\ y \longrightarrow p(x_0) \end{array}$	$ \begin{array}{ccc} \mathfrak{a} > x_0 \\ & & \\ $
$\begin{bmatrix} E \\ p \\ \downarrow \\ B \end{bmatrix}$	loc. hom.	$\begin{array}{c c} x > x_0 \\ & & \\ & & \\ & & \\ & & \\ y \longrightarrow p(x_0) \end{array}$	

Theorem. $p: E \to B$ in **Top** is a local homeomorphism iff



for each pullback p' of p.

Definition. A morphism $f:(X,a)\to (Y,b)$ is called *proper (open)* if the diagram

$$TX \xrightarrow{Tf} TY$$

$$\downarrow b$$

$$X \xrightarrow{f} Y$$

commutes (satisfies the Beck-Chevalley Property).

In addition to our basic situation we assume that

- T commutes with $(_)^{\circ}$ and
- $T(f \cdot r) = Tf \cdot Tr$ for any map f.

Proposition. Proper and open surjections are pb-stable regular epis.

Proposition. The class of proper (open) maps is pullback stable.

Theorem. Open and proper surjections are effective descent.