

Exponentiability in categories of lax algebras

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Exponentiability in categories of lax algebras

- [CH01] Topological features of lax algebras, APCS
- [CT01] Metric, top. and multicat. – a common approach, JPAA
- [CHT02] Exponentiation in categories of (T, \mathbf{V}) -algebras.

(T, e, m) monad in \mathbf{Set}

$\mathbf{Set} \xrightarrow{T} \mathbf{Set}$

(T, e, m) monad in \mathbf{Set}

$\mathbf{Set} \xrightarrow{T} \mathbf{Set}$

$$\begin{array}{ccccc} X & \xrightarrow{e_X} & TX & \xleftarrow{Ta} & T^2X \\ & \searrow 1_X & \downarrow a & & \downarrow m_X \\ & & X & \xleftarrow{a} & TX \end{array}$$

algebra

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ a \downarrow & & \downarrow b \\ X & \xrightarrow{f} & Y \end{array}$$

morphism

(T, e, m) (lax-)extended to Rel

$$\begin{array}{ccc} \mathbf{Set} & \xrightarrow{T} & \mathbf{Set} \\ \downarrow & & \downarrow \\ \mathbf{Rel} & \xrightarrow{T} & \mathbf{Rel} \end{array}$$

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$$\begin{array}{ccccc} X & \xrightarrow{e_X} & TX & \xleftarrow{Ta} & T^2X \\ 1_X \searrow & \leq & \downarrow a & \leq & \downarrow m_X \\ & & X & \xleftarrow{a} & TX \end{array}$$

lax alg.

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ a \downarrow & \leq_f & \downarrow b \\ X & \xrightarrow{f} & Y \end{array}$$

lax morph.

(U, e, m) ultrafilter monad

$$\begin{array}{ccc} \mathbf{Set} & \xrightarrow{U} & \mathbf{Set} \\ \downarrow & & \downarrow \\ \mathbf{Rel} & \xrightarrow{U} & \mathbf{Rel} \end{array}$$

$$\begin{array}{ccccc} X & \xrightarrow{e_X} & UX & \xleftarrow{\quad Ua \quad} & U^2X \\ 1_X \searrow & \leq & \downarrow a & \leq & \downarrow m_X \\ & & X & \xleftarrow{\quad a \quad} & UX \end{array}$$

top. space

$$\begin{array}{ccc} UX & \xrightarrow{Uf} & UY \\ a \downarrow & \leq_f & \downarrow b \\ X & \xrightarrow{f} & Y \end{array}$$

cont. map

(Id, 1, 1) *identity monad*

$$\begin{array}{ccc} \mathbf{Set} & \xrightarrow{\text{Id}} & \mathbf{Set} \\ \downarrow & & \downarrow \\ \mathbf{Rel} & \xrightarrow{\text{Id}} & \mathbf{Rel} \end{array}$$

$$\begin{array}{ccccc} X & \xrightarrow{1_X} & X & \xleftarrow{a} & X \\ \searrow 1_X & \swarrow \leq & \downarrow a & \swarrow \leq & \downarrow 1_X \\ & X & \xleftarrow{a} & X & \end{array}$$

preord. set

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow a & \swarrow \leq & \downarrow b \\ X & \xrightarrow{f} & Y \end{array}$$

monotone map

(T, e, m) (lax-)extended to \mathbf{Rel}

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lax alg.

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ a \downarrow & \leq_f & \downarrow b \\ X & \xrightarrow{f} & Y \end{array}$$

lax morph.

(T, e, m) (lax-)extended to \mathbf{Y}

$$\begin{array}{ccc} \mathbf{Set} & \xrightarrow{T} & \mathbf{Set} \\ \downarrow & & \downarrow \\ \mathbf{Y} & \xrightarrow{\textcolor{blue}{T}} & \mathbf{Y} \end{array}$$

$$\begin{array}{ccccc} X & \xrightarrow{e_X} & TX & \xleftarrow{\textcolor{blue}{Ta}} & T^2X \\ 1_X \searrow & \Rightarrow & \downarrow a & \Rightarrow & \downarrow m_X \\ & & X & \xleftarrow{\textcolor{blue}{a}} & TX \end{array}$$

lax alg.

$$\begin{array}{ccc} TX & \xrightarrow{\textcolor{blue}{Tf}} & TY \\ a \downarrow & \Rightarrow & \downarrow b \\ X & \xrightarrow{\textcolor{blue}{f}} & Y \end{array}$$

lax morph.

The setting

(T, e, m) monad in \mathbf{Set} such that:

$$\begin{array}{ccc} \mathbf{Set} & \xrightarrow{T} & \mathbf{Set} \\ \downarrow & & \downarrow \\ \mathbf{Mat}(\mathbf{V}) & \xrightarrow{T} & \mathbf{Mat}(\mathbf{V}) \end{array}$$

$$\begin{array}{ccc} X & \xrightarrow{e_X} & TX \\ r \downarrow & \Rightarrow & \downarrow Tr \\ Y & \xrightarrow{e_Y} & TY \end{array} \quad \begin{array}{ccc} T^2X & \xrightarrow{m_X} & TX \\ T^2r \downarrow & \Rightarrow & \downarrow Tr \\ T^2Y & \xrightarrow{m_Y} & TY \end{array}$$

lax functor, i.e.

$$TrTs \rightarrow T(rs)$$

e, m **op-lax**

natural transformations

The bicategory $\text{Mat}(\mathbf{V})$

(\mathbf{V} (co)complete symmetric monoidal-closed category)

- objects are **sets** X, Y, \dots
- 1-cells $r : X \rightarrow Y$ are **functors** $r : X \times Y \rightarrow \mathbf{V}$

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- 2-cells $\varphi : r \rightarrow s$ are **natural transformations**
- composition of 1-cells is **matrix multiplication**:

$$(sr)(x, z) = \sum_{y \in Y} r(x, y) \otimes s(y, z).$$

The bicategory $\text{Mat}(\mathbf{V})$

- There is a **pseudofunctor** $\text{Set} \rightarrow \text{Mat}(\mathbf{V})$ with $X \mapsto X$ and $f : X \rightarrow Y \mapsto f : X \times Y \rightarrow \mathbf{V}$,
 $f(x, y) = I$ if $f(x) = y$ and $f(x) = 0$ otherwise.

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transposition: $r^\circ : Y \nrightarrow X$ is defined by
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transposition: $r^\circ : Y \nrightarrow X$ is defined by
 $r^\circ(y, x) := r(x, y)$.
- Set-maps $f : X \rightarrow Y$ are left adjoints to f° ; that is,
 $1_X \rightarrow f^\circ f$ and $ff^\circ \rightarrow 1_Y$.

The category of lax algebras

objects

$$\begin{array}{ccc} TX & & \\ \downarrow a & & \\ X & & \end{array}$$

morphisms

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ \downarrow a & \Rightarrow & \downarrow b \\ X & \xrightarrow{f} & Y \end{array}$$

$\mathbf{Alg}(T, \mathbf{V})$

The category of reflexive lax algebras

objects

$$\begin{array}{ccc} X & \xrightarrow{e_X} & TX \\ & \searrow 1_X \Rightarrow^{\eta} & \downarrow a \\ & & X \end{array}$$

morphisms

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ a \downarrow & \Rightarrow^{\varphi} & \downarrow b \\ X & \xrightarrow{f} & Y \end{array}$$

(φ “preserves” η)

$$\text{Alg}(T, e; \mathbf{V}) \hookrightarrow \text{Alg}(T, \mathbf{V})$$

The cat. of reflexive and transitive lax algebras

objects	morphisms
$ \begin{array}{ccccc} X & \xrightarrow{e_X} & TX & \xleftarrow{Ta} & T^2X \\ & \searrow \begin{smallmatrix} \eta \\ \Rightarrow \\ 1_X \end{smallmatrix} & \downarrow a & \Rightarrow & \downarrow m_X \\ & & X & \xleftarrow{\quad a \quad} & TX \end{array} $	$ \begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ \downarrow a & \Rightarrow & \downarrow b \\ X & \xrightarrow{\quad f \quad} & Y \end{array} $

$((\eta, \mu)$ internal monoid; φ “preserves” η and μ)

$$\mathbf{Alg}(T, e, m; \mathbf{V}) \hookrightarrow \mathbf{Alg}(T, e; \mathbf{V}) \hookrightarrow \mathbf{Alg}(T, \mathbf{V})$$

Examples

$\mathbf{V} \setminus \mathbf{T}$	$\text{Id}_{\mathbf{Set}}$	M	U	T
2	ordered set	multi-ordered set	topological sp	" T -space"
$[0, \infty]$	metric sp	multi-metric sp	approach sp	"fuzzy T -sp"
\mathbf{Set}	category	multicategory	ultracategory	" T -category"
\mathbf{V}	\mathbf{V} -cat	\mathbf{V} -multicat	\mathbf{V} -ultracat	" (T, \mathbf{V}) -cat"

Exponentiation in cats of reflexive lax algebras

\mathbf{V} (co)complete symmetric monoidal-closed category

$(T, e) : \mathbf{Set} \rightarrow \mathbf{Set}$ pointed endofunctor

lax-extended to $\text{Mat}(\mathbf{V})$

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Theorem.

Let $T : \mathbf{Set} \rightarrow \mathbf{Set}$ preserve pullbacks,

or have (BCP), in case \mathbf{V} is a lattice.

If \mathbf{V} is locally cartesian closed, so is $\text{Alg}(T, e; \mathbf{V})$.

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Theorem.

Let $T : \mathbf{Set} \rightarrow \mathbf{Set}$ preserve pullbacks,

or have (BCP), in case \mathbf{V} is a lattice.

If \mathbf{V} is cartesian closed and I is a terminal object of \mathbf{V} , then $\text{Alg}(T, e; \mathbf{V})$ is a quasi-topos.