Multiple groupoids as a non commutative tool for higher dimensional local-to-global problems

Work done with Philip Higgins 1974-93, so quite old. Why talk on it?

1) Aim is to publicise some not well appreciated methods and tools of this linear theory. Hope to get reactions from workers in the areas of this conference (e.g. descent).

2) Just started a book project on

'Crossed complexes and homotopy groupoids'

giving a full account of this material, accessible as a novel account of basic algebraic topology and the cohomology of groups.

3) This year is the 20th anniversary of my first correspondence with Alexander Grothendieck informing him of this interest in n-categories. An extended correspondence led to Pursuing Stacks, whose influence you all know.

Major results: in the following diagram

$$\begin{tikzcd}
 & \text{FTop} \arrow[dl, "\varrho"'] \arrow[dr, "\Pi"] & \\
\begin{array}{c}\text{cubical}\\ \omega\text{-groupoids}\\ +\ \text{connections}\end{array} \arrow[rr, shift left, "\gamma"] & & \begin{array}{c}\text{crossed}\\ \text{complexes}\end{array} \arrow[ll, shift left, "\lambda"]
\end{tikzcd}$$

- $\lambda,\ \gamma$ are inverse adjoint equivalences,
- of monoidal closed categories,
- $\varrho,\ \Pi$ are homotopical functors,
- which preserve certain colimits
- and certain tensor products
- $\gamma\varrho \simeq \Pi$

Filtered Space: $X_* : X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_\infty$
of subspaces of X_∞.

Crossed complex

$$\begin{tikzcd}
\cdots \arrow[r] & C_n \arrow[r, "\delta_n"] \arrow[d, "t"] & C_{n-1} \arrow[r] \arrow[d, "t"] & \cdots \arrow[r] & C_2 \arrow[r, "\delta_2"] \arrow[d, "t"] & C_1 \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\
 & C_0 & C_0 & & C_0 & C_0
\end{tikzcd}$$

e.g. $C_1 = \pi_1(X_1, X_0)$, $C_n = \{\pi_n(X_n, X_{n-1}, x) \mid x \in X_0\}$

(In this talk, I am sticking to the linear theory. Not the most general, but it has certain conveniences!)

Why crossed complexes?

• Generalise groupoids and crossed modules.

• Good for modelling CW-complexes,

• Free crossed resolutions enable calculations with small CW models of $K(G,1)$s and their maps (Whitehead, Wall, Baues)

• Linear model of homotopy types (including all 2-types)

• Convenient for calculation, and the functor Π is classical, involving relative homotopy groups.

• Close to chain complexes with a group(oid) of operators, and related to some classical homological algebra (e.g. chains of syzygies)

• Monoidal structure suggestive of further developments (e.g. crossed differential algebras)

• Good homotopy theory (cylinder object, homotopy colimits)

Why cubical ω-groupoids with connections?

• They are equivalent to crossed complexes

• They have a clear monoidal closed structure

• Cubical (unlike globular or simplicial) methods allow for a simple algebraic inverse to subdivision

• Connections and the equivalence with crossed complexes allow for the sophisticated notion of commutative cube

• Can prove multiple compositions of commutative cubes are commutative

• Current resurgence of cubes, in combinatorics, concurrency, algebraic topology.

Main aim of the work: colimit theorems which give

non abelian tools for higher dimensional
local-to-global problems

giving a variety of new, even non abelian, calculations, which prove (i.e. test) the theory.

Relation with descent:

We suppose given an open cover $\mathcal{U} = \{U^\lambda\}_{\lambda\in\Lambda}$ of X.

This defines a map

$$q : E = \bigsqcup_{\lambda\in\Lambda} U^\lambda \to X$$

and so can form an augmented simplicial space

$$\check{C}(q) : \cdots E \times_X E \times_X E \overset{\textstyle\longrightarrow}{\underset{\textstyle\longrightarrow}{\longrightarrow}} E \times_X E \rightrightarrows E \xrightarrow{q} X$$

where the higher dimensional terms involve disjoint unions of multiple intersections of the U^λ.

We now suppose a filtered situation X_*, and so a corresponding $\check{C}(q_*)$.

Get a diagram as part of $\varrho(\check{\mathsf{C}}(q_*))$

$$\varrho(E_* \times_{X_*} E_*) \rightrightarrows \varrho(E_*) \xrightarrow{\varrho(q_*)} \varrho(X_*). \qquad (\mathbf{c}\varrho)$$

MAIN RESULT (GVKT):
Connectivity conditions imply this is a coequaliser diagram.

By facts stated earlier we get a coequaliser diagram

$$\Pi(E_* \times_{X_*} E_*) \rightrightarrows \Pi(E_*) \xrightarrow{\varrho(q_*)} \Pi(X_*). \qquad (\mathbf{c}\Pi)$$

and so get calculations of the familiar $\Pi(X_*)$.

So we need to understand the definition of
the fundamental cubical ω–groupoid $\varrho(X_*)$ of a filtered space X_*
as a generalisation of the fundamental groupoid on a set of base points.

I^n_*: the n-cube with its skeletal filtration.

Set $R_n(X_*) = \mathsf{FTop}(I^n_*, X_*)$.
This is a cubical set with compositions, connections, and inversions.

For $i = 1, \dots, n$ there are standard:

face maps $\partial^\pm_i : R_n X_* \to R_{n-1} X_*$;

degeneracy maps $\varepsilon_i : R_{n-1} X_* \to R_n X_*$

connections $\Gamma^\pm_i : R_{n-1} X_* \to R_n X_*$

compositions $a \circ_i b$ defined for $a, b \in R_n X_*$ such that $\partial^+_i a = \partial^-_i b$

inversions $-_i : R_n \to R_n$.

Connections are induced by $\gamma^\alpha_i : I^n \to I^{n-1}$ defined using monoid structures $\max, \min : I^2 \to I$. Essential for many reasons, e.g. to discuss the notion of commutative cube.

Obvious geometric properties.

$$p : R_n(X_*) \to \varrho_n(X_*) = (R_n(X_*)/ \equiv)$$

is the quotient map, where $f \equiv g \in R_n(X_*)$ means filter homotopic (i.e. through filtered maps) rel vertices of I^n

Facts (RB-PJH, JPAA 1981)

- The compositions on R are inherited by ϱ to give $\varrho(X_*)$ the structure of cubical multiple groupoid with connections.
- The map $p : R(X_*) \to \varrho(X_*)$ is a Kan fibration of cubical sets.

The second result is almost unbelievable. Its proof has to give a systematic method of deforming a cube with the right faces 'up to homotopy' into a cube with exactly the right faces, using the given homotopies.

Here is an application which is essential in many proofs.

Theorem: Lifting multiple compositions

Let $[\alpha_{(r)}]$ be a multiple composition in $\varrho_n(X_*)$. Then representatives $a_{(r)}$ of the $\alpha_{(r)}$ may be chosen so that the composition $a_{(r)}$ is well defined in $R_n(X_*)$.

Proof: The multiple composition $[\alpha_{(r)}]$ determines a cubical map

$$A : K \to \varrho(X_*)$$

where the cubical set K corresponds to a subdivision of the geometric cube.

Consider the diagram

$$\begin{tikzcd} * \arrow[r] \arrow[d] & R(X_*)\,. \arrow[d, "p"] \\ K \arrow[r, "A"'] \arrow[ur, dashed, "A'"] & \varrho(X_*) \end{tikzcd}$$

Then K collapses to $*$, written $K \searrow *$.
By the fibration result,
A lifts to A', which represents $a_{(r)}$, as required.

So we have to explain collapsing.

Collapsing Let $C \subseteq B \subseteq I^n$ be subcomplexes.

C is an **elementary collapse** of B, $B \overset{e}{\searrow} C$, if for some $s \geqslant 1$ there is an s-cell a of B and $(s-1)$-face b of a, the *free face*, such that

$$B = C \cup a, \qquad C \cap a = \dot{a} \setminus b$$

(where $\dot{a}$ denotes the union of the proper faces of a).

$B_1 \searrow B_r$, B_1 **collapses** to B_r, if there is a sequence

$$B_1 \overset{e}{\searrow} B_2 \overset{e}{\searrow} \cdots \overset{e}{\searrow} B_r$$

of elementary collapses.

If C is a subcomplex of B then

$$B \times I \searrow (B \times \{0\} \cup C \times I)$$

(this is proved by induction on dimension of $B \setminus C$).

I^n collapses to any one of its vertices (this may be proved by induction on n using the first example.)

Partial boxes

Let C be an r-cell in the n-cube I^n.

Two $(r-1)$-faces of C are called opposite if they do not meet.

A partial box in C is a subcomplex B of C generated by one $(r-1)$-face b of C (called a *base* of B) and a number, possibly zero, of other $(r-1)$-faces of C none of which is opposite to b.

The partial box is a box if its $(r-1)$-cells consist of all but one of the $(r-1)$-faces of C.

The proof of the fibration theorem uses a filter homotopy extension property and the following:

Key Proposition: Let B, B' be partial boxes in an r-cell C of I^n such that $B' \subseteq B$. Then there is a chain

$$B = B_s \searrow B_{s-1} \searrow \cdots \searrow B_1 = B'$$

such that

(i) each B_i is a partial box in C

(ii) $B_{i+1} = B_i \cup a_i$ where a_i is an $(r-1)$-cell of C not in B_i;

(iii) $a_i \cap B_i$ is a partial box in a_i.

Proof is quite neat, and follows the pictures.

Methods of collapsing generalise methods of trees in dimension 1.

The proof of the fibration theorem gives a program for carrying out the deformations needed to do the lifting. In some sense, it implies computing a multiple composition can be done using collapsing as the guide.

Another key concept is that of thin element $\alpha \in \varrho_n(X_*)$ for $n \geqslant 2$.

α is geometrically thin if it has a representative $a : I^n_* \to X_*$ such that $a(I^n) \subseteq X_{n-1}$.

α is algebraically thin if it is a multiple composition of degenerate elements or those coming from repeated negatives of connections.
Any composition of algebraically thin elements is thin.

Theorem (i) algebraically thin $\equiv$ geometrically thin.

(ii) In a cubical ω–groupoid with connections, any box has a unique thin filler.

Proof (i) $\Rightarrow$ This uses lifting of multiple compositions, in a stronger form than stated above.

$\Leftarrow$ and (ii) This uses the full algebraic relation between ω–groupoids and crossed complexes.

Back to diagram as part of $\varrho(\check{\mathsf{C}}(q_*))$

$$\begin{tikzcd} \varrho(E_* \times_{X_*} E_*) \arrow[r, shift left] \arrow[r, shift right] & \varrho(E_*) \arrow[r, "\varrho(q_*)"] \arrow[dr, "f"'] & \varrho(X_*) \arrow[d, dashed, "f'"] \\ & & G \end{tikzcd} \qquad (\mathrm{c}\varrho)$$

Proof of GVKT involves checking the universal property for morphisms $f : \varrho(E_*) \to G$ where G is a cubical ω-groupoid with connection.

To get the morphism $f' : \varrho(X_*) \to G$ you subdivide a representative $a = [a_{(r)}]$ of an element $\alpha \in \varrho(X_*)$ so that $a_{(r)}$ lies in an element $U^{(r)}$ of $\mathcal{U}$; use connectivity conditions to deform $a_{(r)}$ into

$$b_{(r)} \in R(U_*^{(r)})$$

and so obtain

$$\beta_{(r)} \in \varrho(U_*^{(r)}).$$

The elements

$$f\beta_{(r)} \in G$$

may be composed in G (by the conditions on f), to give an element

$$\theta(\alpha) = [f\beta_{(r)}] \in G.$$

So the proof of the universal property has to use algebraic inverse to subdivision.

To prove $\theta(\alpha)$ well defined uses crucially properties of thin elements. The key point:
a filter homotopy $h : \alpha \equiv \alpha'$ in $R_n(X_*)$ gives a deficient element of $R_{n+1}(X_*)$.

Do the subdivision and deformation argument on such a homotopy, push the little bits in some

$$\varrho_{n+1}(U_*^\lambda)$$

(now thin) over to G, combine them and get a thin element

$$\tau \in G_{n+1}$$

all of whose faces not involving the direction $(n+1)$ are thin
because h was given to be a filter homotopy.

An inductive argument on unique thin fillers of boxes then shows
τ is degenerate in direction $(n+1)$
so the two ends in direction $(n+1)$ are the same.

Conclusion and questions

• Mirroring the geometry by the algebra is crucial for conjecturing and proving universal properties.

• Thin elements are crucial as modelling commutative cubes, a concept not so easy to define or handle algebraically.

• Colimit theorems give, when they apply, exact information even in non commutative situations. The implications of this for homological algebra could be important.

• One construction inspired eventually by this work, the non abelian tensor product of groups, has a bibliography of 75 papers since it was defined with Loday in 1985.

• Globular methods do not fit easily into this scheme.

• For computations we really need strict structures (although we do want to compute invariants of homotopy colimits).

• In homotopy theory, identifications in low dimensions can affect high dimensional homotopy. So we need structure in a range of dimensions to model homotopical identifications algebraically.

• In this way we calculate say crossed modules modelling homotopy 2-types, whereas the corresponding k-invariant is often difficult to calculate.

• Use of crossed complexes in Čech theory: current project with Tim Porter.

• Question Applications in other contexts where the fundamental groupoid is currently used? Algebraic geometry?

• Question Uses in differential geometry? Is there a non abelian De Rham Theory, using an analogue of crossed complexes?

• Question Is there a truly non commutative integration theory based on limits of compositions of elements of multiple groupoids?