Borel and countably determined reducibility in nonstandard domain

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Blanket agreements:

A nonstandard universe is a structure of the form ${}^*S = \{{}^*S_n\}_{n \in \mathbb{N}}$, where

1) ${}^*S_0 = {}^*\mathbb{N}$, a nonstandard extension of $\mathbb{N}$;

2) ${}^*S_{n+1} \subseteq \mathcal{P}({}^*S_n)$;

3) The whole structure ${}^*S$ models the type theory;

4) ${}^*S$ is countably saturated.

Sets in ${}^*S$ are called *internal*.

In particular, a set $X \subseteq {}^*\mathbb{N}$ is internal iff it belongs to ${}^*S_1$. Other sets $X \subseteq {}^*\mathbb{N}$ are called *external*.

Some external sets

A set $X \subseteq {}^*\mathbb{N}$ is $\boxed{\text{Borel}}$ iff it belongs to the least sigma-algebra containing all internal sets.

$\Sigma^0_0 = \Pi^0_0 =$ all internal sets;

$\Sigma^0_\xi =$ countable unions of sets in $\bigcup_{\alpha<\xi} \Pi^0_\alpha$;

$\Pi^0_\xi =$ complements of sets in Σ^0_ξ.

Borel $= \bigcup_{\xi<\omega_1} \Sigma^0_\xi$ and the hierarchy theorem holds.

Sets of the form $X = \bigcup_{f\in F} \bigcap_m X_{f\restriction m}$, where

1) $F \subseteq \mathbb{N}^{\mathbb{N}}$;

2) every set X_u, $u \in \mathbb{N}^{<\omega}$, is internal;

are called $\boxed{\text{countably determined}}$ (in brief, CD).

Then:

$$\boxed{\text{internal} \subsetneqq \text{Borel} \subsetneqq \text{countably determined}}$$

Warmup: Borel cardinals

Let $X, Y \subseteq {}^*\mathbb{N}$. Define:

$\boxed{X \leq_{\mathtt{B}} Y}$ iff there is a Borel injection $\vartheta : X \to Y$.

$\boxed{X \equiv_{\mathtt{B}} Y}$ iff $X \leq_{\mathtt{B}} Y$ and $Y \leq_{\mathtt{B}} X$
iff there is a Borel bijection $\vartheta : X$ onto Y.

Borel cardinal : a $\equiv_{\mathtt{B}}$-class of a Borel subset of ${}^*\mathbb{N}$.

$X \leq_{\mathtt{CD}} Y$, $X \equiv_{\mathtt{CD}} Y$, *CD cardinal* — similarly.

Theorem (Kalina – Zlatos, 1989: AST).
The structure of Borel cardinals of Borel subsets of ${}^*\mathbb{N}$ under $\leq_{\mathtt{B}}$ is similar to the structure of Borel cuts in ${}^*\mathbb{N}$ modulo the relation:
$U \approx V$ *iff* $\forall x \in U \, \exists y \in V \left(\frac{x}{y} \simeq 1\right)$ *and vice versa (under* $\subseteq$*).*

Theorem (Vopenka – Cuda, 1980: AST).
At least under $\mathbf{CH}$*, there exist* $\leq_{\mathtt{CD}}$*-incomparable countably determined subsets of* ${}^*\mathbb{N}$*.*

It follows that Borel cardinals are linearly ordered by $\leq_{\mathtt{B}}$ while CD cardinals are, perhaps, not linearly ordered by $\leq_{\mathtt{CD}}$.

Reducibility of ERs in nonst. universe

ER = *equivalence relation*

Let E, F be ERs on Borel sets $X, Y \subseteq {}^*\mathbb{N}$.

$\boxed{\mathsf{E} \le_{\mathtt{B}} \mathsf{F}}$ iff there is a Borel map $\vartheta : X \to Y$ s. t. $x \mathrel{\mathsf{E}} x' \Longleftrightarrow \vartheta(x) \mathrel{\mathsf{F}} \vartheta(x')$: Borel reducibility.

Meaning: $\boxed{X/\mathsf{E} \text{ has } \le_{\mathtt{B}} \text{ classes than } Y/\mathsf{F}}$

$\boxed{\mathsf{E} \equiv_{\mathtt{B}} \mathsf{F}}$ iff $\mathsf{E} \le_{\mathtt{B}} \mathsf{F}$ and $\mathsf{F} \le_{\mathtt{B}} \mathsf{E}$

$\boxed{\mathsf{E} <_{\mathtt{B}} \mathsf{F}}$ iff $\mathsf{E} \le_{\mathtt{B}} \mathsf{F}$ but not $\mathsf{F} \le_{\mathtt{B}} \mathsf{E}$

Example: For any X, $\boxed{\mathsf{D}(X) \text{ is the equality on } X}$. Then, for Borel X, Y: $\mathsf{D}(X) \le_{\mathtt{B}} \mathsf{D}(Y)$ iff $X \le_{\mathtt{B}} Y$ in the sense above.

$\boxed{\mathsf{E} \le_{\mathtt{CD}} \mathsf{F},\ \ \mathsf{E} \equiv_{\mathtt{CD}} \mathsf{F},\ \ \mathsf{E} <_{\mathtt{CD}} \mathsf{F}}$ (CD map ϑ)

$\boxed{\mathsf{E} \le_{\mathtt{int}} \mathsf{F},\ \ \mathsf{E} \equiv_{\mathtt{int}} \mathsf{F},\ \ \mathsf{E} <_{\mathtt{int}} \mathsf{F}}$ (internal ϑ)

Program: Study the structure of Borel and CD ERs under all these relations. (Inspired by studies in classical descriptive set theory.)

Smooth and “countable” ERs

ER E is:

“countable” if its equivalence classes are at most countable.

B-smooth if $\mathsf{E} \le_{\mathtt{B}} \mathsf{D}({}^*\mathbb{N})$, *i. e.*, its equiv. classes can be Borel-enumerated by elements of ${}^*\mathbb{N}$.

CD-smooth if $\mathsf{E} \le_{\mathtt{CD}} \mathsf{D}({}^*\mathbb{N})$

A *transversal* is a set having exactly one element in every equivalence class

Theorem . *Any “countable” countably determined equivalence relation* E *on* ${}^*\mathbb{N}$ *admits a CD transversal, hence, is CD-smooth.*

(Jin *JSL* 2001 for the ER: $x \, \mathsf{M}_{\mathbb{N}} \, y$ iff $|x - y| \in \mathbb{N}$.)

Remark: $\mathsf{M}_{\mathbb{N}}$ is a “countable” ER, hence, CD-smooth, but is <u>not</u> B-smooth, therefore, the Borel reducibility is really stronger than the CD one.

Silver − Burgess dichotomy

Infinite internal sets are considered as **large**, in principle, bigger than any fixed external cardinality under a suitable saturation assumption.

It is known (Henson, Cuda–Vopenka) that a CD set $X \subseteq {}^*\mathbb{N}$ either is at most countable (*i. e.*, rather small) or contains an infinite internal subset (*i. e.*, rather large). What about quotients ?

The next theorem resembles theorems of Silver and Burgess in classical descriptive set theory.

Theorem . *Let* E *be a CD equivalence relation on* ${}^*\mathbb{N}$. *Then exactly one of* (I), (II) *holds*:

(I) *there is a number* $h \in {}^*\mathbb{N} \setminus \mathbb{N}$ *and an internal map* $f : {}^*\mathbb{N} \to 2^h$ *s. t.* $f(x){\restriction}\mathbb{N} = f(y){\restriction}\mathbb{N} \Longrightarrow x \,\mathsf{E}\, y$;

(Here 2^h = all internal $f : [0, h) \to 2$.)

(II) *there is an infinite internal pairwise* E*-inequivalent set* $Y \subseteq {}^*\mathbb{N}$.

In Case (I), E has at most $\mathfrak{c}$ equivalence classes, but if E is *Borel* then (by an additional argument) it has either exactly $\mathfrak{c}$ or $\leq \aleph_0$ classes .

Generalization

In the following geleralization of Theorem the distinction between two cases is fotmulated in terms of a given additive *cut* (initial segment) $U \subseteq {}^*\mathbb{N}$.

Theorem . *Let* E *be a CD equivalence relation on* ${}^*\mathbb{N}$, *and* $U \subseteq {}^*\mathbb{N}$ *a countably cofinal additive cut. Then* *<u>at least one</u>* *of* (I), (II) *holds*:

(I) *there is a number* $h \in {}^*\mathbb{N} \setminus U$ *and an internal map* $\vartheta : {}^*\mathbb{N} \to 2^h$ *s. t.* $\vartheta(x){\restriction}U = \vartheta(y){\restriction}U \Longrightarrow x \,\mathsf{E}\, y$;

(II) *there is an internal pairwise* E*-inequivalent set* $Y \subseteq {}^*\mathbb{N}$ *with* $\#Y \notin U$.

Moreover, if (II) *holds and* U *satisfies*

$$x \in U \Longrightarrow 2^x \in U \quad \text{(exponential cut)}$$

then (I) *fails even for CD maps* ϑ.

Monadic ERs

Monadic ERs (Keisler, Leth) are those of the form

$$x\,\mathsf{M}_U\,y \quad \text{iff} \quad |x-y| \in U \qquad (x, y \in {}^*\mathbb{N})$$

where $U \subseteq {}^*\mathbb{N}$ is an additive cut (*i. e.*, an initial segment closed under $+$).

An example: $x\,\mathsf{M}_{\mathbb{N}}\,y$ iff $|x-y|$ is finite.

Examples of cuts. 1) If $c \in {}^*\mathbb{N}$ then $c\mathbb{N} = \bigcup_n [0, cn)$ is a *countably cofinal* additive cut.

2) If $c \in {}^*\mathbb{N} \setminus \mathbb{N}$ then $c/\mathbb{N} = \bigcap_n [0, \frac{c}{n})$ is a *countably coinitial* additive cut.

Accordingly, $\mathsf{M}_{c\mathbb{N}}$ and $\mathsf{M}_{c/\mathbb{N}}$ are monadic ERs.

Remark: If $\emptyset \neq U \subsetneqq {}^*\mathbb{N}$ is a Borel (even *countably determined*) cut then U is either countably cofinal or countably coinitial.

Accordingly, M_U can be called countably cofinal or countably coinitial monadic ER.

Countably cofinal monadic ERs

Let $U = \bigcup_n [0, a_n)$ be a cut, $\{a_n\}$ an increasing ω-sequence in ${}^*\mathbb{N}$.

Define the *rate of growth* of $\{a_n\}$:

$$\mathtt{rate}\,\{a_n\} = \inf_{n\in\mathbb{N}} \sup_{n'>n} \frac{a_{n'}}{a_n} \qquad \text{(a cut in } {}^*\mathbb{N}\text{).}$$

Put $\boxed{\mathtt{rate}\,U = \mathtt{rate}\,\{a_n\}}$ for any increasing sequence cofinal in U. (Independent of the choice of $\{a_n\}$.)

Fact: Countably cofinal additive cuts are densely ordered by $\mathtt{rate}\,U \subseteq \mathtt{rate}\,V$;
cuts $\mathbb{N}$ and $c\mathbb{N}$ are the smallest ($\mathtt{rate}\,c\mathbb{N} = \emptyset$).

Theorem . (i) *Suppose that $U, V \subseteq {}^*\mathbb{N}$ are countably cofinal additive cuts. Then*

$$\mathsf{M}_U \le_{\mathtt{B}} \mathsf{M}_V \quad \text{iff} \quad \mathsf{M}_U \le_{\mathtt{CD}} \mathsf{M}_V \quad \text{iff} \quad \mathtt{rate}\,U \subseteq \mathtt{rate}\,V.$$

(ii) *All those ERs are not smooth, that is, not Borel-reducible to the equality on ${}^*\mathbb{N}$.*

Countably coinitial monadic ERs

Let $U = \bigcap_n [0, a_n)$ be a ctbly coinitial cut, $\{a_n\}$ a **de**creasing ω-sequence in ${}^*\mathbb{N}$.

Define the *rate of decrease* of $\{a_n\}$:

$$\mathtt{rate}\,\{a_n\} = \inf_{n\in\mathbb{N}} \sup_{n'>n} \frac{a_n}{a_{n'}} \qquad \text{(a cut in } {}^*\mathbb{N}\text{)}.$$

Put $\boxed{\mathtt{rate}\,U = \mathtt{rate}\,\{a_n\}}$ for any decreasing sequence coinitial in U. (Independent of the choice of $\{a_n\}$.)

Fact: Countably coinitial additive cuts are densely ordered by $\mathtt{rate}\,U \subseteq \mathtt{rate}\,V$;
cuts $c/\mathbb{N}$ are the smallest ($\mathtt{rate}\,c\mathbb{N} = \emptyset$).

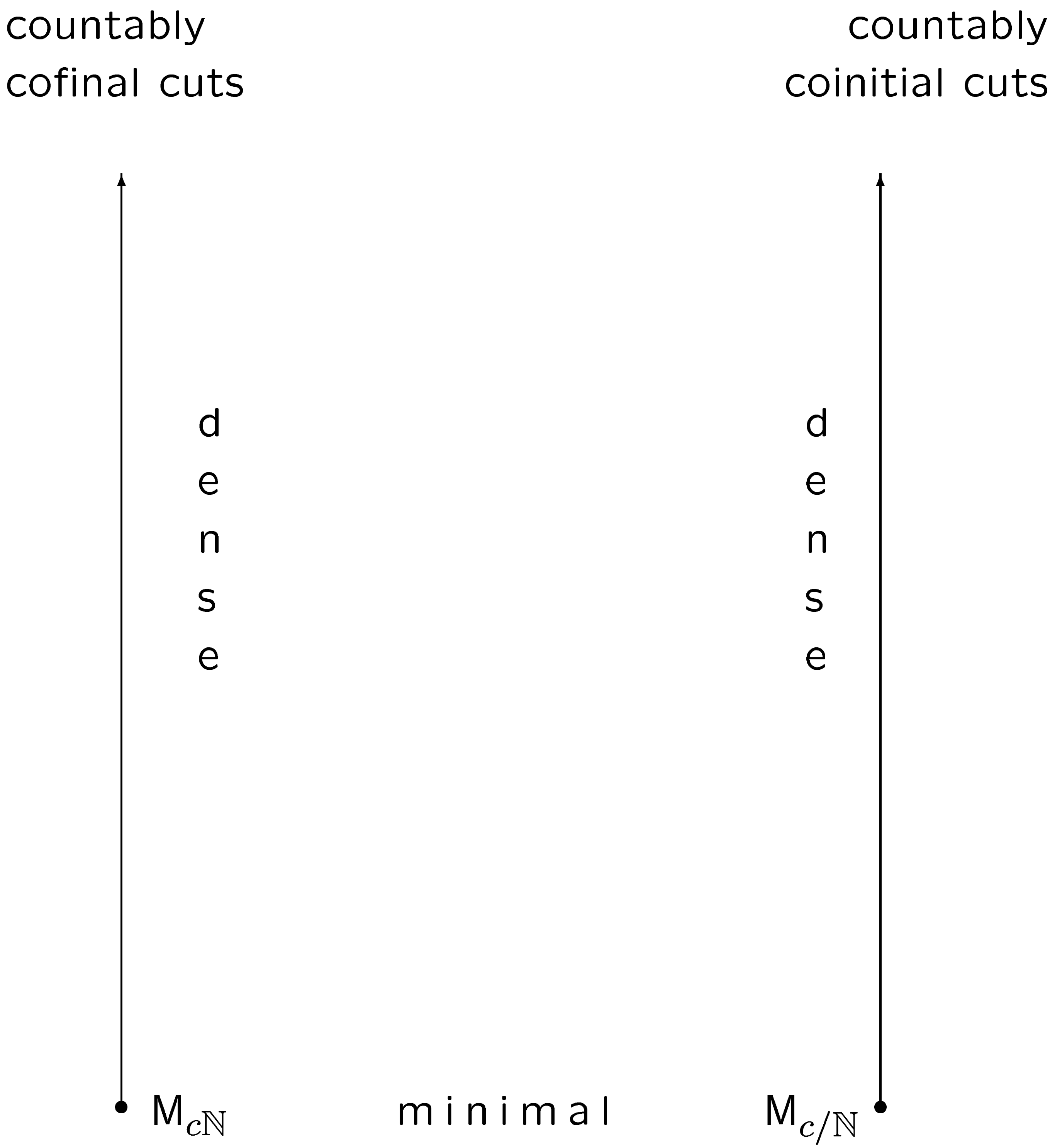
Theorem . (i) *Suppose that $U, V \subseteq {}^*\mathbb{N}$ are countably coinitial additive cuts. Then*

$\mathsf{M}_U \le_{\mathtt{B}} \mathsf{M}_V$ iff $\mathsf{M}_U \le_{\mathtt{CD}} \mathsf{M}_V$ iff $\mathtt{rate}\,U \subseteq \mathtt{rate}\,V$.

(ii) *Countably coinitial ERs are not not smooth.*

(iii) *Countably coinitial monadic ERs are $\le_{\mathtt{B}}$-incomparable with ctbly cofinal monadic ERs.*

Monadic ERs under Borel reducibility



Conjecture: the orders are countably saturated and similar to each other

Upper bound for countably cofinal ERs

Let ${}^*\mathbb{S}$ be the (internal, $*$-countable) set of all internal maps $\xi : {}^*\mathbb{N} \to 2$ such that $\xi(x) = 1$ for all but hyperfinitely many $x \in {}^*\mathbb{N}$.

For ξ, $\eta \in {}^*\mathbb{S}$ define:

$\xi \,\mathsf{FD}\, \eta$ iff $\xi(x) = \eta(x)$ for all but finitely many $x \in {}^*\mathbb{N}$.

(FD from "finite difference".)

Theorem . (i) *If U is a countably cofinal additive cut then* $\mathsf{M}_U <_{\mathtt{B}} \mathsf{FD}$.

(ii) *If $V \subseteq {}^*\mathbb{N}$ is an additive countably coinitial cut then* $\mathsf{M}_V \not\leq_{\mathtt{CD}} \mathsf{FD}$.