

Strong Logics

A strong logic, \vdash_0 , is defined by:

(1) Specifying a collection of *test* structures, these are structures of the form

$$\mathcal{M} = (M, E)$$

where $E \subseteq M \times M$;

(2) Defining

$$ZFC \vdash_0 \phi$$

if for every test structure, \mathcal{M} , if

$$\mathcal{M} \models \mathrm{ZFC}$$

then $\mathcal{M} \models \phi$.

Of course we shall only be interested in the case that there actually exists a test structure, \mathcal{M} , such that

$$\mathcal{M} \models ZFC.$$

- The *smaller* the collection of test structures, the *stronger* the logic.
 - Classical logic is the weakest logic.

Example: β -logic is obtained by simply restricting to *transitive sets*,

$$\mathcal{M} = (M, \in).$$

• The strongest (interesting) logic is when there is only one test structure, V, the universe of sets.

Requirement for a strong logic, \vdash_0 :

• Generic Soundness: Suppose that \mathbb{P} is a partial order, α is an ordinal and that

$$V_{\alpha}^{\mathbb{P}} \models \mathrm{ZFC}.$$

Suppose that

$$ZFC \vdash_0 \phi$$
.

Then

$$V_{\alpha}^{\mathbb{P}} \models \phi.$$

Our context for considering strong logics will require at the very least that there exists a proper class of Woodin cardinals, and so the requirement of *Generic Soundness* is nontrivial. We shall further restrict, in the final analysis, to strong logics that are both

- definable and
- generically invariant.

Thus we shall be considering logics (equivalently, defining notions of mathematical truth) which are completely immune to the effects of forcing.

We begin by defining a specific strong logic

"
$$\Omega$$
-logic".

The definition involves a *transfinite* hierarchy which extends the hierarchy of the projective sets; this is the hierarchy of the *universally Baire sets*.

Definition 1 (Feng-Magidor-Woodin)

A set $A \subseteq \mathbb{R}^n$ is *universally Baire* if for any continuous function,

$$F:\Omega\to\mathbb{R}^n$$
,

where Ω is a compact Hausdorff space, the preimage of A,

$$\left\{ p \in X \,\middle|\, F(p) \in A \right\},$$

has the property of Baire in Ω ; i. e. is open in Ω modulo a meager set.

- Every borel set $A \subseteq \mathbb{R}^n$ is universally Baire.
- The universally Baire sets form a σ -algebra closed under preimages by borel functions

$$f: \mathbb{R}^n \to \mathbb{R}^m$$
.

• The universally Baire sets are Lebesgue measurable etc.

Assuming there is a proper class of Woodin cardinals:

- Every universally Baire set is determined,
 - corollary of the Martin-Steel Theorem;
- The universally Baire sets form a (pre)wellordered hierarchy under Wadge equivalence.
- If $A \subseteq \mathbb{R}$ is universally Baire then every set in

$$L(A,\mathbb{R})\cap\mathcal{P}(\mathbb{R})$$

is universally Baire.

If A and B are universally Baire subsets of P where $P \subset \mathbb{R}$ is compact, perfect, and nowhere dense, then the Wadge order is quite easily defined:

 $A <_{\operatorname{w}} B$ if both A and $P \backslash A$ are preimages of B by functions

$$f: P \to P$$

which satisfy $|f(x) - f(y)| \le |x - y|/2$ for all $x, y \in P$.

Even restricted to the Borel subsets of *P* this order is quite fine.

There is a natural generalization of classical first order logic which is defined from the universally Baire sets. This is Ω -logic;

• "proofs" in Ω -logic are witnessed by universally Baire sets.

 Ω -logic is the natural limit of a hierarchy of logics which begins with first order logic and continues with β -logic etc.

A-closed sets

Suppose that

$$A \subseteq \mathbb{R}$$

is universally Baire and $A \neq \emptyset$

Suppose that V[G] is a set generic extension of V. Then the set A has canonical interpretation as a set

$$A_G \subseteq \mathbb{R}^{V[G]}$$
.

The set A_G is defined as

$$A_G = \bigcup \left\{ \operatorname{ran}(\pi_G) \,\middle|\, \pi : \lambda^\omega \to \mathbb{R}, \pi \in V, \operatorname{ran}(\pi) = A \right\}.$$

In this definition of A_G , π ranges over functions, $\pi: \lambda^{\omega} \to \mathbb{R}$, such that for all $x, y \in \lambda^{\omega}$ with $x \neq y$,

$$|\pi(x) - \pi(y)| \le 1/(n+1)$$

where n is least such that $x(n) \neq y(n)$, and π_G is the function

$$\pi_G: (\lambda^\omega)^{V[G]} \to \mathbb{R}^{V[G]}$$

that π naturally defines in V[G].

It follows that in V[G], the set A_G is universally Baire and if there exists a proper class of Woodin cardinals then

$$\langle H(\omega_1), A \rangle \prec \langle H(\omega_1)^{V[G]}, A_G \rangle.$$

Definition 2 Suppose that $A \subseteq \mathbb{R}$ is universally Baire and that M is a transitive set such that

$$M \models \mathrm{ZFC}.$$

Then M is A-closed if for each partial order

$$\mathbb{P} \in M$$
,

if $G \subseteq \mathbb{P}$ is V-generic then in V[G]:

$$A_G \cap M[G] \in M[G].$$

The definition that M is A-closed actually makes sense if M is simply an ω -model.

Lemma 3 Suppose that (M, E) is an ω -model with

$$(M, E) \models ZFC.$$

Then the following are equivalent.

- (1) (M, E) is wellfounded.
- (2) (M, E) is A-closed for each Π^1_1 set.

So:

• A-closure is a natural generalization of wellfoundedness.

 Ω -logic

Definition 4 Suppose that:

- (i) There exists a proper class of Woodin cardinals.
- (ii) ϕ is a sentence.

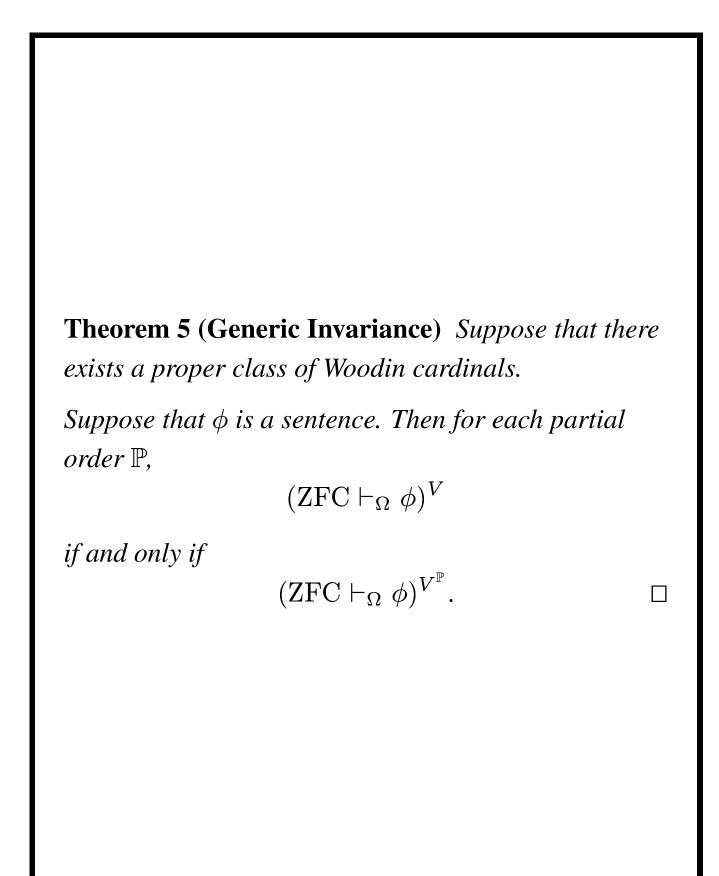
Then

$$ZFC \vdash_{\Omega} \phi$$

if there exists a universally Baire set $A \subseteq \mathbb{R}$ such that if M is any countable transitive set satisfying

- 1. $M \models ZFC$,
- 2. M is A-closed,

then $M \models \phi$.



Theorem 6 (Generic Soundness) Suppose that there exists a proper class of Woodin cardinals.

Suppose that

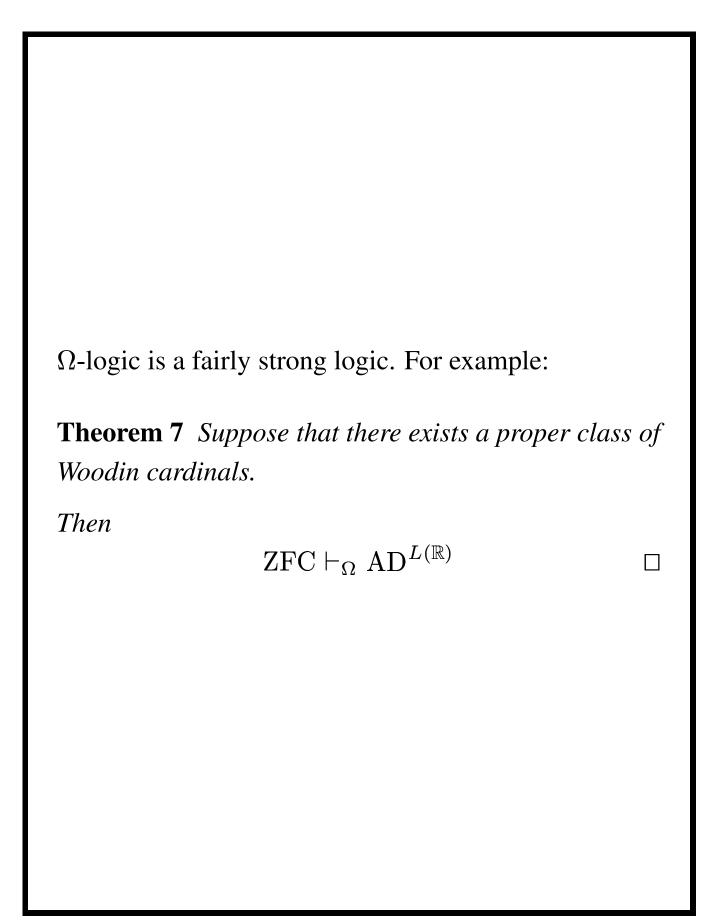
$$V_{\alpha}^{\mathbb{P}} \models \mathrm{ZFC}$$

and that

$$ZFC \vdash_{\Omega} \phi$$
.

Then

$$V_{lpha}^{\mathbb{P}} \models \phi.$$



 Ω^* -logic

Definition 8 (Ω^* -logic) Suppose that:

- (i) There exists a proper class of Woodin cardinals.
- (ii) ϕ is a sentence.

Then

$$ZFC \vdash_{\Omega^*} \phi$$

if for all ordinals α and for all partial orders $\mathbb P$ if

$$V_{\alpha}^{\mathbb{P}} \models \mathrm{ZFC},$$

then $V_{\alpha}^{\mathbb{P}} \models \phi$.

Generic Soundness is immediate for Ω^* -logic.

• Ω^* -logic is the strongest possible logic satisfying this requirement.

The property of generic invariance also holds for Ω^* -logic.

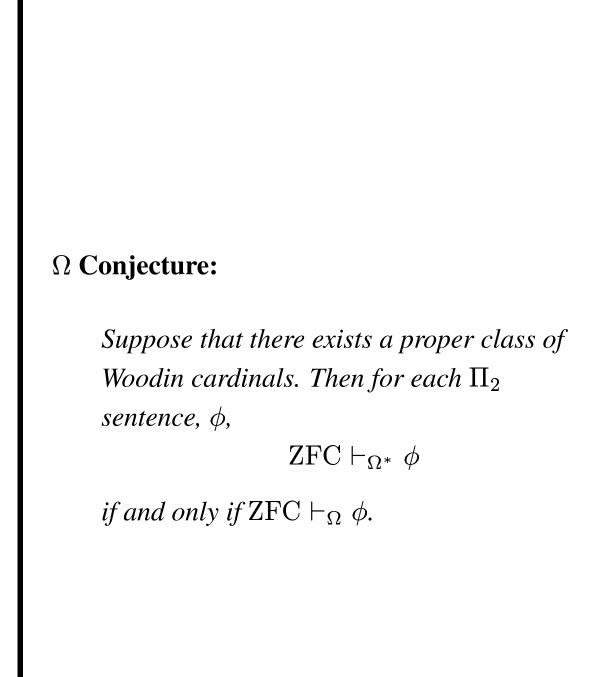
Theorem 9 (Generic Invariance) Suppose that there exists a proper class of Woodin cardinals.

Suppose that ϕ is a sentence. Then for each partial order \mathbb{P} ,

$$(\operatorname{ZFC} \vdash_{\Omega^*} \phi)^V$$

if and only if

$$(\operatorname{ZFC} \vdash_{\Omega^*} \phi)^{V^{\mathbb{P}}}.$$



We define two generalizations of the notion that a set $A \subset \mathbb{R}$ be *recursive*.

Definition 10 Suppose that there exists a proper class of Woodin cardinals. A set $A \subseteq \mathbb{R}$ is Ω -recursive if there exists a formula $\phi(x)$ such that:

1.
$$A = \{r \mid \text{ZFC} \vdash_{\Omega} \phi[r]\};$$

2. For all partial orders, \mathbb{P} , if $G \subset \mathbb{P}$ is V-generic then for each $r \in \mathbb{R}^{V[G]}$, either

$$V[G] \models \mathrm{ZFC} \vdash_{\Omega} \phi[r],$$

or
$$V[G] \models \mathrm{ZFC} \vdash_{\Omega} (\neg \phi)[r]$$
.

Lemma 11 Suppose that there exists a proper class of Woodin cardinals and that $A \subseteq \mathbb{R}$. Then the following are equivalent:

- 1. A is Ω -recursive
- 2. There exists a universally Baire set $B \subseteq \mathbb{R}$ such that the set A is Δ_1 definable in $L(B,\mathbb{R})$ from the parameter $\{\mathbb{R}\}$.

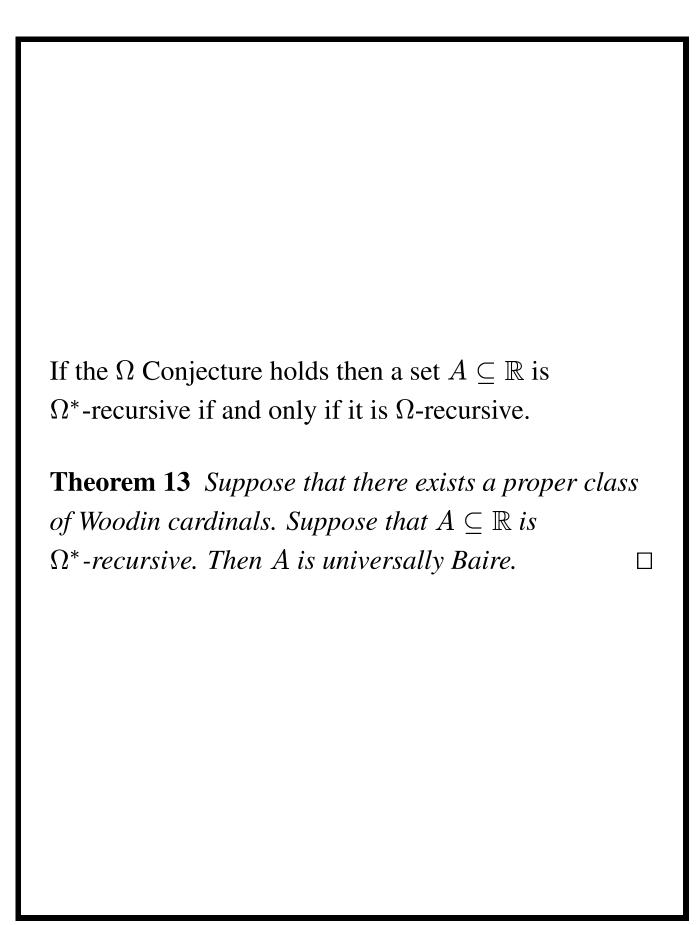
Definition 12 Suppose that there exists a proper class of Woodin cardinals. A set $A \subseteq \mathbb{R}$ is Ω^* -recursive if there exists a formula $\phi(x)$ such that:

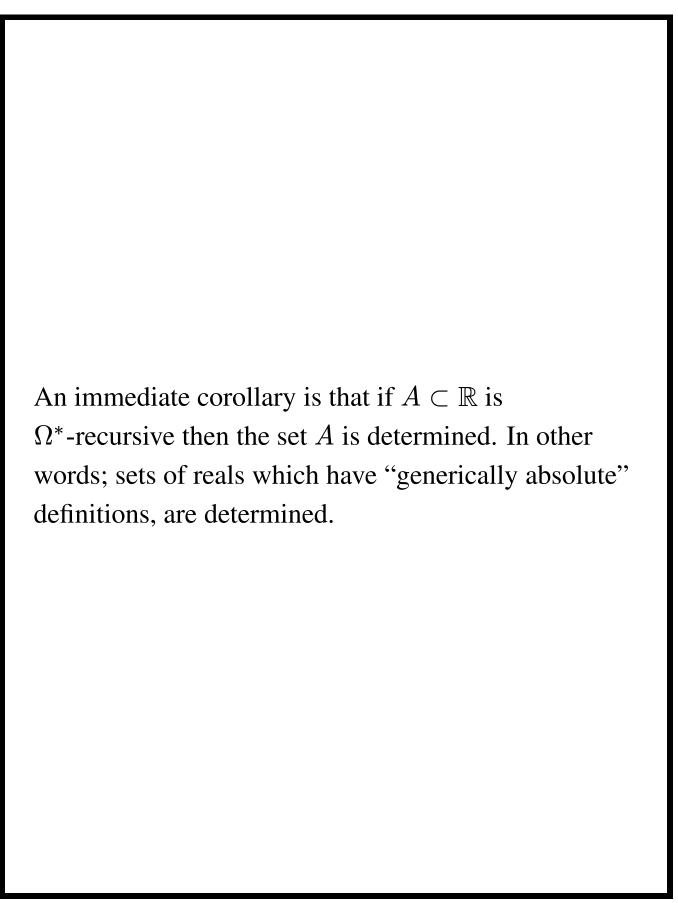
1.
$$A = \left\{ r \, \big| \, \text{ZFC} \vdash_{\Omega^*} \phi[r] \right\};$$

2. For all partial orders, \mathbb{P} , if $G \subset \mathbb{P}$ is V-generic then for each $r \in \mathbb{R}^{V[G]}$, either

$$V[G] \models \mathrm{ZFC} \vdash_{\Omega^*} \phi[r],$$

or
$$V[G] \models \mathrm{ZFC} \vdash_{\Omega^*} (\neg \phi)[r].$$





The question of whether there can exist analogs of determinacy for the structure

$$\langle H(\omega_2), \in \rangle$$

can be given a precise formulation.

Can there exist a sentence Ψ such that for all sentences ϕ either

• ZFC +
$$\Psi \vdash_{\Omega^*}$$
 " $H(\omega_2) \models \phi$ ", or

• ZFC +
$$\Psi \vdash_{\Omega^*}$$
 " $H(\omega_2) \models \neg \phi$ ";

and such that

$$ZFC + \Psi$$

is Ω^* -consistent?

Assuming the Ω Conjecture the answer is "yes" and moreover if Ψ is any such sentence then:

$$ZFC + \Psi \vdash_{\Omega^*} \neg CH.$$

Thus, assuming the Ω Conjecture, a generically absolute theory for $H(\omega_2)$ is *possible* but any such theory implies that CH is false.

This will be discussed further in the next lecture.

Connections with the logic of large cardinal axioms

Definition 14 $(\exists x \phi)$ is a *large cardinal axiom* if

- 1. $\phi(x)$ is a Σ_2 -formula;
- 2. (As a theorem of ZFC) if κ is a cardinal such that

$$V \models \phi[\kappa]$$

then κ is strongly inaccessible and for all partial orders $\mathbb{P} \in V_{\kappa}$,

$$V^{\mathbb{P}}\models\phi[\kappa].$$

Definition 15 Suppose that $(\exists x\phi)$ is a large cardinal axiom.

Then V is ϕ -closed if for every set, X, there exist a transitive set, M, and $\kappa \in M \cap \operatorname{Ord}$ such that

- 1. $M \models ZFC$,
- $2. X \in M_{\kappa}$

3.
$$M \models \phi[\kappa]$$
.

Remark: Suppose that $(\exists x\phi)$ is a large cardinal axiom and there exists a proper class of cardinals κ such that

$$V \models \phi[\kappa].$$

Then V is ϕ -closed.

The following is an easy consequence of the definitions.

Lemma 16 Suppose there there exist a proper class of Woodin cardinals and that Ψ is a Π_2 sentence.

The following are equivalent.

- 1) ZFC $\vdash_{\Omega} \Psi$.
- 2) There is a large cardinal axiom $(\exists x \phi)$ such that
 - (a) ZFC \vdash_{Ω} "V is ϕ -closed",
 - (b) ZFC + "V is ϕ -closed" $\vdash \Psi$.

An immediate corollary of this lemma is that the Ω Conjecture is equivalent to:

Suppose that there exists a proper class of Woodin cardinals. Suppose that $(\exists x \phi)$ is a large cardinal axiom.

The following are equivalent.

- 1. V is ϕ -closed.
- 2. ZFC \vdash_{Ω} "V is ϕ -closed".

Thus the Ω Conjecture implies that Ω -logic is simply the natural logic associated to the set of large cardinal axioms $(\exists x\phi)$ for which V is ϕ -closed.

The Ω Conjecture and the Large Cardinal Hierarchy

Suppose there exists a proper class of Woodin cardinals and let

$$\Gamma^{\infty} = \left\{ A \subseteq \mathbb{R} \,\middle|\, A \text{ is universally Baire} \right\}.$$

The large cardinal axioms $(\exists x\phi)$ such that

ZFC
$$\vdash_{\Omega}$$
 "V is ϕ -closed"

naturally define a wellordered hierarchy.

This is defined as follows.

$$\phi_1 \leq \phi_2$$

if for all $A \in \Gamma^{\infty}$ either:

1. There exists a transitive set M such that M is A-closed and

$$M \models \mathrm{ZFC} + \text{``}V \text{ is not } \phi_2\text{-closed''}$$

or;

2. There exists $x \in \mathbb{R}$ such that if $M \models \mathrm{ZFC}, M$ is A-closed and $x \in M$ then

$$M \models$$
 "V is ϕ_1 -closed".

Thus the rank of ϕ is given by the minimum possible complexity of an Ω -proof,

ZFC
$$\vdash_{\Omega}$$
 "V is ϕ -closed."

- If the Ω Conjecture *holds* in V then this hierarchy includes *all* large cardinal axioms $(\exists x\phi)$ such that V is ϕ -closed;
 - If the Ω Conjecture is *provable*, then this hierarchy is in essence a (coarse) version of the consistency hierarchy.

This, arguably, accounts for the *empirical* fact that all large cardinal axioms are comparable.

The Ω Conjecture and Inner Model Theory

Definition 17 Suppose that $(\exists x\phi)$ is a large cardinal axiom. $(\exists x\phi)$ admits a weak inner model theory if there exists a formula $\Phi(x,y)$ such that the following three conditions hold where for each transitive set, M,

$$I_{\Phi}^{M} = \left\{ (a,b) \,\middle|\, M \models \Phi[a,b] \right\}.$$

Suppose that M is a transitive model of ZFC and that in M there is a proper class of Woodin cardinals and a proper class of cardinals for which ϕ holds.

(1) I_{Φ}^{M} is a function,

$$I_{\Phi}^{M}: M \cap \mathcal{P}(M \cap \mathrm{Ord}) \to M,$$

such that for all $a \in M \cap \mathcal{P}(M \cap \text{Ord})$,

- a) $|N|^M = |a \cup \omega|^M$,
- b) N is transitive, $a \in N_{\delta}$, and $N \models \phi[\delta]$,
- c) $N \models ZFC$,

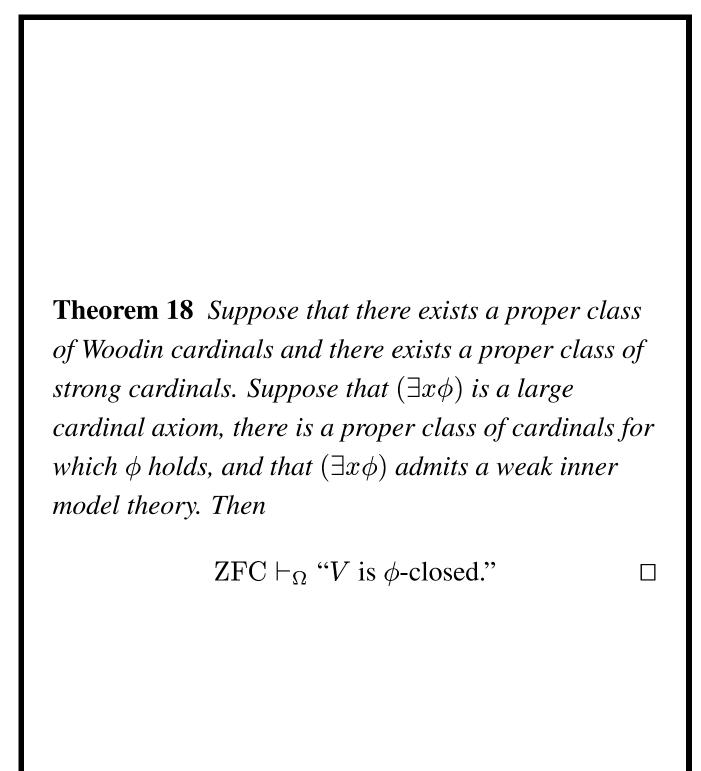
where $(\delta, N) = I_{\Phi}^{M}(a)$.

(2) If $\mathbb{P}\in M$ and $G\subseteq \mathbb{P}$ is M-generic, then $I_{\Phi}^{M}=I_{\Phi}^{M[G]}\cap M.$

(3) Suppose that κ is a measurable cardinal in M such that in M, κ is a limit of Woodin cardinals and a limit of cardinals for which ϕ holds in M_{κ} .

Then
$$I_{\Phi}^M \cap M_{\kappa} = I_{\Phi}^{M_{\kappa}}$$
.

Here is an example. Let $(\exists x \phi_0)$ be the large cardinal axiom where $\phi_0(x)$ asserts: "x is a measurable cardinal". Let $\Phi_0(x, y)$ assert: "x is a set of ordinals and y is the the ω -model of x^{\dagger} ". Then Φ_0 witnesses that the large cardinal axiom $(\exists x \phi_0)$ admits a weak inner model theory.



There is also an approximate converse.

Theorem 19 Suppose that there exists a proper class of Woodin cardinals, $(\exists x\phi)$ is a large cardinal axiom and that

ZFC \vdash_{Ω} "V is ϕ -closed."

Then there is a large cardinal axiom $(\exists x \psi)$ such that

- (1) ZFC \vdash "If V is ψ -closed then V is ϕ -closed.".
- (2) V is ψ -closed.
- (3) $(\exists x\psi)$ admits a weak inner model theory.

Actually one can , in certain conditions, define a wellordering on all large cardinal axioms $(\exists x\phi)$ such that V is ϕ -closed even if the Ω Conjecture fails to hold in V .
The reason lies in the following lemma.

Lemma 20 Assume that there exists a proper class inaccessible limits of Woodin cardinals and let Γ^{∞} be the set of all $A \subseteq \mathbb{R}$ such that A is universally Baire. Suppose that

$$L(\Gamma^{\infty}, \mathbb{R}) \not\models AD$$

and suppose that $(\exists x\phi)$ is a large cardinal axiom such that V is ϕ -closed. Then there exists $A \in \Gamma^{\infty}$ such that for all sets X there exists a transitive set M such that

- (1) $M \models ZFC +$ "There is a proper class of Woodin cardinals",
- (2) $X \in M$ and $M \models$ "V is ϕ -closed",

and such that M is not A-closed.

Now suppose that there exists a proper class of Woodin cardinals and that

$$L(\Gamma^{\infty}, \mathbb{R}) \not\models AD$$

where Γ^{∞} is the set of all universally Baire subsets of \mathbb{R} . For each large cardinal axiom, $(\exists x\phi)$, such that V is ϕ -closed let $A_{\phi} \subseteq \mathbb{R}$ be a witness to the lemma of minimum rank in the Wadge order. Now define $\phi_1 \leq \phi_2$ by comparing the Wadge ranks of A_{ϕ_1} and A_{ϕ_2} .

If the Ω Conjecture is provable then this order is simply a coarser version of the order defined above by comparing the minimum possible lengths of Ω -proofs that V is ϕ -closed.

Suppose there exists a proper class of Woodin cardinals. Let 0^Ω be the set of pairs $(\phi(x),r)$ such that $r\in\mathbb{R}$ and

$$ZFC \vdash_{\Omega} \phi[r].$$

We naturally regard $0^{\Omega} \subset \mathbb{R}$. Clearly

$$0^{\Omega} \in L(\Gamma^{\infty}, \mathbb{R}).$$

The set, 0^{Ω} , is a generalization of 0' to Ω -logic.

There is a version of the theorem on CH which does not require the Ω Conjecture.

Theorem 21 Suppose that there exists a proper class of Woodin cardinals and that Q^{Ω} is not universally Baire. Suppose that Ψ is a sentence such that for all partial orders \mathbb{P} , for all formulas $\phi(x)$, and for all $r \in (\mathbb{R})^{V^{\mathbb{P}}}$, either

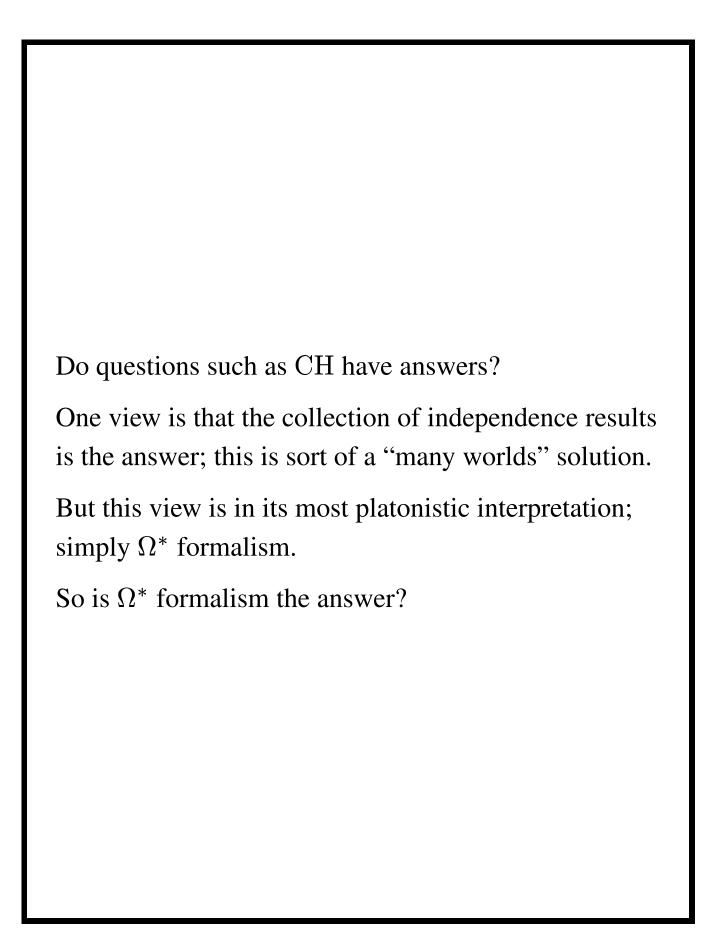
$$(\operatorname{ZFC} + \Psi \vdash_{\Omega^*} "H(\omega_2) \models \phi[r]")^{V^{\mathbb{P}}}$$

or

$$(\operatorname{ZFC} + \Psi \vdash_{\Omega^*} "H(\omega_2) \models (\neg \phi)[r]")^{V^{\mathbb{P}}}$$

П

Then
$$ZFC + \Psi \vdash_{\Omega^*} \neg CH$$
.



It could be that the Ω Conjecture fails badly and in fact that the set

$$\{\phi \,|\, \mathrm{ZFC} \vdash_{\Omega^*} \phi\}$$

is recursively equivalent to the complete Π_2 definable subset of \mathbb{N} .

In this case Ω^* formalism is arguably a reasonable position (no "complexity" is sacrificed).

