The Continuum Hypothesis and the Ω Conjecture

W. Hugh Woodin

Strong Logics

A strong logic, $\vdash_0$, is defined by:

(1) Specifying a collection of *test* structures, these are structures of the form

$$\mathcal{M} = (M, E)$$

where $E \subseteq M \times M$;

(2) Defining

$$\mathrm{ZFC} \vdash_0 \phi$$

if for every test structure, $\mathcal{M}$, if

$$\mathcal{M} \models \mathrm{ZFC}$$

then $\mathcal{M} \models \phi$.

Of course we shall only be interested in the case that there actually exists a test structure, $\mathcal{M}$, such that

$$\mathcal{M} \models \text{ZFC}.$$

- The *smaller* the collection of test structures, the *stronger* the logic.
  - Classical logic is the weakest logic.

Example: β-logic is obtained by simply restricting to *transitive sets*,

$$\mathcal{M} = (M, \in).$$

- The strongest (interesting) logic is when there is only one test structure, V, the universe of sets.

Requirement for a strong logic, $\vdash_0$:

- **Generic Soundness:** Suppose that $\mathbb{P}$ is a partial order, α is an ordinal and that

  $$V_\alpha^{\mathbb{P}} \models \mathrm{ZFC}.$$

  Suppose that

  $$\mathrm{ZFC} \vdash_0 \phi.$$

  Then

  $$V_\alpha^{\mathbb{P}} \models \phi.$$

Our context for considering strong logics will require at the very least that there exists a proper class of Woodin cardinals, and so the requirement of *Generic Soundness* is nontrivial.

We shall further restrict, in the final analysis, to strong logics that are both

- *definable* and
- *generically invariant*.

Thus we shall be considering logics (equivalently, defining notions of mathematical truth) which are completely immune to the effects of forcing.

We begin by defining a specific strong logic

“ Ω-logic ”.

The definition involves a *transfinite* hierarchy which extends the hierarchy of the projective sets; this is the hierarchy of the *universally Baire sets*.

Definition 1 (Feng–Magidor–Woodin)
A set $A \subseteq \mathbb{R}^n$ is *universally Baire* if for any continuous function,

$$F : \Omega \to \mathbb{R}^n,$$

where Ω is a compact Hausdorff space, the preimage of A,

$$\left\{ p \in X \,\middle|\, F(p) \in A \right\},$$

has the property of Baire in Ω; i. e. is open in Ω modulo a meager set. $\square$

- Every borel set $A \subseteq \mathbb{R}^n$ is universally Baire.
- The universally Baire sets form a σ-algebra closed under preimages by borel functions

$$f : \mathbb{R}^n \to \mathbb{R}^m.$$

- The universally Baire sets are Lebesgue measurable etc.

Assuming there is a proper class of Woodin cardinals:

- Every universally Baire set is determined,
  - corollary of the Martin-Steel Theorem;
- The universally Baire sets form a (pre)wellordered hierarchy under Wadge equivalence.
- If $A \subseteq \mathbb{R}$ is universally Baire then every set in

$$L(A, \mathbb{R}) \cap \mathcal{P}(\mathbb{R})$$

is universally Baire.

If A and B are universally Baire subsets of P where $P \subset \mathbb{R}$ is compact, perfect, and nowhere dense, then the Wadge order is quite easily defined:

$A <_{\mathrm{w}} B$ if both A and $P \backslash A$ are preimages of B by functions

$$f : P \to P$$

which satisfy $|f(x) - f(y)| \leq |x - y|/2$ for all $x, y \in P$.

Even restricted to the Borel subsets of P this order is quite fine.

There is a natural generalization of classical first order logic which is defined from the universally Baire sets.

This is Ω-logic;

- "proofs" in Ω-logic are witnessed by universally Baire sets.

Ω-logic is the natural limit of a hierarchy of logics which begins with first order logic and continues with β-logic etc.

$\boxed{A\text{-closed sets}}$

Suppose that

$$A \subseteq \mathbb{R}$$

is universally Baire and $A \neq \emptyset$

Suppose that $V[G]$ is a set generic extension of V.

Then the set A has canonical interpretation as a set

$$A_G \subseteq \mathbb{R}^{V[G]}.$$

The set A_G is defined as

$$A_G = \cup \left\{ \operatorname{ran}(\pi_G) \,\middle|\, \pi : \lambda^\omega \to \mathbb{R}, \pi \in V, \operatorname{ran}(\pi) = A \right\}.$$

In this definition of A_G, π ranges over functions, $\pi : \lambda^\omega \to \mathbb{R}$, such that for all $x, y \in \lambda^\omega$ with $x \neq y$,

$$|\pi(x) - \pi(y)| \leq 1/(n+1)$$

where n is least such that $x(n) \neq y(n)$, and π_G is the function

$$\pi_G : (\lambda^\omega)^{V[G]} \to \mathbb{R}^{V[G]}$$

that π naturally defines in $V[G]$.

It follows that in $V[G]$, the set A_G is universally Baire and if there exists a proper class of Woodin cardinals then

$$\langle H(\omega_1), A \rangle \prec \langle H(\omega_1)^{V[G]}, A_G \rangle.$$

Definition 2 Suppose that $A \subseteq \mathbb{R}$ is universally Baire and that M is a transitive set such that

$$M \models \text{ZFC}.$$

Then M is *A-closed* if for each partial order

$$\mathbb{P} \in M,$$

if $G \subseteq \mathbb{P}$ is V-generic then in $V[G]$:

$$A_G \cap M[G] \in M[G]. \qquad \square$$

The definition that M is A-closed actually makes sense if M is simply an ω-model.

Lemma 3 *Suppose that (M, E) is an ω-model with*

$$(M, E) \models \mathrm{ZFC}.$$

Then the following are equivalent.

(1) (M, E) *is wellfounded.*

(2) (M, E) *is A-closed for each Π^1_1 set.* $\square$

So:

- A-closure is a natural generalization of wellfoundedness.

Ω-logic

Definition 4 Suppose that:

(i) There exists a proper class of Woodin cardinals.

(ii) ϕ is a sentence.

Then

$$\mathrm{ZFC} \vdash_{\Omega} \phi$$

if there exists a universally Baire set $A \subseteq \mathbb{R}$ such that if M is any countable transitive set satisfying

1. $M \models \mathrm{ZFC}$,
2. M is A-closed,

then $M \models \phi$. $\square$

Theorem 5 (Generic Invariance) *Suppose that there exists a proper class of Woodin cardinals.*

Suppose that ϕ is a sentence. Then for each partial order $\mathbb{P}$,

$$(\mathrm{ZFC} \vdash_\Omega \phi)^V$$

if and only if

$$(\mathrm{ZFC} \vdash_\Omega \phi)^{V^{\mathbb{P}}}. \qquad \square$$

Theorem 6 (Generic Soundness) *Suppose that there exists a proper class of Woodin cardinals.*

Suppose that

$$V_\alpha^{\mathbb{P}} \models \mathrm{ZFC}$$

and that

$$\mathrm{ZFC} \vdash_\Omega \phi.$$

Then

$$V_\alpha^{\mathbb{P}} \models \phi. \qquad \square$$

Ω-logic is a fairly strong logic. For example:

Theorem 7 *Suppose that there exists a proper class of Woodin cardinals.*

Then

$$\mathrm{ZFC} \vdash_{\Omega} \mathrm{AD}^{L(\mathbb{R})} \qquad \square$$

$\boxed{\Omega^*\text{-logic}}$

Definition 8 (Ω^*-logic) Suppose that:

(i) There exists a proper class of Woodin cardinals.

(ii) ϕ is a sentence.

Then

$$\mathrm{ZFC} \vdash_{\Omega^*} \phi$$

if for all ordinals α and for all partial orders $\mathbb{P}$ if

$$V_\alpha^{\mathbb{P}} \models \mathrm{ZFC},$$

then $V_\alpha^{\mathbb{P}} \models \phi$. □

Generic Soundness is immediate for Ω^*-logic.

- Ω^*-logic is the strongest possible logic satisfying this requirement.

The property of generic invariance also holds for Ω^*-logic.

Theorem 9 (Generic Invariance) *Suppose that there exists a proper class of Woodin cardinals.*

Suppose that ϕ is a sentence. Then for each partial order $\mathbb{P}$,

$$(\mathrm{ZFC} \vdash_{\Omega^*} \phi)^V$$

if and only if

$$(\mathrm{ZFC} \vdash_{\Omega^*} \phi)^{V^{\mathbb{P}}}. \qquad \square$$

Ω Conjecture:

Suppose that there exists a proper class of Woodin cardinals. Then for each Π_2 *sentence,* ϕ*,*

$$\mathrm{ZFC} \vdash_{\Omega^*} \phi$$

if and only if $\mathrm{ZFC} \vdash_{\Omega} \phi$.

We define two generalizations of the notion that a set $A \subset \mathbb{R}$ be *recursive*.

Definition 10 Suppose that there exists a proper class of Woodin cardinals. A set $A \subseteq \mathbb{R}$ is Ω-*recursive* if there exists a formula $\phi(x)$ such that:

1. $A = \left\{ r \,\middle|\, \mathrm{ZFC} \vdash_\Omega \phi[r] \right\}$;
2. For all partial orders, $\mathbb{P}$, if $G \subset \mathbb{P}$ is V-generic then for each $r \in \mathbb{R}^{V[G]}$, either

$$V[G] \models \mathrm{ZFC} \vdash_\Omega \phi[r],$$

or $V[G] \models \mathrm{ZFC} \vdash_\Omega (\neg\phi)[r]$. $\square$

Lemma 11 *Suppose that there exists a proper class of Woodin cardinals and that $A \subseteq \mathbb{R}$. Then the following are equivalent:*

1. *A is* Ω-recursive
2. *There exists a universally Baire set $B \subseteq \mathbb{R}$ such that the set A is Δ_1 definable in $L(B, \mathbb{R})$ from the parameter $\{\mathbb{R}\}$.* $\square$

Definition 12 Suppose that there exists a proper class of Woodin cardinals. A set $A \subseteq \mathbb{R}$ is Ω^*-*recursive* if there exists a formula $\phi(x)$ such that:

1. $A = \left\{ r \,\middle|\, \mathrm{ZFC} \vdash_{\Omega^*} \phi[r] \right\}$;
2. For all partial orders, $\mathbb{P}$, if $G \subset \mathbb{P}$ is V-generic then for each $r \in \mathbb{R}^{V[G]}$, either
$$V[G] \models \mathrm{ZFC} \vdash_{\Omega^*} \phi[r],$$
or $V[G] \models \mathrm{ZFC} \vdash_{\Omega^*} (\neg\phi)[r]$. $\square$

If the Ω Conjecture holds then a set $A \subseteq \mathbb{R}$ is Ω^*-recursive if and only if it is Ω-recursive.

Theorem 13 *Suppose that there exists a proper class of Woodin cardinals. Suppose that $A \subseteq \mathbb{R}$ is Ω^*-recursive. Then A is universally Baire.* $\square$

An immediate corollary is that if $A \subset \mathbb{R}$ is Ω^*-recursive then the set A is determined. In other words; sets of reals which have "generically absolute" definitions, are determined.

The question of whether there can exist analogs of determinacy for the structure

$$\langle H(\omega_2), \in \rangle$$

can be given a precise formulation.

Can there exist a sentence Ψ such that for all sentences ϕ either

- $\mathrm{ZFC} + \Psi \vdash_{\Omega^*}$ *"$H(\omega_2) \models \phi$", or*
- $\mathrm{ZFC} + \Psi \vdash_{\Omega^*}$ *"$H(\omega_2) \models \neg\phi$";*

and such that

$$\mathrm{ZFC} + \Psi$$

is Ω^*-consistent?

Assuming the Ω Conjecture the answer is "yes" and moreover if Ψ is any such sentence then:

$$\mathrm{ZFC} + \Psi \vdash_{\Omega^*} \neg\mathrm{CH}.$$

Thus, assuming the Ω Conjecture, a generically absolute theory for $H(\omega_2)$ is *possible* but any such theory implies that CH is false.

This will be discussed further in the next lecture.

Connections with the logic of large cardinal axioms

Definition 14 $(\exists x \phi)$ is a *large cardinal axiom* if

1. $\phi(x)$ is a Σ_2-formula;
2. (As a theorem of ZFC) if κ is a cardinal such that

$$V \models \phi[\kappa]$$

then κ is strongly inaccessible and for all partial orders $\mathbb{P} \in V_\kappa$,

$$V^{\mathbb{P}} \models \phi[\kappa]. \qquad \square$$

Definition 15 Suppose that $(\exists x \phi)$ is a large cardinal axiom.

Then V is *ϕ-closed* if for every set, X, there exist a transitive set, M, and $\kappa \in M \cap \mathrm{Ord}$ such that

1. $M \models \mathrm{ZFC}$,
2. $X \in M_\kappa$,
3. $M \models \phi[\kappa]$. $\square$

Remark: Suppose that $(\exists x \phi)$ is a large cardinal axiom and there exists a proper class of cardinals κ such that

$$V \models \phi[\kappa].$$

Then V is ϕ-closed.

The following is an easy consequence of the definitions.

Lemma 16 *Suppose there there exist a proper class of Woodin cardinals and that Ψ is a Π_2 sentence.*

The following are equivalent.

1) ZFC $\vdash_\Omega \Psi$.
2) *There is a large cardinal axiom* $(\exists x \phi)$ *such that*
   (a) ZFC $\vdash_\Omega$ "V is ϕ-closed",
   (b) ZFC + "V is ϕ-closed" $\vdash \Psi$. $\square$

An immediate corollary of this lemma is that the Ω Conjecture is equivalent to:

Suppose that there exists a proper class of Woodin cardinals. Suppose that $(\exists x \phi)$ *is a large cardinal axiom.*

The following are equivalent.

1. *V is ϕ-closed.*
2. ZFC $\vdash_\Omega$ *"V is ϕ-closed"*.

Thus the Ω Conjecture implies that Ω-logic is simply the natural logic associated to the set of large cardinal axioms $(\exists x \phi)$ for which V is ϕ-closed.

The Ω Conjecture and the Large Cardinal Hierarchy

Suppose there exists a proper class of Woodin cardinals and let

$$\Gamma^{\infty} = \left\{ A \subseteq \mathbb{R} \middle| \; A \text{ is universally Baire} \right\}.$$

The large cardinal axioms $(\exists x \phi)$ such that

$$\text{ZFC} \vdash_{\Omega} \text{“}V \text{ is } \phi\text{-closed”}$$

naturally define a wellordered hierarchy.

This is defined as follows.

$$\phi_1 \leq \phi_2$$

if for all $A \in \Gamma^\infty$ either:

1. There exists a transitive set M such that M is A-closed and

   $$M \models \text{ZFC} + \text{“}V \text{ is not } \phi_2\text{-closed”}$$

   or;

2. There exists $x \in \mathbb{R}$ such that if $M \models$ ZFC, M is A-closed and $x \in M$ then

   $$M \models \text{“}V \text{ is } \phi_1\text{-closed”}.$$

Thus the rank of ϕ is given by the minimum possible *complexity* of an Ω-proof,

$$\mathrm{ZFC} \vdash_\Omega \text{“}V \text{ is } \phi\text{-closed.”}$$

- If the Ω Conjecture *holds* in V then this hierarchy includes *all* large cardinal axioms $(\exists x \phi)$ such that V is ϕ-closed;
  - If the Ω Conjecture is *provable*, then this hierarchy is in essence a (coarse) version of the consistency hierarchy.

This, arguably, accounts for the *empirical* fact that all large cardinal axioms are comparable.

Thus if the Ω Conjecture is true then the large cardinal axioms which admit an inner model analysis are "cofinal" and one has a precise definition of the hierarchy of large cardinal axioms.

The Ω Conjecture and Inner Model Theory

Definition 17 Suppose that $(\exists x\phi)$ is a large cardinal axiom. $(\exists x\phi)$ *admits a weak inner model theory* if there exists a formula $\Phi(x, y)$ such that the following three conditions hold where for each transitive set, M,

$$I_{\Phi}^{M} = \left\{(a, b) \,\middle|\, M \models \Phi[a, b]\right\}.$$

Suppose that M is a transitive model of ZFC and that in M there is a proper class of Woodin cardinals and a proper class of cardinals for which ϕ holds.

(1) I^M_Φ is a function,

$$I^M_\Phi : M \cap \mathcal{P}(M \cap \mathrm{Ord}) \to M,$$

such that for all $a \in M \cap \mathcal{P}(M \cap \mathrm{Ord})$,

a) $|N|^M = |a \cup \omega|^M$,

b) N is transitive, $a \in N_\delta$, and $N \models \phi[\delta]$,

c) $N \models \mathrm{ZFC}$,

where $(\delta, N) = I^M_\Phi(a)$.

(2) If $\mathbb{P} \in M$ and $G \subseteq \mathbb{P}$ is M-generic, then
$I_\Phi^M = I_\Phi^{M[G]} \cap M$.

(3) Suppose that κ is a measurable cardinal in M such that in M, κ is a limit of Woodin cardinals and a limit of cardinals for which ϕ holds in M_κ.

Then $I_\Phi^M \cap M_\kappa = I_\Phi^{M_\kappa}$. $\square$

Here is an example.

Let $(\exists x \phi_0)$ be the large cardinal axiom where $\phi_0(x)$ asserts: "x is a measurable cardinal". Let $\Phi_0(x, y)$ assert: "x is a set of ordinals and y is the the ω-model of $x^\dagger$". Then Φ_0 witnesses that the large cardinal axiom $(\exists x \phi_0)$ admits a weak inner model theory.

Theorem 18 *Suppose that there exists a proper class of Woodin cardinals and there exists a proper class of strong cardinals. Suppose that* $(\exists x \phi)$ *is a large cardinal axiom, there is a proper class of cardinals for which* ϕ *holds, and that* $(\exists x \phi)$ *admits a weak inner model theory. Then*

$$\text{ZFC} \vdash_{\Omega} \text{“}V \text{ is } \phi\text{-closed.”} \qquad \square$$

There is also an approximate converse.

Theorem 19 *Suppose that there exists a proper class of Woodin cardinals,* $(\exists x \phi)$ *is a large cardinal axiom and that*

$$\text{ZFC} \vdash_\Omega \text{“}V \text{ is } \phi\text{-closed.”}$$

Then there is a large cardinal axiom $(\exists x \psi)$ *such that*

(1) ZFC $\vdash$ “If V is ψ-closed then V is ϕ-closed.”.

(2) *V is ψ-closed.*

(3) $(\exists x \psi)$ *admits a weak inner model theory.* $\square$

Actually one *can*, in certain conditions, define a wellordering on all large cardinal axioms $(\exists x\phi)$ such that V is ϕ-closed even if the Ω Conjecture fails to hold in V.

The reason lies in the following lemma.

Lemma 20 *Assume that there exists a proper class inaccessible limits of Woodin cardinals and let* Γ^∞ *be the set of all* $A \subseteq \mathbb{R}$ *such that* A *is universally Baire. Suppose that*

$$L(\Gamma^\infty, \mathbb{R}) \not\models \mathrm{AD}$$

and suppose that $(\exists x \phi)$ *is a large cardinal axiom such that* V *is* ϕ*-closed. Then there exists* $A \in \Gamma^\infty$ *such that for all sets* X *there exists a transitive set* M *such that*

(1) $M \models \mathrm{ZFC}\, +$
"There is a proper class of Woodin cardinals",

(2) $X \in M$ *and* $M \models$ *"*V *is* ϕ*-closed",*

and such that M *is not* A*-closed.* $\square$

Now suppose that there exists a proper class of Woodin cardinals and that

$$L(\Gamma^\infty, \mathbb{R}) \not\models \mathrm{AD}$$

where Γ^∞ is the set of all universally Baire subsets of $\mathbb{R}$. For each large cardinal axiom, $(\exists x \phi)$, such that V is ϕ-closed let $A_\phi \subseteq \mathbb{R}$ be a witness to the lemma of minimum rank in the Wadge order. Now define $\phi_1 \leq \phi_2$ by comparing the Wadge ranks of A_{ϕ_1} and A_{ϕ_2}.

If the Ω Conjecture is provable then this order is simply a coarser version of the order defined above by comparing the minimum possible lengths of Ω-proofs that V is ϕ-closed.

Suppose there exists a proper class of Woodin cardinals. Let $\underset{\sim}{0}^{\Omega}$ be the set of pairs $(\phi(x), r)$ such that $r \in \mathbb{R}$ and

$$\mathrm{ZFC} \vdash_{\Omega} \phi[r].$$

We naturally regard $\underset{\sim}{0}^{\Omega} \subset \mathbb{R}$. Clearly

$$\underset{\sim}{0}^{\Omega} \in L(\Gamma^{\infty}, \mathbb{R}).$$

The set, $\underset{\sim}{0}^{\Omega}$, is a generalization of $0'$ to Ω-logic.

There is a version of the theorem on CH which does not require the Ω Conjecture.

Theorem 21 *Suppose that there exists a proper class of Woodin cardinals and that* $\underset{\sim}{0}^{\Omega}$ *is not universally Baire. Suppose that* Ψ *is a sentence such that for all partial orders* $\mathbb{P}$*, for all formulas* $\phi(x)$*, and for all* $r \in (\mathbb{R})^{V^{\mathbb{P}}}$*, either*

$$\left(\mathrm{ZFC} + \Psi \vdash_{\Omega^*} \text{“}H(\omega_2) \models \phi[r]\text{”}\right)^{V^{\mathbb{P}}}$$

or

$$\left(\mathrm{ZFC} + \Psi \vdash_{\Omega^*} \text{“}H(\omega_2) \models (\neg\phi)[r]\text{”}\right)^{V^{\mathbb{P}}}$$

Then $\mathrm{ZFC} + \Psi \vdash_{\Omega^*} \neg\mathrm{CH}$. $\square$

Do questions such as CH have answers?

One view is that the collection of independence results is the answer; this is sort of a “many worlds” solution.

But this view is in its most platonistic interpretation; simply Ω^* formalism.

So is Ω^* formalism the answer?

It could be that the Ω Conjecture fails badly and in fact that the set

$$\left\{\phi \,\middle|\, \mathrm{ZFC} \vdash_{\Omega^*} \phi\right\}$$

is recursively equivalent to the complete Π_2 definable subset of $\mathbb{N}$.

In this case Ω^* formalism is arguably a reasonable position (no "complexity" is sacrificed).

On the other hand if the Ω Conjecture is true then Ω logic is definable in the structure; $\langle H(c^+), \in\rangle$.

In this case Ω^* formalism is no more reasonable than formalism itself.

Both views ultimately reject an unambiguous conception of the transfinite.