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Lecture III
The central derivative of the genus 2 Eisenstein series

Today, I want to discuss the identity

$$\text{(A)} \qquad \langle\, \hat{\phi}_1(\tau_1), \hat{\phi}_1(\tau_2)\,\rangle \;\equiv\; \mathcal{E}_2'(\begin{pmatrix} \tau_1 & \\ & \tau_2 \end{pmatrix}, 0, B),$$

which played a key role in the proof of the formula for the inner product $<\hat{\theta}(f), \hat{\theta}(f)>$ described in Lecture II. Recall that $\equiv$ means the two sides differ by a combination of theta functions coming from $O(1)$'s. In fact, the two sides should agree exactly, but the proof of this is not finished.

§1. The genus 2 Eisenstein series.

Let

$$\begin{aligned}
G''_{\mathbb{A}} &= \widetilde{\mathrm{Sp}_2(\mathbb{A})} = \text{metaplectic cover of } \mathrm{Sp}_2(\mathbb{A})\\
P &= NM \ \subset \mathrm{Sp}_2 = \text{Siegel parabolic}\\
M &= \{\ m(a) = \begin{pmatrix} a & \\ & {}^t a^{-1} \end{pmatrix} \mid a \in \mathrm{GL}_2\ \}\\
N &= \{\ n(b) = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \mid b \in \mathrm{Sym}_2\ \},\\
I(s) &= \mathrm{Ind}_{P_{\mathbb{A}}}^{G''_{\mathbb{A}}}(|\ |^s) = \text{global degenerate principal series}\\
&= \{\ \Phi(s) : G''_{\mathbb{A}} \longrightarrow \mathbb{C} \ \mid\ \Phi(n(b)m(a)g, s) = \chi^{\psi}(a)|a|^{s+\frac{3}{2}}\, \Phi(g,s)\ \}.
\end{aligned}$$

A section $\Phi(s) \in I(s)$, defines a Siegel Eisenstein series

$$E(g'', s, \Phi) = \sum_{\gamma \in P_{\mathbb{Q}} \backslash G''_{\mathbb{Q}}} \Phi(\gamma g'', s),$$

convergent in the half plane $\mathrm{Re}(s) > \frac{3}{2}$, with a meromorphic continuation in s and functional equation

$$E(g'', s, \Phi) = E(g'', -s, M(s)\Phi)$$

for the intertwining operator $M(s) : I(s) \longrightarrow I(-s)$. We will only need sections of the form

$$\Phi(s) = \Phi_{\infty}^{\frac{3}{2}}(s) \otimes \Phi_f(s)$$

where $\Phi_{\infty}^{\frac{3}{2}}(s)$ has weight $\frac{3}{2}$ for the action of $K''_{\infty} = \widetilde{U(2)}$, and the associated Eisenstein series can be pushed down to a function of $\tau \in \mathfrak{H}_2$ by the usual procedure:

$$E(\tau, s, \frac{3}{2}, \Phi_f) := \det(v)^{\frac{3}{4}}\, E(g''_{\tau}, s, \Phi_{\infty}^{\frac{3}{2}} \otimes \Phi_f).$$

§2. The Eisenstein series $\mathcal{E}_2(\tau, s, B)$.

Recall from Lecture II that, for each prime p, we have ternary quadratic spaces $V_p^{\pm}$ over $\mathbb{Q}_p$ with Hasse invariants $\epsilon_p(V_p^{\pm}) = \pm 1$. The local metaplectic cover

$$G''_p = \widetilde{\mathrm{Sp}_2(\mathbb{Q}_p)}$$

acts in the space $S((V_p^{\pm})^2)$ via the Weil representation ω_p, and we get a map

$$\lambda_p : S((V_p^{\pm})^2) \longrightarrow I_p(0), \qquad \varphi \mapsto \lambda_p(\varphi_p),$$

$$\lambda_p(\varphi_p)(g) = \big(\omega(g)\varphi_p\big)(0).$$

Here $I_p(s)$ is the local induced representation. In fact, if

$$\Pi(V_p^{\pm}) = \lambda_p(S((V_p^{\pm})^2))$$

is the image of $\lambda_p^{\pm}$, then this image is an irreducible G_p'' submodule of $I_p(0)$ and

$$I_p(0) = \Pi(V_p^+) \oplus \Pi(V_p^-).$$

This decomposition is closely connected with local theta dichotomy.

We define local sections as follows.
Let

$$\begin{aligned}
O_p^{\pm} &= \text{maximal order in } B_p^{\pm} \\
O_p^{e} &= \text{Eichler order of level } p \text{ in } B_p^{+} = M_2(\mathbb{Q}_p) \\
R_p^{\pm} &= V_p^{\pm} \cap O_p^{\pm} \\
R_p^{e} &= V_p^{+} \cap O_p^{e} \\
\varphi_p^{0} &= \text{char}((R_p^{+})^2) \in S((V_p^{+})^2) \\
\varphi_p^{e} &= \text{char}((R_p^{e})^2) \in S((V_p^{+})^2) \\
\varphi_p^{-} &= \text{char}((R_p^{-})^2) \in S((V_p^{-})^2) \\
\Phi_p^{\bullet}(s) &= \text{standard section with} \\
\Phi_p^{\bullet}(0) &= \lambda_p^{\pm}(\varphi_p^{\bullet}), \qquad \bullet \in \{\ 0,\ -,\ e\ \}.
\end{aligned}$$

Finally, let

$$\tilde{\Phi}_p(s) = \Phi_p^{-}(s) + A_p(s)\,\Phi_p^{0}(s) + B_p(s)\,\Phi_p^{e}(s)$$

where $A_p(s)$ and $B_p(s)$ are entire functions of s with

$$A_p(0) = B_p(0) = 0, \qquad \text{and} \qquad A_p'(0) = -\frac{2}{p^2-1}\log(p), \quad B_p'(0) = \frac{1}{2}\,\frac{p+1}{p-1}\log(p).$$

As we will see, the peculiar choice of the section $\tilde{\Phi}_p(s)$ is dictated by the intersection theory of vertical cycles in the fibers of bad reduction of the arithmetic surface $\mathcal{M}$.

Next, recall some notation

$$\begin{aligned} B &= \text{indefinite quaternion algebra over } \mathbb{Q} \\ D(B) &= \text{product of ramified primes, } D(B) > 1, \\ V = V^B &= \{x \in B \mid \operatorname{tr}(x) = 0\}, \qquad Q(x) = -x^2 \end{aligned}$$

Finally, we define a section $\tilde{\Phi}^B(s) \in I_f(s)$ by

$$\tilde{\Phi}^B(s) = \Big(\otimes_{p|D(B)} \tilde{\Phi}_p(s) \Big) \otimes \Big(\otimes_{p\nmid D(B)} \Phi^0_p(s) \Big),$$

and the associated (normalized) Eisenstein series

$$\mathcal{E}_2(\tau, s, B) = \eta(s, B)\, \zeta(2s+2)\, E(\tau, s, \frac{3}{2}, \tilde{\Phi}^B).$$

for a certain factor $\eta(s, B)$, with $\eta(0, B) \neq 0$, whose definition we omit.

By the construction,

$$\mathcal{E}_2(\tau, 0, B) = 0.$$

§3. Nonsingular Fourier coefficients.

In order to prove identity I, we want to compare the individual Fourier coefficients of the two sides. The following observation is crucial for the structure of the non-singular coefficients.

Lemma. *Suppose that $T \in \mathrm{Sym}_2(\mathbb{Q})$ with $\det(T) \neq 0$. Then, up to isomorphism, there is a unique quaternion algebra B^T over $\mathbb{Q}$ such that T is represented by ternary quadratic space V^T of trace zero elements in B^T. Explicitly, the matrix for the quadratic form on V^T is*

$$\begin{pmatrix} T & \\ & \det(V)/\det(T) \end{pmatrix}.$$

Definition.

$$\mathrm{Diff}(T,B) = \{\ p \mid \mathrm{inv}_p(B^T) = -\mathrm{inv}_p(B)\ \}.$$

Note that

$$\begin{aligned} |\mathrm{Diff}(T,B)| &= \text{the `distance' between } B^T \text{ and } B \\ &\equiv 0 \mod (2) \end{aligned}$$

For the nonsingular Fourier coefficients of $\mathcal{E}_2(\tau,s,B)$, we have a product formula

$$\mathcal{E}_{2,T}(\tau,s,B) = W_{T,\infty}(\tau,s)\cdot\prod_p W_{T,p}(s),$$

where the local factors $W_{T,p}(s)$ are determined by the local factors of the section $\tilde{\Phi}^B(s)$. There are local vanishing results at $s=0$:

$$W_{T,\infty}(\tau,0) = \begin{cases} q^T & \text{if } T>0, \\ 0 & \text{if } \mathrm{sig}(T) = (1,1) \text{ or } (0,2). \end{cases}$$

For a finite prime p, with $p \nmid 2D(B)\det(T)$,

$$W_{T,p}(s) = \begin{cases} 1 & \text{if } T \in \mathrm{Sym}_2(\mathbb{Z}_p) \\ 0 & \text{otherwise.} \end{cases}$$

In general

$$W_{T,p}(0) \neq 0 \iff T \in \mathrm{Sym}_2(\mathbb{Z}_p) \text{ and } p \notin \mathrm{Diff}(T,B).$$

Thus, for $T \in \mathrm{Sym}_2(\mathbb{Z})$,

$$\mathcal{E}'_{2,T}(\tau,0,B) = 0$$

except in the cases:

(i) $T>0$ and $\mathrm{Diff}(T,B) = \{\infty, p\}$ for a unique finite prime p.

(ii) $B^T = B$, so $\mathrm{sig}(T) = (1,1)$ or $(0,2)$.

In case (i), we have

$$\mathcal{E}'_{2,T}(\tau, 0, B) = q^T \cdot W'_{T,p}(0) \cdot \prod_{\ell \neq p} W_{T,\ell}(0).$$

In case (ii), we have

$$\mathcal{E}'_{2,T}(\tau, 0, B) = W'_{T,\infty}(\tau, 0) \cdot \prod_{\ell} W_{T,\ell}(0).$$

The Fourier coefficients for singular T's do not have a product structure and are more difficult to calculate.

§4. Heights and Fourier coefficients.

Returning to the main identity (A), the idea is to compare the Fourier coefficients of the two sides. On the geometric side, we have:

$$\langle\, \hat{\phi}_1(\tau_1), \hat{\phi}(\tau_2)\ \rangle = \sum_{t_1,\, t_2} \langle\, \widehat{\mathcal{Z}}(t_1, v_1), \widehat{\mathcal{Z}}(t_2, v_2)\ \rangle\ q_1^{t_1}\, q_2^{t_2}.$$

For the Eisenstein series we have

$$\mathcal{E}'_2(\begin{pmatrix} \tau_1 & \\ & \tau_2 \end{pmatrix}, 0, B) = \sum_{t_1,\, t_2} \sum_{\substack{T \\ \mathrm{diag}(T)=(t_1,t_2)}} \mathcal{E}'_{2,T}(\begin{pmatrix} \tau_1 & \\ & \tau_2 \end{pmatrix}, 0, B).$$

Thus (A) amounts to a family of identities:

$$(\mathrm{A}') \qquad \langle\, \widehat{\mathcal{Z}}(t_1, v_1), \widehat{\mathcal{Z}}(t_2, v_2)\ \rangle\ q_1^{t_1}\, q_2^{t_2} \quad = \quad \sum_{\substack{T \\ \mathrm{diag}(T)=(t_1,t_2)}} \mathcal{E}'_{2,T}(\begin{pmatrix} \tau_1 & \\ & \tau_2 \end{pmatrix}, 0, B).$$

Theorem. *Suppose that t_1t_2 is not a square and that the density identity below holds for $p = 2$. Then* (A$'$) *holds.*

The proof comes down to an explicit computation of the quantities on the two sides. First note that the condition t_1t_2 is not a square implies that

(i) For all T with $\mathrm{diag}(T) = (t_1, t_2)$, $\det(T) \neq 0$, so that only nonsingular T's occur on the right side of (A$'$).

(ii) The cycles $\mathcal{Z}(t_1)$ and $\mathcal{Z}(t_2)$ on $\mathcal{M}$ do not meet on the generic fiber, i.e., $\mathcal{Z}(t_1)_{\mathbb{Q}} \cap \mathcal{Z}(t_2)_{\mathbb{Q}} = \emptyset$.

The disjointness of $\mathcal{Z}(t_1)$ and $\mathcal{Z}(t_2)$ on the generic fiber means that these cycles do not have common horizontal components and, hence,

$$\langle\, \widehat{\mathcal{Z}}(t_1, v_1), \widehat{\mathcal{Z}}(t_2, v_2)\,\rangle = \langle\, \widehat{\mathcal{Z}}(t_1, v_1), \widehat{\mathcal{Z}}(t_2, v_2)\,\rangle_\infty + \sum_{p<\infty} \langle\, \mathcal{Z}(t_1), \mathcal{Z}(t_2)\,\rangle_p,$$

where $\langle\, \widehat{\mathcal{Z}}(t_1, v_1), \widehat{\mathcal{Z}}(t_2, v_2)\,\rangle_\infty$ is the contribution of the Green's functions, and $\langle\, \mathcal{Z}(t_1), \mathcal{Z}(t_2)\,\rangle_p$ is the contribution of the intersection of the cycles supported in the fiber $\mathcal{M}_p$.

A more precise version of the previous result is then:

Theorem. *(i)*

$$\langle\, \widehat{\mathcal{Z}}(t_1, v_1), \widehat{\mathcal{Z}}(t_2, v_2)\,\rangle_\infty\, q_1^{t_1} q_2^{t_2} = \sum_{\substack{T \\ \mathrm{diag}(T)=(t_1,t_2) \\ B^T = B}} \mathcal{E}'_{2,T}(\begin{pmatrix} \tau_1 & \\ & \tau_2 \end{pmatrix}, 0, B).$$

(ii) Suppose that the density identity holds for $p = 2$. Then

$$\langle\, \mathcal{Z}(t_1), \mathcal{Z}(t_2)\,\rangle_p\, q_1^{t_1} q_2^{t_2} = \sum_{\substack{T \\ \mathrm{diag}(T)=(t_1,t_2) \\ B^T = B^{(p)}}} \mathcal{E}'_{2,T}(\begin{pmatrix} \tau_1 & \\ & \tau_2 \end{pmatrix}, 0, B).$$

The product formulas described in section 3 can be used to compute the right hand side. Since we have said very little about the Green functions involved in (i), we will only discuss case (ii).

§4. Computation of intersection multiplicities.

Recall that $\mathcal{Z}(t)$ is the locus of triples (A,ι,x) where $x\in V(A,\iota)$ is a special endomorphism with $Q(x)=t$. The fiber product of two such cycles over $\mathcal{M}$ decomposes as follows:

Lemma.

$$\mathcal{Z}(t_1)\times_{\mathcal{M}}\mathcal{Z}(t_2)=\coprod_{\substack{T\\ \mathrm{diag}(T)=(t_1,t_2)\\ T\geq 0}}\mathcal{Z}(T),$$

where

$$\mathcal{Z}(T)=\text{locus of }(A,\iota,\mathbf{x}),\quad \mathbf{x}\in V(A,\iota)^2,\quad Q(\mathbf{x})=T.$$

Note that, when t_1t_2 is not a square, only positive definite T's occur in the decomposition.

Proposition. *Suppose that* $T\in\mathrm{Sym}_2(\mathbb{Z})$ *is positive definite. Then*
(i)

$$\mathcal{Z}(T)_{\mathbb{Q}}=\emptyset.$$

(ii)

$$|\mathrm{Diff}(T,B)|>2\qquad\Longrightarrow\qquad\mathcal{Z}(T)=\emptyset.$$

(iii)

$$\mathrm{Diff}(T,B)=\{\infty,p\}\qquad\Longrightarrow\qquad\mathrm{supp}(\mathcal{Z}(T))\ \subset\ \mathcal{M}_p.$$

Proof. The key point is that only a supersingular abelian surface in characteristic p can support such an additional pair of special endomorphisms $\mathbf{x}$ with $\det Q(\mathbf{x}) \neq 0$. But, for such an A,

$$V(A, \iota) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq V^{(p)},$$

where

$$\mathrm{Diff}(B^{(p)}, B) = \{\infty, p\}.$$

Since $V(A, \iota) = V^{(p)}$ must represent T, we have $B^T = B^{(p)}$. □

When $\mathrm{Diff}(T, B) = \{\infty, p\}$, so that $V^T = V^{(p)}$, there are two cases:

(a) $\quad p \nmid D(B) \quad \Longrightarrow \quad \mathcal{Z}(T) = 0$–cycle in $\mathcal{M}_p$.

(b) $\quad p \mid D(B) \quad \Longrightarrow \quad \mathcal{Z}(T)$ can be a curve in $\mathcal{M}_p$.

In case (a), the support of $\mathcal{Z}(T)$ is a finite collection of points, each occuring with the same multiplicity. This multiplicity can be calculated via the deformation theory of the corresponding p–divisible formal groups, by specializing a result of Gross and Keating.

Theorem. (Gross–Keating) *The 'multiplicity', $\delta_p(T)$, depends only on the $\mathrm{GL}_2(\mathbb{Z}_p)$–equivalence class of T. Explicitly,*

$$\delta_p(T) = \begin{cases} \sum_{j=0}^{(\alpha-1)/2} (\alpha + \beta - 4j)\, p^j & \text{if } \alpha \text{ is odd,} \\ \sum_{j=0}^{\alpha/2-1} (\alpha + \beta - 4j)\, p^j + \frac{1}{2}(\beta - \alpha + 1) p^{\alpha/2} & \text{if } \alpha \text{ is even.} \end{cases}$$

where, for $p \neq 2$,

$$T \simeq \begin{pmatrix} \epsilon_1 p^\alpha & \\ & \epsilon_2 p^\beta \end{pmatrix},$$

with $0 \leq \alpha \leq \beta$ and ϵ_1, $\epsilon_2 \in \mathbb{Z}_p^\times$. The same formula holds for $p = 2$ for the appropriate definition of invariants α and β for $T \in \mathrm{Sym}_2(\mathbb{Z}_2)$.

Thus, the contribution to the height pairing of $\mathcal{Z}(T) = \mathrm{Spec}\,(R(T))$ is

$$\begin{aligned}\widehat{\mathcal{Z}}(T) : t = \widehat{\deg}\ (\mathcal{Z}(T)) &= \log|R(T)| \\ &= \Bigg(\text{ number of points in } \mathrm{supp}(\mathcal{Z}(T)) \Bigg) \cdot \delta_p(T) \cdot \log(p).\end{aligned}$$

On the other hand, by using results of Kitaoka on representation densities of quadratic forms in the case $p \neq 2$, we find the **p-adic density formula**:

$$W'_{T,p}(0) = \delta_p(T) \cdot \log(p). \tag{B}$$

We expect this formula to hold for $p = 2$ as well, but the computations are not yet complete.

Next, up the some elementary constants[1], independent of T, the Siegel formula for the space $V^{(p)}$ together with a parametrization of the supersingular points in the fiber $\mathcal{M}_p$ yields

$$\Bigg(\text{ number of points in } \mathrm{supp}(\mathcal{Z}(T)) \Bigg) \mathrel{\mathring{=}} \prod_{\ell \neq p} W_{T,\ell}(0). \tag{C}$$

This yields the identity

$$\widehat{\mathcal{Z}}(T) \ = \ W'_{T,p}(0) \cdot \prod_{\ell \neq p} W_{T,\ell}(0), \tag{D}$$

where, for $p = 2$, we must assume the 2-adic density formula (B).

In case (b), where $p \mid D(B)$, the cycle $\mathcal{Z}(T)$ is, in general, a union of configurations of components of the fiber $\mathcal{M}_p$ of bad reduction. Their contribution, $\widehat{\mathcal{Z}}(T)$, to the total intersection multiplicity can be computed using p-adic uniformization, and is again a product of a "multiplicity" $\delta_p(T)$, which is independent of the configuration, times

[1] Hence the notation $\mathrel{\mathring{=}}$

the number of configurations. The analogues of (B), (C) and (D) then hold, where, (B) is again obtained via a coincidence of the intersection multiplicity $\delta_p(T)$, now computed using Drinfeld's p-adic upper half plane, and a combination of derivatives and values of representation densities of quadratic forms. This coincidence has not yet been proved in the case $p = 2$, so that we must again assume (B) in this case. Finally, summing on T, we obtain the equality

$$\begin{aligned} \langle\, \mathcal{Z}(t_1), \mathcal{Z}(t_2)\,\rangle_p\, q_1^{t_1} q_2^{t_2} &= \sum_{\substack{T \\ \mathrm{diag}(T)=(t_1,t_2) \\ B^T = B^{(p)}}} \widehat{\mathcal{Z}}(T)\, q^T \\ &= \sum_{\substack{T \\ \mathrm{diag}(T)=(t_1,t_2) \\ B^T = B^{(p)}}} \mathcal{E}'_{2,T}(\begin{pmatrix} \tau_1 & \\ & \tau_2 \end{pmatrix}, 0, B). \end{aligned}$$

claimed in (ii).