Coxeter Lectures

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Arithmetic theta functions

Lecture I

Among the most classical of modular forms are the theta functions, which arise as generating functions for the representation numbers $r_Q(n)$ of quadratic forms.

For example, if (L, Q) is a lattice of rank m with an integer valued quadratic form Q, then the formal power series

$$\theta(\tau, L) = \sum_{n=0}^{\infty} r_Q(n)\, q^n = \sum_{x \in L} q^{Q(x)}$$

is the q–expansion of a modular form $\theta(\tau, L)$, $q = e(\tau) = e^{2\pi i \tau}$ of weight $\frac{1}{2}m$ for a subgroup $\Gamma_0(N) \subset \mathrm{SL}_2(\mathbb{Z})$, where the level N is determined by (L, Q).

The main theme of these lectures is that certain generating series constructed using quantities in geometry/arithmetic geometry turn out to be the q-expansions of modular forms.

In this lecture, I will discuss two types of examples:

I. generating functions for divisors or 0-cycles on certain complex surfaces $S/\mathbb{C}$. (joint work with John Millson)

II. generating functions for divisors or 0-cycles on certain arithmetic surfaces $\mathcal{M}/\mathrm{Spec}\,(\mathbb{Z})$. (joint work with Michael Rapoport and Tonghai Yang)

The modular forms coming from complex geometry are indeed theta functions attached to indefinite quadratic forms, and the methods of proof have their roots in the classical work of Siegel on such functions.

The modular forms coming from arithmetic geometry are more exotic. They behave in many ways like the theta functions in the complex case, however, hence the terminology 'arithmetic theta functions'.

I. Cycles on complex surfaces

The idea of constructing generating functions for the (cohomology classes of) curves on a Hilbert modular surface goes back, of course, to Hirzebruch and Zagier.

§1. Complex surfaces.

The complex surfaces in question are constructed as follows:

$$\begin{aligned} V &= \mathbb{Q}\text{–vector space with } \dim_{\mathbb{Q}} V = 4, \\ (\ ,\ ) &= \text{symmetric bilinear form on } V \text{ of signature } (2,2), \\ H &= SO(V), \qquad H(\mathbb{R}) \simeq SO(2,2) \end{aligned}$$

The associated symmetric space is

$$\begin{aligned} D &= \{\ z \subset V(\mathbb{R}) \ \mid\ z = \text{orient. 2-plane},\ (\ ,\ )|_z < 0\ \} \quad \subset \quad \mathrm{Gr}_2(V(\mathbb{R})) \\ &\simeq SO(2,2)/SO(2)\times SO(2) \\ &\simeq (\mathfrak{H}^{\pm}\times\mathfrak{H}^{\pm})_0 \end{aligned}$$

Here $\mathfrak{H}^{\pm} = \mathbb{P}^1(\mathbb{C})\backslash\mathbb{P}^1(\mathbb{R})$, and the subscript 0 indicates that either both components lie in $\mathfrak{H}^+$ or both components lie in $\mathfrak{H}^-$.

Let

$$
\begin{aligned}
Q(x) &= \frac{1}{2}(x,x),\\
L &= \text{lattice on which } Q \text{ is } \mathbb{Z}\text{–valued},\\
\Gamma_L &:= \{ \gamma \in H(\mathbb{Q}) \mid \gamma L = L \} \quad \subset H(\mathbb{Q}),\\
\Gamma &\subset \Gamma_L, \qquad \text{finite index}.
\end{aligned}
$$

The quotient

$$S = S_\Gamma = \Gamma \backslash D$$

is a quasi-projective surface.

Assume: (i) V is anisotropic over $\mathbb{Q}$. This implies that S is projective.
(ii) Γ is sufficiently small. This implies that S is smooth.
Thus we have the usual cohomology groups $H^\bullet(S)$.

§2. Curves on S.

Certain algebraic cycles on S can be described in terms of lattice vectors.
For $x \in L$ with $Q(x) > 0$, there is a curve $D_x \subset D$,

$$D_x = \{ z \in D \mid (x,z) = 0 \} \simeq \mathfrak{H}.$$

Let Γ_x be the stabilizer of x in Γ. Then, we get a curve

$$
\begin{array}{ccc}
D_x & \hookrightarrow & D \\
\downarrow & & \downarrow \\
i_x : \Gamma_x \backslash D_x & \longrightarrow & S
\end{array}
\qquad \text{with image } Z(x) \subset S.
$$

Note: $Z(x)$ depends only on the Γ–orbit of x.

For $t \in \mathbb{Z}_{>0}$, let

$$Z(t) = \sum_{\substack{x \in L \\ Q(x)=t \\ \bmod \Gamma}} Z(x),$$

and let

$$[Z(t)] \in H^2(S),$$

be the corresponding cohomology class.

Define the generating series

$$\phi_1(\tau) := \; [Z(0)] \; + \; \sum_{t>0} [Z(t)] \; q^t \quad \in \; H^2(S)[[q]],$$

where

$$[Z(0)] = [\Omega], \qquad \Omega = \text{a suitable Kähler form.}$$

The analogue of the Hirzebruch–Zagier result is the following:

Theorem. (K.-Millson) *The generating function $\phi_1(\tau)$ for the cohomology classes of the divisors $Z(t)$ on S is a holomorphic modular form of weight 2 valued in $H^2(S)$.*

Here, $\tau = u + iv \in \mathfrak{H}$ and $q = e(\tau) = e^{2\pi i \tau}$.

§3. 0–cycles on S.

Cycles of codimension 2 are associated to pairs of vectors. For

$$x = [x_1, x_2] \in L^2, \qquad \text{with } Q(x) = \frac{1}{2}\big((x_i, x_j)\big) > 0,$$

$$U_x = \text{span of } x_1 \text{ and } x_2 \text{ in } V$$

$$D_x = \{\ z \in D \mid (x, z) = 0\ \} = U^{\perp} = \text{point}$$

and let

$$Z(x) = \text{image of } D_x \text{ in } S.$$

Again, $Z(x)$ depends only on the Γ–orbit of x.
Then, for a fixed $T \in \mathrm{Sym}_2(\mathbb{Z})_{>0}$, let

$$Z(T) = \sum_{\substack{x \in L^2 \\ Q(x)=T \\ \bmod \Gamma}} Z(x).$$

Then $Z(T)$ is a 0–cycle on S with class $[Z(T)] \in H^4(S)$. Of course, under the degree isomorphism $H^4(S) \simeq \mathbb{C}$,

$$[Z(T)] = \deg(Z(T)) = \#\text{ of points in } Z(T).$$

We can form part of a generating function

$$\sum_{T>0} [Z(T)]\, q^T \quad \in \ H^4(S)[[q]],$$

but terms associated to singular T's must still be added.

If $T \in \mathrm{Sym}_2(\mathbb{Z})_{\geq 0}$ with $\det(T) = 0$ has rank 1, the definitions just given yield:

$$\begin{aligned} Q(x) = T &\implies x_1 \text{ and } x_2 \text{ span a line} \\ &\implies Z(x) = \text{curve on } S, \text{ as before,} \\ &\implies Z(T) = \text{curve on } S, \text{ so } [Z(T)] \ \in H^2(S), \\ &\implies [Z(T)] \cup [\Omega] \ \in H^4(S), \end{aligned}$$

Of course, under the degree isomorphism,

$$[Z(T)] \cup [\Omega] = \mathrm{vol}(Z(T), \Omega) = \int_{Z(T)} \Omega,$$

is the volume of the curve $Z(T)$.

Finally, for $T = 0$, we set

$$[Z(0)] = [\Omega^2] = \mathrm{vol}(S, \Omega^2) = \int_S \Omega^2,$$

and we obtain the complete generating function

$$\begin{aligned} \phi_2(\tau) &= \sum_{T \in \mathrm{Sym}_2(\mathbb{Z})_{\geq 0}} [Z(T)] \ q^T \\ &= \mathrm{vol}(S) + \sum_{\substack{T \geq 0 \\ \mathrm{rank}(T) = 1}} \mathrm{vol}(Z(T))\, q^T + \sum_{T > 0} \deg(Z(T)) \ q^T. \end{aligned}$$

Let $\tau = u + iv \in \mathfrak{H}_2$, the Siegel space of genus 2 and let

$$q^T = e(\mathrm{tr}(T\tau)).$$

Theorem. (K.-Millson) *The generating function $\phi_2(\tau)$ for the family of cohomology classes $[Z(T)]$ on S is a holomorphic Siegel modular form of weight 2 and genus 2.*

II. Cycles on arithmetic surfaces

We now turn to the generating functions for cycles on arithmetic surfaces associated to Shimura curves.

§4. Arithmetic surfaces for Shimura curves.

The arithmetic theta function of the title will be a generating function for curves on the arithmetic surface attached to a Shimura curve over $\mathbb{Q}$.

To define the Shimura curve over $\mathbb{C}$, let

$$\begin{aligned}
B &= \text{indefinite quaternion algebra over } \mathbb{Q} \\
D(B) &= \text{product of ramified primes} \\
O_B &= \text{a maximal order in } B \\
V &= \{x \in B \mid \operatorname{tr}(x) = 0\}, \qquad Q(x) = -x^2 \\
\operatorname{sig}(V) &= (1,2) \\
D &= \{w \in V(\mathbb{C}) \mid (w,w) = 0,\ (w,\bar{w}) < 0\ \}/\mathbb{C}^\times \\
&\simeq \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R}) \\
H &= B^\times = \operatorname{GSpin}(V) \\
K &= \widehat{O}_B^\times \subset H(\mathbb{A}_f) \\
M(\mathbb{C}) &\simeq H(\mathbb{Q})\backslash \Big(D \times H(\mathbb{A}_f)/K \ \Big) \\
M &= \text{the canonical model of the Shimura curve over } \mathbb{Q} \\
&\qquad \text{attached to } B.
\end{aligned}$$

In this setup, one has classical modular forms, the Shimura-Shintani-Niwa-Waldspurger (theta) correspondence with modular forms of half integral weight, etc.

A model for M over $\mathrm{Spec}\,(\mathbb{Z})$ is defined via moduli. Let

$$\begin{aligned} \mathcal{M} &= \text{moduli stack over } \mathrm{Spec}\,(\mathbb{Z}) \text{ for } (A,\iota)\text{'s} \\ A &= \text{abelian surface} \\ \iota &: O_B \hookrightarrow \mathrm{End}(A) \\ &\qquad \text{an action of } O_B \text{ on } A \\ &\qquad \text{satisfying Drinfeld's special condition.} \end{aligned}$$

Then

$$M = \mathcal{M}_{\mathbb{Q}} = \mathcal{M} \times_{\mathbb{Z}} \mathbb{Q}.$$

$$p \nmid D(B) \implies \mathcal{M} \text{ has good reduction at } p.$$

$$p \mid D(B) \implies \mathcal{M} \text{ has bad reduction at } p \text{ and the fiber } \mathcal{M}_p$$
$$\text{for such } p \text{ has a } p\text{-adic uniformization.}$$

§5. Curves on $\mathcal{M}$.

The arithmetic cycles in $\mathcal{M}$ are defined by imposing additional endomorphisms:

Define: Special endomorphisms of (A,ι):

$$V(A,\iota) = \left\{ x \in \mathrm{End}(A) \,\middle|\, \begin{array}{c} x\,\iota(b) = \iota(b)\,x, \quad \forall b \in O_B \\ \mathrm{tr}(x) = 0. \end{array} \right\}$$

This $\mathbb{Z}$-module has a quadratic form

$$Q : V(A,\iota) \longrightarrow \mathbb{Z}$$

defined by

$$x^2 = -Q(x) \cdot 1_A.$$

Define: For an integer $t > 0$,

$$\begin{aligned} \mathcal{Z}(t) := \ & \text{locus in } \mathcal{M} \text{ of } (A, \iota, x)\text{'s} \\ & x \in V(A, \iota) \text{ with } Q(x) = t \end{aligned}$$

A few features of the geometry of $\mathcal{Z}(t)$ are as follows:

$$\begin{aligned} \mathcal{Z}(t) &= \text{a (possibly reducible) curve on } \mathcal{M}. \\ \mathcal{Z}(t)(\mathbb{C}) &= \text{set of } (A, \iota) \text{ over } \mathbb{C} \text{ with CM by } \mathbb{Z}[\sqrt{-t}] \\ & \qquad \text{i.e., CM points on the Shimura curve } M. \end{aligned}$$

The horizontal part of $\mathcal{Z}(t)$ is the closure in $\mathcal{M}$ of these points.

One interesting consequence of the modular definition of $\mathcal{Z}(t)$ is that vertical components can occur in the fibers of bad reduction.

Example:

- If the field $\boldsymbol{k}_t = \mathbb{Q}(\sqrt{-t})$ does not split B, then $\mathcal{Z}(t)_{\mathbb{Q}} = \emptyset$.
- If there is a unique prime $p \mid D(B)$ which is split in $\boldsymbol{k}_t$, then $\mathcal{Z}(t)$ is a *purely vertical cycle* consisting of components of $\mathcal{M}_p$. Replacing t by $t\,p^{2r}$ 'thickens' this vertical cycle!!
- In general, if $\boldsymbol{k}_t$ splits $B^{(p)}$, then $\mathcal{Z}(t)$ has vertical components $\iff$ $\mathrm{ord}_p(t) \geq 2$.

§6. Arithmetic Chow groups and Green functions.

We would like to define a generating function for the $\mathcal{Z}(t)$'s. To do this we need to take their classes in the arithmetic Chow group $\widehat{CH}^1(\mathcal{M})$ of the arithmetic surface $\mathcal{M}$, which plays the role of $H^\bullet(S)$.

To define the classes of the $\mathcal{Z}(t)$'s in the arithmetic Chow group, we need to add Green functions. Recall that $\widehat{CH}^1(\mathcal{M})$ is defined as the quotient of:

$$\hat{Z}^1(\mathcal{M}) = \left\{ \begin{array}{c} (\mathcal{Z}, g) \mid \\ \mathcal{Z} = \text{divisor with } \mathbb{R}\text{-coefficients on } \mathcal{M} \\ g = \text{Green function for } \mathcal{Z} \end{array} \right\}$$

by the $\mathbb{R}$ span of the relations

$$\widehat{\operatorname{div}}(f) = (\operatorname{div}(f), -\log|f|^2),$$

for $f \in \mathbb{Q}(\mathcal{M})^\times$, a nonzero rational function on $\mathcal{M}$.
Here, g is a function on $\mathcal{M}(\mathbb{C})$, smooth except for a log–singularity along $\mathcal{Z}(\mathbb{C})$, and with

$$dd^c g + \delta_{\mathcal{Z}} = [\omega].$$

- For a vertical divisor $\mathcal{Z}$, there is a class

$$(\mathcal{Z}, 0) \in \widehat{CH}^1(\mathcal{M}).$$

- For a smooth function ϕ on $\mathcal{M}(\mathbb{C})$, there is a class

$$(0, \phi) \in \widehat{CH}^1(\mathcal{M}).$$

For the cycles $\mathcal{Z}(t)$ on $\mathcal{M}$, Green's functions can be constructed as follows:
Recall

$$V = \{x \in B \mid \operatorname{tr}(x) = 0\}, \qquad Q(x) = -x^2$$

$$D = \{w \in V(\mathbb{C}) \mid (w, w) = 0,\ (w, \bar{w}) < 0\ \}/\mathbb{C}^\times$$

For $z \in D$ and $x \in V$, let

$$R(x,z) = |(x,w)|^2 |(w,\bar{w})|^{-1}.$$

For $t \neq 0$ and $v \in \mathbb{R}_+^\times$, let

$$\Xi(t,v)(z) = \sum_{\substack{x \in O_B \cap V \\ Q(x)=t}} \beta_1(2\pi v R(x,z)),$$

where

$$\beta_1(r) = \int_1^\infty e^{-ru}\, u^{-1}\, du$$

is the exponential integral.

Proposition. *(i) For $t > 0$, $\Xi(t,v)$ is a Green function for $\mathcal{Z}(t)$, so*

$$\widehat{\mathcal{Z}}(t,v) = (\mathcal{Z}(t), \Xi(t,v)) \in \widehat{CH}^1(\mathcal{M}).$$

(ii) For $t < 0$, $\Xi(t,v)$ is a smooth function on $\mathcal{M}(\mathbb{C})$, so

$$\widehat{\mathcal{Z}}(t,v) = (0, \Xi(t,v)) \in \widehat{CH}^1(\mathcal{M}).$$

§7. The arithmetic theta function $\hat{\phi}_1(\tau)$.

The arithmetic theta function is now the generating function for the classes $\widehat{\mathcal{Z}}(t,v)$: for $\tau = u + iv \in \mathfrak{H}$, $\qquad q = e(\tau) = e^{2\pi i \tau}$,

$$\hat{\phi}_1(\tau) = \sum_{t \in \mathbb{Z}} \widehat{\mathcal{Z}}(t,v)\, q^t \quad \in \ \widehat{CH}^1(\mathcal{M}),$$

with the constant term

$$\widehat{\mathcal{Z}}(0,v) = -\hat{\omega} - (0, \log(v)) + (0, \mathbf{c})$$

$$\hat{\omega} = \text{metrized Hodge line bundle on } \mathcal{M},$$

$$= \epsilon^* \Omega^2_{\mathcal{A}/\mathcal{M}}, \qquad \mathcal{A} \longrightarrow \mathcal{M} \quad \text{univ. ab.sch.}$$

and $\mathbf{c}$ is the real constant:

$$\langle \hat{\omega}, \hat{\omega} \rangle - \zeta_{D(B)}(-1) \left[2\frac{\zeta'(-1)}{\zeta(-1)} + 1 - \log(4\pi) - \gamma - \sum_{p|D(B)} \frac{p\log(p)}{p-1} \right],$$

where $\zeta_{D(B)}(s) = \zeta(s) \prod_{p|D(B)} (1-p^{-s})$.

The constant $\mathbf{c}$ arises because we do not, at present, know the quantity $\langle \hat{\omega}, \hat{\omega} \rangle$, for the height pairing $\langle\ ,\ \rangle$ on $\widehat{CH}^1(\mathcal{M})$.

Conjecture: $\mathbf{c} = 0$.

A basic result is then:

Theorem. *$\hat{\phi}_1(\tau)$ is a (non-holomorphic) modular form of weight $\frac{3}{2}$ valued in $\widehat{CH}^1(\mathcal{M})_{\mathbb{C}}$.*

This result, which is obtained by analyzing the various components of $\hat{\phi}_1(\tau)$ for the decomposition of the arithmetic Chow group, will be discussed in Lecture II.

§8. 0–cycles on $\mathcal{M}$.

To define cycles of codimension 2 on $\mathcal{M}$, we impose a pair of special endomorphisms. Thus, for $T \in \mathrm{Sym}_2(\mathbb{Z})_{>0}$, let

$$\mathcal{Z}(T) = \text{ locus of } (A, \iota, \mathbf{x}), \qquad \mathbf{x} = [x_1, x_2] \in V(A,\iota)^2, \qquad Q(\mathbf{x}) = T.$$

For all $T > 0$, $\mathcal{Z}(T)$ is either empty or is a cycle supported in a single fiber $\mathcal{M}_p$ for a prime p determined by T in a simple way. We consider only T for which the cycle is nonempty. If $p \nmid D(B)$, then $\mathcal{Z}(T)$ is a 0–cycle in $\mathcal{M}_p$, and we obtain an associated class

$$\widehat{\mathcal{Z}}(T) = (\mathcal{Z}(T), 0) \in \widehat{CH}^2(\mathcal{M}),$$

the second arithmetic Chow group of $\mathcal{M}$ with real coefficients. The arithmetic degree map defines an isomorphism

$$\widehat{\deg} \ : \ \widehat{CH}^2(\mathcal{M}) \xrightarrow{\sim} \mathbb{R},$$

and

$$\widehat{\deg} \ (\widehat{\mathcal{Z}}(T)) = \log |R(T)|,$$

where

$$\mathcal{Z}(T) = \mathrm{Spec}\,(R(T))$$

for an Artin ring $R(T)$. Such T's will be called regular. In addition, there are positive definite T's where $\mathcal{Z}(T)$ is supported in a fiber with $p \mid D(B)$, and where $\mathcal{Z}(T)$ is a combination of components of $\mathcal{M}_p$. In such an irregular case, the definition of $\widehat{\mathcal{Z}}(T)$ is more complicated.

For $\tau = u + iv \in \mathfrak{H}_2$, we can then form a generating series

$$\begin{aligned}\hat{\phi}_2(\tau) = \widehat{\mathcal{Z}}(0,v) \; + & \sum_{\substack{T\in \mathrm{Sym}_2(\mathbb{Z}) \\ \mathrm{rank}(T)=1}} \widehat{\mathcal{Z}}(T,v)\; q^T \\ & + \sum_{\substack{T\in \mathrm{Sym}_2(\mathbb{Z}) \\ \mathrm{sig}(T)=(1,1) \text{ or } (0,2)}} \widehat{\mathcal{Z}}(T,v)\; q^T \\ & + \sum_{\substack{T\in \mathrm{Sym}_2(\mathbb{Z}) \\ T>0}} \widehat{\mathcal{Z}}(T)\, q^T,\end{aligned}$$

where $q^T = e(\mathrm{tr}(T\tau))$.

Here, the terms for T of signature $(1,1)$ and $(0,2)$ as well as the terms for all singular T's, including $T=0$ must be defined. More will be said about these terms in Lecture III.

In any case, the series $\hat{\phi}_2(\tau)$ is the analogue of the generating function for 0–cycles on the complex surface S.

Conjecture. *$\hat{\phi}_2(\tau)$ is a Siegel modular form of weight $\frac{3}{2}$.*

A more precise version of this conjecture is the following:
There is a Siegel Eisenstein series $\mathcal{E}_2(\tau, s, B)$ of weight $\frac{3}{2}$ and genus 2 associated to B. In the Langlands normalization, this series converges in the halfplane $\mathrm{Re}(s) > \frac{3}{2}$, has a meromorphic analytic continuation to the whole s–plane with a functional equation relating s and $-s$. Moreover,

$$\mathcal{E}_2(\tau, 0, B) = 0.$$

Conjecture.

$$\hat{\phi}_2(\tau) = \mathcal{E}_2'(\tau, 0, B).$$

This conjecture is almost proved. The point is that it is possible to compare the two sides one Fourier coefficient at a time and to check that:

$$\widehat{\mathcal{Z}}(T, v)\, q^T \overset{??}{=} \mathcal{E}_{2,T}'(\tau, 0, B)$$

for all T. For example, the following cases have been checked:

(i) T of signature $(1,1)$ and $(0,2)$,

(ii) $T > 0$ with $p \neq 2$,

(iii) T of rank 1.

Some of these results will be described in more detail in Lecture III.